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Inverse problems of determining the order of fractional derivatives of partial differential equations

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ABSTRACT

When considering fractional differential equations as a model equation in the analysis of various anomalous processes, the order of fractional derivatives is often unknown and difficult to directly measure, which requires a discussion of the inverse problem of identifying this physical quantity from some indirectly observed information about the solutions. Inverse problems of determining these unknown parameters are not only of theoretical interest but are also necessary for finding a solution to the initial-boundary value problem and studying the properties of solutions. This series of lectures discusses methods for solving such inverse problems for the equations of mathematical physics. It is assumed that students are familiar with the theory of partial differential equations and elements of functional analysis.

Topics:
1. Fractional derivatives. Mittag-Leffler functions. Sobolev's embedding theorem. Fractional powers of elliptic operators.
2. Classical solution of forward problems for subdiffusion equations and the first method for solving inverse problems.
3. Generalized solutions of forward problems and the second method for solving inverse problems.

Introduction.

The main object of study in these lectures are differential equations of fractional order. The theory of such equations has gained considerable popularity mainly in the past few decades, due to its applications in numerous fields of science and technology.

[1] J.A.T. Machado, Editor: Handbook of fractional calculus with applications. V. 1 - 8. DeGruyter, (2019).
[2] S. Umarov, M. Hahn, K. Kobayashi, Beyond the Triangle: Brownian Motion, Itô Calculas, and Fokker-Plank Equation - Fractional Generalizations. World Scientific (2018).
[3] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rev. 339. No 1 (2000), 1–77.
[4] R. Hilfer, Editor: Applications of fractional calculas in physics. Singapore, World Scientific (2000).
[5] V.V. Uchaikin, Fractional derivatives for Physicists and Engineers, 1, Background and Theory. 2, Application, Springer (2013).

In the theory of differential equations (of integer or fractional order) there are two types of problems: forward (or direct) and inverse problems. How are these tasks defined? Consider this in the following example of an initial-boundary value problem:

$$u_t(x,t) - a(x,t)\Delta u(x,t) = f(x,t), \quad x \in \Omega \subset R^N, \ t > 0,$$

$$u(x,t) = \psi(x,t), \quad x \in \partial\Omega, \ t \geq 0,$$

$$u(x,0) = \varphi(x), \quad x \in \overline{\Omega},$$

where $\Delta u(x,t) = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} u(x,t)$- the Laplace operator.

Note that the interest in studying inverse problems for equations of mathematical physics is due to the importance of their applications in many branches of modern science, including mechanics, seismology, medical tomography, epidemics, and geophysics, just to mention a few. A significant number of studies are devoted to inverse problems of determining the right-hand side of subdiffusion equations.

The present lectures are devoted to the other important type of inverse problems, namely to determining of the order of fractional derivative. Suppose that when modeling some process from real life, we come to a fractional order equation

$$\partial_t^\rho u(x,t) = a(x,t)\Delta u(x,t) + f(x,t), \ \ \rho \in (0,1).$$

One of the practical example is a modeling of COVID-19 outbreak. The data presented by Johns Hopkins University about the outbreak from different countries seem to show fractional order dynamical processes.

Assume that ρ is unknown. There are many ways and equipment to measure and determine $a(x,t)$ and $f(x,t)$. But unfortunately there is no equipment for measuring the order of the derivative ρ.

The problem of identification of fractional order of the model was considered by many researchers. Note that all the publications assumed the fractional derivative of order $0 < \rho < 1$ in the sense of Caputo and studied mainly the uniqueness problem.

One more note. Since ρ is an additional unknown, one more condition is needed to find it. So in all known works, the authors used the following condition as an additional condition:

$$u(x_0, t) = h(t), \;\; 0 < t < T,$$

at a monitoring point $x_0 \in \Omega$.

This condition seems to be natural: since we are looking for the order of the derivative with respect to t, then we must know the nature of the solution over a long period of time.

However, using this condition, as a rule, it is possible to prove only the uniqueness of the parameter ρ. An exception is the work [15] where both uniqueness and existence are proved.
The complexity of the proof of the existence can be seen from the statement of corresponding theorem (Theorem 7.2 is formulated on more than one journal page, p. 16-17).

[15] Janno, J. Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time-fractional diffusion equation. Electronic J. Differential Equations V. 216(2016), pp. 1-28,

In this regard, in the survey paper [16] (p. 440) by Z. Li et al. in the Open Problems section the following problem is formulated:

Problem 1. Is it possible to identify the order of fractional derivatives if an additional information has the form

$$F(u(\cdot, t_0)) = d_0, \tag{1}$$

with some linear functional F? $(u(x_0, t) = d_0(t),\ 0 < t < T)$

[16] Z. Li, Y. Liu, M. Yamamoto, Inverse problems of determining parameters of the fractional partial differential equations. Handbook of fractional calculus with applications. 2, DeGruyter (2019).

In this series of lectures, we will propose two methods for solving the inverse problem of determining the order of fractional derivatives, based on new over-determination (additional) conditions in the form (1). Thus we give a positive solution to Problem 1.

The advantage of these methods is that these methods can prove both the uniqueness of an unknown parameter and its existence. Moreover, this method can be applied to subdiffusion equations involving Riemann-Liouville derivatives. One can also apply this method to fractional order wave equations (that is $\rho \in (1,2)$) and to some equations of mixed type.

A few words about how we came to these problems. In 2020, when the pandemic began, the Academy of Sciences of Uzbekistan turned to our institute with a request to predict the spread of coronavirus in our republic using mathematical modeling. At that time, the chief researcher of our laboratory, Umarov, who is also a professor at the University of New Haven, USA, was working on this problem together with colleagues from the University of California. They have compiled a mathematical model of this problem and since April 2020 have been providing a daily forecast for the spread of coronavirus in our republic and in neighboring countries. It turned out that the system of differential equations modeling this process is a system of non-linear differential equations of fractional order.

As an example of a model equation, we will consider an initial-boundary value problem for a single linear differential equation of fractional order:

$$\partial_t^\rho u(x,t) = \Delta u(x,t), \quad x \in \Omega \subset R^N, \ t > 0, \tag{2}$$

$$Bu(x,t) \equiv \frac{\partial u(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \ t \geq 0, \tag{3}$$

$$\lim_{t \to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \overline{\Omega}, \tag{4}$$

where n is the unit outward normal vector of $\partial\Omega$. This is also called Direct or Forward problem.

In this problem $u(x,t)$ means the number of people infected with the coronavirus at the time t in the geographic location x. Condition (3) means that the boundaries of the region Ω are closed (i.e., there is no flow through the boundaries).

