On the Pinsky Phenomenon

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For an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$, consider a positive self-adjoint extension of the Laplace operator $L = -\Delta$ with discrete spectrum consisting of eigenvalues$\lambda_k \to +\infty$ that correspond to a complete orthonormal system of eigenfunctions $\{u_k(x)\}$ in $L_2(\Omega)$. The Riesz means of order $s \geq 0$ of the spectral expansion of an arbitrary function $f \in L_2(\Omega)$ are given by the relation

$$E_\lambda^s f(x) \;=\; \sum_{\lambda_k < \lambda} \left(1 - \lambda_k/\lambda\right)^s (f, u_k) u_k(x). \tag{1}$$

It is well known that the convergence or summability of spectral expansions is provided by a sufficiently high smoothness of the function to be expanded; moreover, the larger the dimension of the domain, the higher the required degree of smoothness is (see [1]).

A problem on converse theorems was posed in the monograph [8] (Sec. 18.14): what kind of regularity of a function in a neighborhood of some point does follow from the convergence or Riesz summability of its spectral expansion at that point? Obviously, the convergence at some point does not imply even the continuity of the function at that point. To justify this, it suffices to consider an arbitrary odd function: all terms of its expansion in a multiple Fourier series are zero at the origin; therefore, the series is convergent, but the function is not necessarily continuous. Nevertheless, in this situation one can speak of continuity and even smoothness of the function in some generalized sense.

For an arbitrary $x_0 \in \Omega$ consider the ball of radius R with center a point $x_0 \in \Omega$ where R is a positive real not exceeding the distance of the point x to the boundary $\partial\Omega$ of the domain Ω. Let $f_0(x)$ be a function which is smooth in the closed ball $\{x \in \Omega \;:\; |x - x_0| \leq R\}$ and vanishes outside this ball. Then, according to the Pinsky phenomenon, $E_\lambda f_0(x_0) \to f(x_0)$ if and only if the following condition

$$\int_{|x-x_0|=R} f(x)\, d\sigma(x) \;=\; 0$$

is satisfied. Moreover, if the function f is equal to the sum of the functions f_k of such kind with different R_k and the same x_0, the result is similar: interference cannot remove the divergence of the spectral expansion (see [5],[6], where spectral decomposition of Laplace-Beltrami operator in homogeneous Riemannian manifolds was studied). These investigations were continued in many works, see, e. g. [7], [2]-[4].

For each $\alpha > 0$, we define the weighted mean of order α of a function $f \in L(\Omega)$ over a ball of radius r with center a point $x \in \Omega$ (see [8, Sec. 18.14) as follows:

$$S_r^\alpha f(x) \;=\; \frac{1}{\omega(n,\alpha)\, r^n} \int_{|y|\leq r} \left(1 - \frac{|y|^2}{r^2}\right)^{\alpha-1} f(x+y)\, dy, \tag{2}$$

where $\omega(n,\alpha)$ is a normalizing factor. In addition, we introduce the mean value over the sphere with radius r and center the point $x \in \Omega$,

$$S_r f(x) \;=\; \frac{1}{\omega_n} \int_\theta f(x + r\theta)\, d\theta, \quad \text{where} \quad \omega_n \;=\; 2\pi^{n/2}/\Gamma(n/2). \tag{3}$$

Throughout the following, $S_r^\alpha f(x)$ with $\alpha = 0$ stands for the average (3) over the sphere. We discuss the connection of the following results with the Pinsky phenomenon.

Let $\alpha \geq 0$ and $s \geq 0$ be numbers satisfying the condition $s - \alpha < (n-3)/2$; in addition, let the expansion of a function $f \in L_2(\Omega)$ at some point $x \in \Omega$ be summable by Riesz means of order s:

$$\lim_{\lambda\to\infty} E_\lambda^s f(x) \;=\; f(x). \tag{4}$$

Then there exists a function $\phi_\alpha(r)$ continuous on the interval $0 < r < \mathrm{dis}\{x, \partial\Omega\}$ and such that $\phi_\alpha(r) = S_r^\alpha f(x)$ almost everywhere on that interval and the relation $\lim_{r\to 0} \phi_\alpha(r) \;=\; f(x)$ is valid. Therefore, if the spectral expansion of a function $f \in L_2(\Omega)$ is summable by Riesz means of some order $s \geq 0$ at some point, then there always exists an exponent $\alpha \geq 0$ such that the weighted means of this function of order α over the ball with center at that point depend continuously on the radius.