Data presented by Johns Hopkins University on outbreaks of COVID-19 from different countries show that the order of the fractional derivative ρ in this problem lies in the interval $(0, 1)$. Unfortunately, there is no device that could accurately measure this parameter ρ. Umarov and his colleagues tried various parameters ρ, but did not get the desired result. Then we started looking for an analytical way to find this parameter. In this case, we come to the so-called inverse problem of determining the order of the fractional derivative ρ. Since this is an additional unknown, we need to set one more additional condition. The complexity of this inverse problem lies precisely in finding this additional condition. On the one hand, this condition must be easily verifiable and, on the other hand, it must ensure both the uniqueness and the existence of the parameter ρ. It is this condition that we will find in our lectures.

Chapter 1. Fractional derivatives. Mittag-Leffler functions. Sobolev's embedding theorem. Fractional powers of elliptic operators.

Section 1.1. Fractional derivatives. Mittag-Leffler functions.

Subsection 1.1.1. The first attempts to determine the derivative of a fractional order.

As soon as the concept of the derivative $f'(x)$ was introduced, the question immediately arose: is it possible to define the derivative $f^{\left(\frac{1}{2}\right)}(x)$? In the year 1695 G. Leibniz (1646-1716) in his letters to Lopital (Marquis de L'Hopital (1661-1704)) made several remarks about the possibility of considering derivatives of order 1/2.

The first step was taken by L. Euler, in 1738, who noticed that the result of calculating the derivative of a power function

$$(x^n)^{(k)} = \frac{n!}{(n-k)!}x^{n-k}$$

can be given meaning for a non-integer k. In this regard, he introduced a generalization of the factorial, which is now called the Euler gamma function

$$\Gamma(s+1) = s\Gamma(s), \quad \Gamma(n) = (n-1)!$$

Since a wide class of functions can be written in the form of power series, it is possible to define a fractional derivative for such functions in this way.

The next step was taken by J. Fourier (1768-1830) in 1822, who suggested using the equalities

$$f(x) = \sum_{k=0}^{\infty} f_k e^{ikx}, \quad f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy,$$

$$f^{(k)}(x) = \sum_{k=0}^{\infty} (ik)^k f_k e^{ikx},$$

to determine the derivative of a non-integer order. This was the first definition of the derivative of any positive order and of any (enough "good") function.

The facts given above are only the prehistory of fractional calculus. The real story began with the works of N. Abel (1802-1829) and J. Liouville (1809-1882). In works published in 1823 and 1826, N. G. Abel investigated the problem of the tautochrone. To solve this problem, he obtained the integral equation

$$A_\alpha \varphi(x) \equiv \int_a^x \frac{\varphi(t)dt}{(x-t)^\alpha} = f(x), \ x > a, \ 0 < \alpha < 1,$$

where the unknown $\varphi(t)$ is the curve that gives the solution to the tautochrone problem. Abel solved this equation and found the operator inverse to A_α:

$$\varphi(x) = R_\alpha f(x).$$

As it turns out later, A_α is the operation of fractional integration of order $1-\alpha$, and R_α is the operation of fractional differentiation. However, these definitions appeared later.

Let us remind the problem of tautochrone. It is a problem to find a curve along which the ball, placed in any place, will fall to the bottom at the same time. (Such a curve exists because gravity is different at different points on the curve.)

Acquainted with the works of Abel in 1832-1837 J. Liouville wrote a number of works. In these works, he created a fairly complete theory of fractional integro-differentiation.

Next to the works of J. Liouville in importance should be put the work of B. Riemann (1826-1866). B. Riemann created the theory of integro-differentiation, which we use today: he defined fractional integrals and derivatives and studied their properties. He completed this series of works in 1847 during his student years, but they were published in 1876, 10 years after his death.

Thus the first attempt at a systematic development of fractional calculus was made by Liouville (1832) and Riemann (1847) in first half of the nineteenth century.

Subsection 1.1.2. The Riemann-Liouville and Caputo derivatives.

There are different types of fractional derivatives, not always equivalent. We will consider only the most common of them: the Riemann-Liouville and Caputo derivatives.

Let $D = \frac{d}{dt}$ be the classical derivative, and

$$Jf(t) = \int_0^t f(\tau)d\tau.$$

Then $DJf = f$ (but $JDf = f(t) - f(0)$), i.e. J right inverse operator: $J = D^{-1}$. We have

$$J^n f(t) = \int_0^t \cdots \int_0^{\tau_2} f(\tau_1)d\tau_1 \cdots d\tau_n =$$

$$= \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)d\tau.$$

Obviously, $J^n = (D^n)^{-1}$, i.e. $D^n J^n f(t) = f(t)$.

Replacing n with an arbitrary real $\alpha > 0$ and taking into account the relationship $\Gamma(n) = (n-1)!$, we obtain the definition of Riemann - Liouville fractional integrals of an arbitrary order $\alpha > 0$:

$$J^{\alpha} f(t) = \left(\partial_t^{-\alpha} f(t)\right) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Recall that the Euler gamma-function is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \;\; Rez > 0. \tag{5}$$

$\Gamma(z)$ can be analytically extended to the whole complex plain $\mathbb{C}$ except points $z = 0, -1, -2, ...$, which are simple poles of the gamma-function. Using the integration by parts, one can show that $\Gamma(1+z) = z\Gamma(z)$. Obviously, $\Gamma(1) = 1$. These two facts immediately imply $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$ (with the convention $0! = 1$).

Let us remind also the Euler beta function, which is determined by the formula:

$$B(\gamma, \beta) = \int_0^1 \xi^{\gamma-1}(1-\xi)^{\beta-1} d\xi, \ \gamma, \beta \in \mathbb{C}, \ \Re(\gamma) > 0, \ \Re(\beta) > 0. \tag{6}$$

This function is related to the gamma function by the formula:

$$B(\gamma, \beta) = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}. \tag{7}$$

Based on the definition (6) and the property (7), it can be shown that the operator J^α has the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \quad \alpha, \ \beta > 0. \tag{8}$$

By analogy with the case of α integers, the left inverse of J^α is the fractional derivative of order α: $D^\alpha = (J^\alpha)^{-1}$. But even if $0 < \alpha < 1$, we need to give some meaning to the integral $(J^\alpha)^{-1}$.

Let $0 < \alpha < 1$. Consider the operator $DJ^{1-\alpha}$. Let us show that this particular operator is the left inverse of the operator J^α. Indeed, $DJ^{1-\alpha}J^\alpha = DJ = I$.
So, the fractional Riemann-Liouville derivative of order $\alpha, 0 < \alpha < 1$, is defined by the formula:

$$\partial_t^\alpha f(t) = DJ^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau,$$

provided that the right side of the equality exists.

This operator, as we checked above, satisfies $\partial_t^\alpha J^\alpha = DJ^{1-\alpha}J^\alpha = I$, extending the relation $D^n J^n = I$ to any real number $\alpha \in (0, 1)$.

Fractional derivative in the Riemann-Liouville sense of order α, $k-1<\alpha\le k$, $k\in\mathbb{N}$, of the function f defined on $[0,\infty)$, is determined by the formula ($\partial_t^\alpha f(t) = D^k J^{k-\alpha}$)

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha+1-k}}, \quad t>0.$$

If $\alpha=k$, then the fractional derivative coincides with the usual classical derivatives: $\partial_t^k f(t) = \frac{d^k}{dt^k}f(t)$ (In fact, there is an equality $\lim\limits_{\alpha\to k}\partial_t^\alpha f(t) = \frac{d^k}{dt^k}f(t)$).

It is interesting to note that, in general (e.g. $f(t)=t^{\alpha-1}$, $\beta\in(0,\alpha)$ and $\alpha-\beta$ is not integer),

$$\partial_t^\alpha\partial_t^\beta f(t)\ne\partial_t^\beta\partial_t^\alpha f(t), \quad \partial_t^\alpha\partial_t^\beta f(t)\ne\partial_t^{\alpha+\beta} f(t).$$

Fractional derivatives are not as easy to calculate as classical ones. Consider examples.

1. Let $f(t) = t^s$, $s > -1$. Then for any $\alpha < 1$

$$\partial_t^\alpha f(t) = \frac{\Gamma(1+s)}{\Gamma(1+s-\alpha)}\, t^{s-\alpha},\;\; t > 0.$$

Indeed, we have

$$\partial_t^\alpha t^s = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\tau^s}{(t-\tau)^\alpha} d\tau =$$

(use the substitution $\tau = t\xi,\; d\tau = td\xi$)

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^t \xi^s (1-\xi)^{-\alpha} d\xi \;\frac{d}{dt} t^{1+s-\alpha} = \frac{1}{\Gamma(1-\alpha)} B(s+1, 1-\alpha)(1+s-\alpha) t^{s-\alpha} =$$

(apply (7) and formula $\Gamma(2+s-\alpha) = (1+s-\alpha)\Gamma(1+s-\alpha)$)

$$= \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(s+1)\Gamma(1-\alpha)}{\Gamma(2+s-\alpha)}\;(1+s-\alpha) t^{s-\alpha} = \frac{\Gamma(1+s)}{\Gamma(1+s-\alpha)}\, t^{s-\alpha}.$$

2. If $s = 0$, then

$$\partial_t^\alpha 1 = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}.$$

3. Since $\Gamma(0) = \infty$, then

$$\partial_t^\alpha t^{\alpha-1} = 0.$$

Moreover, the general solution of the equation

$$\partial_t^\alpha u(t) = 0$$

has the form: $u(t) = Ct^{\alpha-1}$.

Therefore, if $0 < \alpha < 1$, then the Cauchy problem

$$\begin{cases} \partial_t^\alpha u(t) = 0, \quad t > 0, \\ u(0) = a_0, \end{cases} \tag{9}$$

where $a_0 \neq 0$ has no solutions. (In this case, the initial condition is usually given in the form: $\lim\limits_{t\to+0} t^{1-\alpha} u(t) = a_0$).

It was in connection with this fact that the 60s of the last century were introduced derivatives in the sense of Caputo (Dzhrbashyan, Gerasimov):

$$D_t^\alpha f(t) = J^{1-\alpha} D f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{d}{d\tau} f(\tau)}{(t-\tau)^\alpha} d\tau.$$

Here is the Cauchy problem

$$\begin{cases} D_t^\alpha u(t) = 0, \quad t > 0, \\ u(0) = a_0, \end{cases} \tag{10}$$

has a unique solution $u(t) = a_0$.

The active use of fractional derivatives began in the 60s of the last century. Two points played an important role here:
1. Some processes propagate more slowly (subdiffusion processes), and some faster (superdiffusion processes) than heat;
2. To determine the classical derivatives, information about the function under study is needed only on a very small neighborhood of the function under consideration:

$$f'(t_0) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

And the fractional derivative depends on the behavior of the function in some neighborhood of the point under consideration. Therefore, to model some processes that have a "memory effect", fractional derivatives are used.

Today, fractional differential equations are an effective tool for modeling various processes that arise in science and technology. Fractional models adequately reflect subtle intrinsic properties, such as memory or hereditary properties of complex processes, that classical integer-order models neglect.

1. The first international conference on Fractional Calculus was held in 1974 at the University of New Haven.
2. In MSC (Mathematics Subject Classifications) 2010 there was only one classification number: 26A33, and now there are more than 40!
3. More than 20 international journals included in the SCOPUS database (Fractinal calculus and applied analysis, IF 3.74).
4. If you type "fractinal calculus" on google.com, you'll get over 4 million links!
5. More than 100 books on fractional calculus!

Subsection 1.1.3. The Mittag-Leffler functions.

The solution of the Cauchy problem

$$y'(t) + \lambda y(t) = 0, \;\; y(0) = y_0,$$

has the form

$$y(t) = y_0 e^{-\lambda t}.$$

If we replace the first derivative with a fractional derivative of order $\alpha \in (0, 1)$, then instead of the exponent we get a function named after the great Swedish mathematician Gosta Magnus Mittag-Leffler (1846-1927). He defined it by a power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + 1)}, \;\; \alpha > 0, \;\; z \in C,$$

and studied its properties in 1902-1905 in connection with his summation method for divergent series. This function provides a simple generalization of the exponential function

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

The simplest (and for applications most important) generalizations of the Mittag-Leffler function, namely the two-parametric Mittag-Leffler function has the form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha\, k + \beta)}, \;\; z, \beta \in C, \;\; \alpha > 0. \qquad (11)$$

We recall some estimates for the Mittag-Leffler function. At large t and for arbitrary complex numbers μ and $0 < \rho < 1$ there is an asymptotic estimate

$$E_{\rho,\mu}(-t) = \frac{1}{t}\left(1 + O\left(\frac{1}{t}\right)\right), \quad t > 1, \tag{12}$$

and in particular we have the estimate

$$0 < |E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0. \tag{13}$$

Lemma 1

Let $0 < \rho < 1$ and $\lambda > 0$. If $f(t)$ is continuous, then the Cauchy problem

$$\partial_t^\rho T(t) + \lambda T(t) = f(t), \quad \lim_{t\to 0} \partial_t^{\rho-1} T(t) = \varphi, \quad (\text{or } \lim_{t\to 0} J_t^{1-\rho} T(t) = \varphi) \tag{14}$$

has unique solution and this solution has the form

$$T(t) = \varphi t^{\rho-1} E_{\rho,\rho}(-\lambda t^\rho) + \int_0^t \xi^{\rho-1} E_{\rho,\rho}(-\lambda \xi^\rho) f(t-\xi) d\xi.$$

Proof of this formula can be found e.g. in [6]. When the derivative is in the sense of Caputo, then the solution to this problem can be found, for example in [7].