The following assertion is the generalization of the previous result. Let real numbers $s \geq 0$ and $\alpha \geq 0$ and a positive integer m satisfy the condition $s+m-\alpha < (n-3)/2$. If relation (4) holds, then the function $\phi_\alpha(r) = S_r^\alpha f(x)$ is m times continuously differentiable on the interval $0 < r < \mathrm{dist}\{x, \partial\Omega\}$.

The proof uses the integral representation of the weighted means (2) via the Riesz means (1) obtained for the case in which the latter are convergent. Namely, in this case the relation

$$S_r^\alpha f(x) = \frac{C(n,\alpha,s)}{r^{\,n/2+\alpha-s-2}} \int_0^\infty (\sqrt{\lambda})^{s-\alpha-n/2} J_{n/2+\alpha+s}(r\sqrt{\lambda}) E_\lambda^s f(x)\, d\lambda \tag{5}$$

holds for an arbitrary point $x \in \Omega$ and for each r in the interval $0 < r < \mathrm{dist}\{x, \partial\Omega\}$, where $J_\nu(z)$ is a Bessel function of the first kind and $C(n, \alpha, s)$ is some positive constant. Using the theorem of Abelian type and taking into consideration (5) we obtain the above-mentioned results.

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Derivations on Operator Algebras

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Given an algebra A, a linear operator $D : A \to A$ is called a *derivation*, if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$. Each element $a \in A$ implements a derivation D_a on A as $D_a(x) = ax - xa,\ x \in A$. Such derivations are said to be *inner derivations.* If the element implementing the derivation D_a belongs to a larger algebra B containing A. Then D_a is called a *spatial derivation* on A. In this talk we discuss derivations on algebras of operators on a Hilbert space, emphasizing their properties such as innerness and spatiality. These notions are very important in the structure theory and cohomology of abstract rings and algebras and at the same time they have deep applications in mathematical physics, in particular in the problem of constructing the dynamics in quantum statistical mechanics (see [4,5]).

Therefore we also discuss a physical background of derivations on operator algebras. After expositions of some well-known results on derivation on C^*- algebras and von Neumann algebras we consider open problems concerning derivations on algebras of measurable operators affiliated with von Neumann algebras, posed and partially solved in [1-3].

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Analytic quasiperiodic Schrodinger operators and matrix cocycles

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Analytic quasiperiodic matrix cocycles is a simple dynamical system, where analytic and dynamical properties are related in an unexpected and remarkable way. We will focus on this relation, leading to a new approach to the proof of joint continuity of Lyapunov exponents in frequency and cocycle, at irrational frequency, first proved for $SL(2, C)$ cocycles in Bourgain-Jitomirskaya, 2002. The approach is powerful enough to handle singular and multidimensional cocycles, thus establishing the above continuity in full generality. This has important consequences including a dense open version of Bochi-Viana theorem in this setting, with a completely different underlying mechanism of the proof. A large part of the talk is a report on a joint work with A. Avila and C. Sadel.

Counting nodal domains on surfaces with concave boundary

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It is conjectured that any Riemannian manifold has a sequence of eigenfunctions for which the number of nodal domains tends to infinity. But there are almost no Riemannian manifolds for which this has been proved. Recently there have been some results on non-positively curved surfaces. I will present one result, joint with Junehuyk Jung, that for a surface with concave boundary and ergodic billiards, the number of nodal domains tends to infinity for almost the full sequence of eigenfunctions.

The Semiclassical Limit of Thomas Fermi Theory - Forty Years After

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In Commemoration of the 40th Anniversary of our work, I talk about the history of and proof of the fact that Thomas Fermi is an exact approximation in the limit of infinite Z atoms.

Counting nodal domains on surfaces with concave boundary

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Infinite Dimensional Superalgebras

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We will discuss structure and representations of infinite dimensional Lie and Jordan superalgebras that are close to the so called Superconformal Algebras.