[6] R. Ashurov, A. Cabada, B. Turmetov, Operator method for construction of solutions of linear fractional differential equations with constant coefficients. Frac. Calculus Appl. Anal. 1 (2016), 229-252.
[7] R. Ashurov, Yu. Fayziev, On construction of solutions of linear fractional differential equations with constant coefficients and the fractional derivatives, Uzb.Math.Journ. 3 (2017), 3–21, (in Russian).

Lemma 2

Let $0 < \rho < 1$ and $\lambda > 0$. If $f(t)$ is continuous, then the Cauchy problem

$$D_t^\rho T(t) + \lambda T(t) = f(t), \quad T(0) = \varphi \tag{15}$$

has unique solution and this solution has the form

$$T(t) = \varphi E_{\rho,1}(-\lambda t^\rho) + \int\limits_0^t \xi^{\rho-1} E_{\rho,\rho}(-\lambda \xi^\rho) f(t-\xi) d\xi.$$

When solving the Cauchy problem, the following lemma is very helpful (see [8] p.53).

Lemma 3

Let $T_0(t) = t^{1-\rho} T(t) \in C[0,1]$ and $0 < \rho < 1$. Then

$$\lim_{t \to 0} \partial_t^{\rho-1} T(t) = \Gamma(\rho) \lim_{t \to 0} t^{1-\rho} T(t).$$

[8] A.V. Pskhu, Fractional partial differential equations (in Russian), M. Nauka (2005).

Proof of Lemma 3. We have

$$\lim_{t\to 0}\partial_t^{\rho-1}T(t)=\frac{1}{\Gamma(1-\rho)}\lim_{t\to 0}\int\limits_0^t\frac{T(s)}{(t-s)^\rho}ds=\frac{1}{\Gamma(1-\rho)}\lim_{t\to 0}\int\limits_0^t\frac{T_0(s)s^{\rho-1}}{(t-s)^\rho}ds=$$

$$=\frac{1}{\Gamma(1-\rho)}\lim_{t\to 0}\int\limits_0^1\frac{T_0(tp)p^{\rho-1}}{(1-p)^\rho}dp=$$

(by virtue of equalities (6)-(7))

$$=\lim_{t\to 0}T_0(t)\frac{1}{\Gamma(1-\rho)}\int\limits_0^1\frac{p^{\rho-1}}{(1-p)^\rho}=\lim_{t\to 0}T_0(t)\frac{1}{\Gamma(1-\rho)}B(\rho,1-\rho)=\Gamma(\rho)\lim_{t\to 0}T_0(t).$$

Using this lemma, the initial condition in (14) can be rewritten in the form

$$\lim_{t\to 0}t^{1-\rho}T(t)=\frac{\varphi}{\Gamma(\rho)},\qquad (16)$$

which is undoubtedly simpler.

Subsection 1.1.4. Logarithmic derivative of $\Gamma(z)$.

Denote by $\Psi(z)$ the lagorithmic derivative of the gamma function ([9], p.15):

$$\Psi(z) = \big(\ln\Gamma(z)\big)' = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Then

$$\Gamma'(z) = \Gamma(z)\Psi(z).$$

It follows from the property of the gamma function (see [9], p.1)

$$\Psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)},$$

where $\gamma \approx 0.577215\cdots$ is the Euler-Mascheroni constant.
The $\Psi(z)$ function is meromorphic with simple poles at $z = 0, -1, -2, \cdots$. Clearly

$$\Psi(1) = -\gamma. \tag{17}$$

Since $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma'(z+1) = z\Gamma'(z) + \Gamma(z)$, then

$$\Psi(z) = \Psi(z+1) - \frac{1}{z}. \tag{18}$$

[9] H. Bateman, Higher transcendental functions, McGraw-Hill (1953).

Section 1.2. Sobolev's embedding theorem.

Subsection 1.2.1 The generalized derivative. The Sobolev space.

Let α - the multiindex and $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ and $D = (D_1, ..., D_N)$. Then

$$D^\alpha u(x) = D_1^{\alpha_1} \cdots D_N^{\alpha_N} u(x) = \left(\frac{1}{i}\right)^{|\alpha|} \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

The convenience of dividing by i in the definition of D_j is that with such a definition we have $D^\alpha e^{inx} = n^\alpha e^{inx}$, $n \in \mathbb{Z}^N$.
Let Ω be an arbitrary domain in $\mathbb{R}^N$. As usual, the space of infinitely differentiable finite functions is denoted by $C_0^\infty(\Omega)$.
If a function $u(x)$ has a sufficient number of derivatives, then it satisfies the identities

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u(x) \varphi(x) dx \tag{19}$$

where $\varphi(x) \in C_0^\infty(\Omega)$. Suppose that for a function $u(x)$ one can indicate an integrable function $v(x)$ such that

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x) \varphi(x) dx \tag{20}$$

where $\varphi(x) \in C_0^\infty(\Omega)$, $\alpha = \{\alpha_1, ..., \alpha_N\}$.

Then we will say that $u(x)$ has a generalized (in the sense of S.L. Sobolev) derivative $D^{\alpha}u\,(x)$, which is defined by the equality

$$D^{\alpha}u(x) = v(x). \tag{21}$$

The generalized derivative is uniquely determined (up to values on a set of measure zero). Identities (21) mean that for smooth functions generalized functions coincide with the usual ones. In what follows, when using generalized derivatives, the word "generalized" will often be omitted.

Example 1. Let

$$h(x) = \begin{cases} 5, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then

$$\int_{-\infty}^{\infty} h(x)\varphi'(x)dx = 5\varphi(0) = \int_{-\infty}^{\infty} 5\delta(x)\varphi(x)dx,$$

that is $h'(x) = 5\delta(x)$.

We denote by $W_2^k = W_2^k(\Omega)$, where $k = 0, 1, 2, ...$, the set of such functions defined on Ω, for which the norm

$$\|u\,(x)\|^2_{W_2^k(\Omega)} = \|u\|^2_{L_2(\Omega)} + \sum_{|\alpha|=k} \|D^\alpha u\|^2_{L_2(\Omega)}\,. \qquad (22)$$

is meaningful and finite. The spaces W_2^k are complete normed spaces; the spaces W_2^0 obviously coincide with the spaces $L_2\,(\Omega)$.
In addition to spaces W_2^k, below we will also use the spaces $C^k = C^k\,(\Omega)$ of $k \geq 0$ times continuously differentiable on Ω functions $u(x)$ with the norm

$$\|u(x)\|_{C^k(\Omega)} = \sum_{0\leq|\alpha|\leq k} \max_{x\in\Omega} |D^\alpha u(x)|\,. \qquad (23)$$

The class $C^0(\Omega)$ will also be denoted as $C(\Omega)$.

The Parseval equality. If $f \in L_2(\Omega)$ and $\{v_k(x)\}$ is a complete orthonormal system in the space $L_2(\Omega)$, then the following Parseval equality is valid:

$$\int_\Omega |f(x)|^2 dx = \sum_{k=1}^\infty |f_k|^2,$$

here f_k are the Fourier coefficients, i.e.

$$f_k = \int_\Omega f(x) v_k(x) dx.$$

Now let $\Omega = \mathbb{T}^N$, i.e. N-dimensional torus: $\mathbb{T}^N = \left\{x \in \mathbb{R}^N : -\pi < x_k \le \pi\right\}$. The classes $W_2^k\left(\mathbb{T}^N\right)$ can also be described in terms of the Fourier coefficients of the function f:

$$f_n = (2\pi)^{-N} \int_{\mathbb{T}^N} f(y) e^{iny} dy.$$

Namely, $f \in W_2^k\left(\mathbb{T}^N\right)$ if and only if the following norm is finite:

$$\|f\|_{W\,_2^k(\mathbb{T}^N)} = \sum_{n \in Z^N} \left(1 + |n|^2\right)^k |f_n|^2. \tag{24}$$

The fact that the (22) and (24) norms are equivalent can be easily checked. Indeed, since $(D^\alpha f)_n = (n)^\alpha f_n$, then the Parseval equality implies that the norm (22) is equivalent to the norm

$$\|f\|_{W_2^k} = \|\sum_n f_n e^{inx}\|^2_{L_2(T^N)} + \|\sum_n \sum_{|\alpha|=k} n^\alpha f_n e^{inx}\|_{L_2(T^N)} =$$

$$\sum_{n \in Z^N} \left(1 + (\sum_{|\alpha|=k} n^\alpha)^2\right) |f_n|^2.$$

This norm is equivalent to the norm (24), since

$$c_1 \left(1 + |n|^2\right)^k \le 1 + (\sum_{|\alpha|=k} n^\alpha)^2 \le c_2 \left(1 + |n|^2\right)^k.$$

Note that in the (22) norm, the exponent k must be natural, since it is not entirely clear what meaning can be attached to this norm when k is noninteger. The situation changes when looking at the equivalent norm (24) – in this definition, k can be any positive number. The class of functions $f \in L_2\left(\mathbb{T}^N\right)$ for which the norm (24) is finite for a given fixed real $a > 0$ is called the Liouville class $L_2^a\left(\mathbb{T}^N\right)$. Thus,

$$\|f\|_{L_2^a(\mathbb{T}^N)} = \sum_{n \in Z^N} \left(1 + |n|^2\right)^a |f_n|^2, \quad a > 0. \tag{25}$$

The classes $L_2^a\left(\mathbb{T}^N\right)$ coincide for natural a with the Sobolev classes $W_2^a\left(\mathbb{T}^N\right)$ and are their natural extensions to fractional values of a.

Subsection 1.2.2. Sobolev's embedding theorem.

Back in 1950, S.L. Sobolev proved [10] that if a is a sufficiently large number, i.e. $a > \frac{N}{2}$, then each function from $L_2^a(\Omega)$, $\Omega \subset \mathbb{R}^N$, (definition see e.g. [11]) is continuous. For classes $L_2^a\left(\mathbb{T}^N\right)$, Sobolev's theorem is proved quite simply. We present the corresponding result with proof.

Teorema 4

(Sobolev) Let $a > \frac{N}{2}$. Then every function $f \in L_2^a\left(\mathbb{T}^N\right)$ is continuous on $\mathbb{T}^N$ and the inequality

$$||f||_{C(\mathbb{T}^N)} \leq C||f||_{L_2^a(\mathbb{T}^N)}$$

holds.

[10] Sobolev, S. L. (1950): Applications of functional analysis in mathematical physics. Leningrad: LGU (256 pp.). English translation: Providence, Rhode Island. Am. Math. Soc. 1963 (239 pp.)
[11] Nikolskii, S. M. (1977): Approximation of functions of several variables and imbedding theorems. 2nd ed. Moscow: Nauka (456 pp.). English translation: New York: Springer- Verlag 1975.

Proof. We have

$$||\sum f_n e^{inx}||_{C(\mathbb{T}^N)} \leq \sum |f_n| = \sum |f_n|(1+|n|^2)^{a/2}(1+|n|^2)^{-a/2} \leq$$

(by Chauchy-Bunyakovsky inequality)

$$\leq \left(\sum \left(1+|n||^2\right)^a \, |f_n|^2 \right)^{1/2} \left(\sum \frac{1}{(1+|n|^2)^a} \right)^{1/2}.$$

Since $a > \frac{N}{2}$, then the last series converges.

It should be noted that for $a = \frac{N}{2}$ the assertion of the theorem is already false: there are examples of functions from the class $L_2^a\left(\mathbb{T}^N\right)$ that are unbounded (see, for example, [12]).

[12] Alimov Sh.A., Ashurov R.R., Pulatov A.K.: Multiple Fourier Series and Fourier Integrals.Commutative Harmonic Analysis, Springer, Berlin (1992)

Section 2.3. Fractional powers of elliptic operators.

Subsection 2.3.1. The powers of elliptic operators and their domains.

An operator A defined in a Hilbert space H with the scalar product $(\cdot, \cdot)$ and the norm $||\cdot||$ is called self-adjoint if it coincides with its adjoint $A^\star$, that is, the corresponding domains $D(A)$ and $D(A^\star)$ coincide and $A = A^\star$ for all $u \in D(A)$. Thus, the operator A is self-adjoint if and only if

$$(Au, v) = (u, Av)\,, \tag{26}$$

for all $u, v \in D\,(A)$.

An operator A is called symmetric (formally self-adjoint) if equality (26) holds on an everywhere dense set $K \subset H$ $(K \subset D(A))$. If $H = L_2(\Omega)$ and A is a differential operator, then K can be taken as $C_0^\infty(\Omega)$.

Operator B with a domain of definition $D(B)$ is an extension of operator A with a domain of definition $D(A) \subset D(B)$ if equality $Au = Bu$ holds for all $u \in D(A)$.

Let A be symmetric semibounded from below by the number μ, i.e. $(Au, u) \geq \mu(u, u)$, for all $u \in D\,(A)$. Then, by the Friedrichs theorem, the operator A has at least one self-adjoint extension $\hat{A}$.

Let $A(D)=\sum_{|\alpha|=m} a_\alpha D^\alpha$ be an elliptic differential operator with constant coefficients, defined on the class of functions $C^\infty(\mathbb{T}^N)$. The class $C^\infty(\mathbb{T}^N)$ as usual consists of infinitely differentiable functions that are 2π-periodic with respect to each argument x_j. The ellipticity of the differential expression $A(D)$ means the fulfillment of inequality

$$A(\xi)>0 \text{ for all } \xi\in\mathbb{R}^N,\ \ \xi\neq 0, \tag{27}$$

where the polynomial $A(\xi)=\sum_{|\alpha|=m} a_\alpha \xi^\alpha$ is called the symbol of the operator $A(D)$. Note that it follows from ellipticity, in particular, that the operator $A(D)$ has an even order: $m=2k$.

It can be verified by a direct computation that the system of eigenfunctions of this operator has the form $\{\gamma e^{inx}\}$, where the constant $\gamma = \gamma(N) = (2\pi)^{-\frac{N}{2}}$ is necessary for the norms of the eigenfunctions to be equal to one. The corresponding eigenvalues have the form $A(n) = \sum_{|\alpha|=m} a_\alpha n^\alpha$. In other words

$$A(D)\, e^{inx} = A(n)e^{inx}.$$

It is not hard to see, that operator $A(D)$ is symmetric and positive: $(Au, u) \geq 0$, for all $u \in C_0^\infty(\mathbb{R}^N)$. Therefore, by the Friedrichs theorem, the operator $A(D)$ has at least one self-adjoint extension $\hat{A}$.

Let us introduce the operator A_T with the domain of definition

$$D(A_T) = \{f(x) \in L_2(\mathbb{T}^N) : \sum A^2(n)|f_n|^2 < \infty\}, \tag{28}$$

acting for any element $f \in D(A_T)$ as

$$A_T f(x) = \sum_{n \in \mathbb{Z}^N} A(n) f_n e^{inx}.$$

Obviously A_T is a self-adjoint operator.

Let us verify that A_T is a self-adjoint extension of $A(D)$, that is, let us show that $D(A(D)) = C_0^\infty(\mathbb{T}^N) \subset D(A_T)$, and that $A\left(D\right)f\left(x\right) = A_T f(x)$ for any function $f \in C_0^\infty(\mathbb{T}^N)$.
Let $f\left(x\right) \in C_0^\infty(\mathbb{T}^N)$. Then

$$\left(A(D)f, e^{inx}\right) = \int_{\mathbb{T}^N} \left[A\left(D\right)f\left(x\right)\right] e^{-inx} dx,$$

and since $A\left(D\right)$ is formally self-adjoint,

$$\left(A(D)f, e^{inx}\right) = \int_{\mathbb{T}^N} f\left(x\right) \overline{A\left(D\right)e^{inx}} dx = \int_{\mathbb{T}^N} f\left(x\right) A(n) e^{-inx} dx = A(n) f_n.$$

Since $A\left(D\right)f\left(x\right) \in L_2(\mathbb{T}^N)$, by Parseval's equality we have $f \in D\left(A\right)$, and since the functions $A\left(D\right)f\left(x\right)$ and $A_T f$ have the same Fourier coefficients $A(n)f_n$ and the system $\left\{e^{inx}\right\}$ is complete, the required equality $A\left(D\right)f = A_T f$, understood as an equality of elements of $L_2(\mathbb{T}^N)$, holds.

According to the von Neumann theorem for any real number τ, we determine the degree of the operator and the domain of definition as

$$A_T^\tau f(x) = \sum_{n\in Z^N} A^\tau(n) f_n e^{inx}, \tag{29}$$

and

$$D(A_T^\tau) = \{f(x) \in L_2(\mathbb{T}^N) : \sum_{n\in\mathbb{Z}^N} A^{2\tau}(n)|f_n|^2 < \infty\}. \tag{30}$$

Let $A(x, D)$ is an arbitrary symmetric semibounded elliptic operator and $\hat{A}$ its self-adjoint extension, then the question naturally arises: how to check whether a given function belongs to the domain of the power $\hat{A}^\tau$. As far as we know, in the general case of an elliptic operator $A(x, D)$ defined in an arbitrary N-dimensional domain Ω, this question remains open. But in some special cases of Ω domains, for example, in the case of an N-dimensional torus: $\mathbb{T}^N = (\pi, \pi]^N$, we can give a positive answer to this question. Indeed, let A_T be the operator defined above. Then the domain $D(A_T^\tau)$ has the form (30). Since $f \in L_2\left(\mathbb{T}^N\right)$ means the convergence of the series $\sum |f_n|^2 < \infty$, the domain of definition can be written as

$$D(A_T^\tau) = \{f \in L_2(\mathbb{T}^N) : \sum \left(1 + A^{2\tau}(n)\right)|f_n|^2 < \infty\}.$$

Teorema 5

For any $\tau > 0$ one has
$$D(A_T^\tau) = L_2^{\tau m}(\mathbb{T}^N).$$

This assertion is well known and its proof is based on the following lemma.

Lemma 6

There exist positive constants c_1 and c_2 such that
$$c_1(1+|n|^2)^{\tau m} \leq 1 + A(n)^{2\tau} \leq c_2(1+|n|^2)^{\tau m}.$$

Since the fact that the function f belongs to the space $L_2^{\tau m}(\mathbb{T}^N)$ means that the estimate
$$\|f\|_{L_2^{\tau m}(\mathbb{T}^N)} = \sum_{n\in Z^N} (1+|n|^2)^{\tau m} \, |f_n|^2 < \infty$$

holds, then the equality $D(A_T^\tau) = L_2^{\tau m}(\mathbb{T}^N)$ follows from the definition of the set $D(A_T^\tau)$ and the Lemma 6.
Theorem 5 is proved.

Proof of Lemma 6. Since $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ is an elliptic operator, then

$$A(\xi) > 0 \text{ for all } \xi \neq 0. \tag{31}$$

Therefore, we obtain the estimate

$$\min_{|\xi|=1} A(\xi) = a_0 > 0,$$

since the unit sphere $\{\xi \in \mathbb{R}^N : |\xi| = 1\}$ is a compact set, and the function $A(\xi)$ reaches its minimum on any compact.

It should be noted that the polynomial $A(\xi)$ is an m-order homogeneous function, that is, $A(t\xi) = t^m A(\xi)$ for all $t > 0$. Hence,

$$A(\xi) = A\left(\frac{\xi}{|\xi|}\right)|\xi|^m \geq a_0|\xi|^m.$$

For any $n \in \mathbb{Z}^N$ and τ this inequality implies

$$c(1 + |n|^{2\tau m}) \leq 1 + A(n)^{2\tau},\ c > 0.$$

On the other hand,

$$\frac{(1+|n|^2)^{\tau m}}{1+|n|^{2\tau m}} \leq C,$$

that is

$$(1+|n|^2)^{\tau m} \leq C(1+|n|^{2\tau m}).$$

Hence, there is a positive constant c_1, such that

$$c_1(1+|n|^2)^{\tau m} \leq 1+A(n)^{2\tau}. \tag{32}$$

It is easy to verify that

$$\frac{1+A(n)^{2\tau}}{(1+|n|^2)^{\tau m}} \equiv \frac{1+(\sum_{|\alpha|=m} a_\alpha n^\alpha)^{2\tau}}{(1+|n|^2)^{\tau m}} \leq c_2,$$

since the highest power of n_j is the same for both the numerator and denominator, Therefore,

$$1+A(n)^{2\tau} \leq c_2(1+|n|^2)^{\tau m}.$$

Finally, if we combine this inequality and (32), we obtain

$$c_1(1+|n|^2)^{\tau m} \leq 1+A(n)^{2\tau} \leq c_2(1+|n|^2)^{\tau m}.$$

Subsection 1.3.2. Lemma of M.A. Krasnosel'skii et al.

The following lemma from the monograph by Krasnoselsky et al. [13] plays a key role in our method for studying initial-boundary value problems. This lemma allows us to reduce the study of the uniform convergence of Fourier series to the study of convergence in the $L_2(\Omega)$ norm, and the convergence of orthogonal series in $L_2(\Omega)$, due to the Parseval equality, is equivalent to the convergence of the numerical series of Fourier coefficients.

For a general elliptic operator, the proof of this lemma is rather complicated. But in the case of the operator A_T it is proved rather simply and actually follows from the Sobolev embedding theorem (see Theorem 4).

[13] Krasnoselskii M.A., Zabreyko P.P., Pustylnik E.I., Sobolevski P.S., Integral Operators in Spaces of integrable Functions, Nauka, Moscow, 1966, 499 p. (in Russian)

Lemma 7

Let $\tau > \frac{|\alpha|}{m} + \frac{N}{2m}$. Then for any $|\alpha| \le m$, the operator $D^\alpha \left(A_T + \mathrm{I}\right)^{-\tau}$ maps (completely) continuously from $L_2\left(\mathbb{T}^N\right)$ into $C\left(\mathbb{T}^N\right)$, moreover the following estimate holds true:

$$\left\|D^\alpha \left(A_T + \mathrm{I}\right)^{-\tau} g\right\|_{C\left(\mathbb{T}^N\right)} \le C \left\|g\right\|_{L_2\left(\mathbb{T}^N\right)}. \tag{33}$$

Proof of Lemma 7. Since the Sobolev embedding theorem (see Theorem 4) one has

$$||D^\alpha(A_T + \mathrm{I})^{-\tau} g||_{C\left(\mathbb{T}^N\right)} \le C||D^\alpha(A_T + \mathrm{I})^{-\tau} g||_{L_2^a\left(\mathbb{T}^N\right)}$$

for any $a > \frac{N}{2}$. Therefore it is sufficient to prove the inequality

$$||D^\alpha(A_T + \mathrm{I})^{-\tau} g||_{L_2^a\left(\mathbb{T}^N\right)} \le C||g||_{L_2\left(\mathbb{T}^N\right)}.$$

In turn, to prove this inequality, it suffices to prove the validity of estimate

$$\sum_{n\in Z^n} |g_n|^2 \, |n|^{2|\alpha|} \left(1 + A\left(n\right)\right)^{-2\tau} \left(1 + |n|^2\right)^a \le C \sum_{n\in Z^n} |g_n|^2 \,, \tag{34}$$

for $\frac{N}{2} < a \le \tau m - |\alpha|$. But this is a consequence of the estimate

$$|n|^{2|\alpha|} \left(1 + A\left(n\right)\right)^{-2\tau} \left(1 + |n|^2\right)^a \le C, \tag{35}$$

which immediately follows from estimates (32) and $a \le \tau m - |\alpha|$:

$$\frac{|n|^{2|\alpha|} \left(1 + |n|^2\right)^a}{(1 + A\left(n\right))^{2\tau}} \le \frac{|n|^{2|\alpha|} \left(1 + |n|^2\right)^{\tau m}}{c_1 (1 + |n|^2)^{\tau m} (1 + |n|^2)^{|\alpha|}} \le C.$$

Lemma is proved.

Chapter 2. Classical solution of forward problems for subdiffusion equations and the first method for solving inverse problems.

Subsection 2.1. Problem formulation.

Let $\rho \in (0,1)$ be a constant number and $A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ - elliptic operator. Consider the initial-boundary value problem (forward problem)

$$\partial_t^\rho u(x,t) + A(D)u(x,t) = 0, \quad x \in \mathbb{T}^N, \quad 0 < t \leq T, \tag{36}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \mathbb{T}^N. \tag{37}$$

Instead of boundary conditions we consider the 2π-periodic in each argument x_j functions and suppose that $\varphi(x)$ is also 2π-periodic functions in x_j.

Definition 8

A 2π-periodic function $u(x,t)$ with the properties

1. $\partial_t^\rho u(x,t), A(D)u(x,t) \in C(\mathbb{T}^N \times (0,\infty))$,
2. $\partial_t^{\rho-1} u(x,t) \in C(\mathbb{T}^N \times [0,\infty))$

and satisfying all the conditions of problem (36) - (37) in the classical sense is called a classical solution (or simply a solution) of forward problem (36) - (37).

Here is the main result for the forward problem.

Teorema 9

Let $a > \frac{N}{2m}$ and $\varphi \in D(A_T^a)$. Then there exists a solution of initial-boundary value problem (36) - (37) and it has the form

$$u(x,t) = \sum_{n \in \mathbb{Z}^N} \varphi_n t^{\rho-1} E_{\rho,\rho}(-A^2(n)\, t^\rho)\, e^{inx}, \tag{38}$$

which absolutely and uniformly converges on $x \in \mathbb{T}^N$ and for each $t \in (0,T]$, where φ_n are Fourier coefficients of φ. Moreover, the series obtained after applying term-wise the operators ∂_t^ρ and $A(D)$ also converge absolutely and uniformly on $x \in \mathbb{T}^N$ and for each $t \in (0,T]$.

Subsection 2.2. Uniqueness.

Suppose that the condition of the theorem is satisfied and let initial-boundary value problem (36) - (37) have two solutions $u_1(x,t)$ and $u_2(x,t)$. Our aim is to prove that $u(x,t) = u_1(x,t) - u_2(x,t) \equiv 0$. Since the problem is linear, then we have the following homogenous problem for $u(x,t)$:

$$\partial_t^\rho u(x,t) + A(D)u(x,t) = 0, \quad x \in \mathbb{T}^N, \quad t > 0; \tag{39}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = 0, \quad x \in \mathbb{T}^N; \tag{40}$$

Let $u(x,t)$ be a solution of problem (39)-(40). Consider the function

$$w_n(t) = \int\limits_{\mathbb{T}^N} u(x,t)\, e^{-inx} dx, \;\; n \in \mathbb{Z}^N. \tag{41}$$

By definition of the solution we may write

$$\partial_t^\rho w_n(t) = \int\limits_{\mathbb{T}^N} \partial_t^\rho u(x,t) e^{-inx} dx = -\int\limits_{\mathbb{T}^N} A(D) u(x,t) e^{-inx} dx, \quad t > 0,$$

or, integrating by parts (note $A(-n) = A(n)$),

$$\partial_t^\rho w_n(t) = -\int\limits_{\mathbb{T}^N} u(x,t) A(D) e^{-inx} dx = -A(n) \int\limits_{\mathbb{T}^N} u(x,t) e^{-inx} dx = -A(n) w_n(t), \quad t > 0.$$

Using in (41) the homogenous initial condition (40), we have the following Cauchy problem for $w_n(t)$:

$$\partial_t^\rho w_n(t) + A(n) w_n(t) = 0, \quad t > 0; \quad \lim_{t\to 0} \partial_t^{\rho-1} w_n(t) = 0.$$

This problem has the unique solution; therefore, the function defined by (41), is identically zero: $w_n(t) \equiv 0$ (see Lemma 2). From completeness in $L_2(\mathbb{T}^N)$ of the system of eigenfunctions $\{e^{inx}\}$, we have $u(x,t) = 0$ for all $x \in \mathbb{T}^N$ and $t > 0$. Hence the uniqueness is proved.

Subsection 2.3. Existence.

In accordance with the Fourier method, we will look for a solution to problem (36) - (37) in the form of a series:

$$u(x,t) = \sum_{n\in\mathbb{Z}^N} T_n(t)e^{inx}, \quad t>0,\ x\in\mathbb{T}^N, \tag{42}$$

where functions $T_n(t)$ are solutions to the Cauchy type problem

$$\partial_t^\rho T_n + A(n)T_n = 0, \quad \lim_{t\to 0}\partial_t^{\rho-1}T_n(t) = \varphi_n, \quad \forall n\in\mathbb{Z}^N. \tag{43}$$

The unique solution of problem (43) has the form

$$T_n(t) = \varphi_n t^{\rho-1}E_{\rho,\rho}(-A(n)t^\rho), \tag{44}$$

where $E_{\rho,\mu}$ is the Mittag-Leffler function.

Let $|\alpha| \leq m$. First we prove that one can validly apply the operators D^α and ∂_t^ρ to the series in (38) term-by-term. Suppose that the function $\varphi(x)$ satisfies the condition of the theorem. Then for $\tau > \frac{N}{2m}$ one has

$$\sum_{\mathbb{Z}^N} (A(n)+1)^{2\tau} |\varphi_n|^2 \leq C_\varphi < \infty. \tag{45}$$

Consider the sum

$$S_k(x,t) = \sum_{A(n)<k} \varphi_n t^{\rho-1} E_{\rho,\rho}(-A(n)t^\rho) e^{inx}. \tag{46}$$

Since $(A_T + I)^{-\tau-1} e^{inx} = (A(n)+1)^{-\tau-1} e^{inx}$, we can rewrite the latter in the form

$$S_k(x,t) = (A_T + I)^{-\tau-1} \sum_{A(n)<k} (A(n)+1)^{\tau+1} \varphi_n t^{\rho-1} E_{\rho,\rho}(-A(n)t^\rho) e^{inx}.$$

Therefore, by virtue of Lemma 7, one has

$$\left\|D^\alpha S_k^1\right\|_{C(\mathbb{T}^N)} = \left\| D^\alpha (A_T + I)^{-\tau-1} \sum_{A(n)<k} (A(n)+1)^{\tau+1} \varphi_n t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) e^{inx} \right\|_{C(\mathbb{T}^N)}$$

$$\leq C \left\| \sum_{A(n)<k} (A(n)+1)^{\tau+1} \varphi_n t^{\rho-1} E_{\rho,\rho}(-A(n) t^\rho) e^{inx} \right\|_{L_2(\mathbb{T}^N)} . \qquad (47)$$

Using the orthonormality of the system $\{\gamma_N e^{inx}\}$, we have

$$\|D^\alpha S_k\|^2_{C(\mathbb{Z}^N)} \leq C \sum_{A(n)<k} \left|(A(n)+1)^{\tau+1} \varphi_n t^{\rho-1} E_{\rho,\rho}(-A(n)t^\rho)\right|^2 . \qquad (48)$$

For the Mittag-Leffler function with a negative argument we have an estimate (see (13))

$$|E_{\rho,\rho}(-t)| \leq \frac{C}{1+t}, \quad t > 0.$$

Applying this inequality we have

$$\sum_{A(n)<k} \left|(A(n)+1)^{\tau+1}\varphi_n t^{\rho-1}E_{\rho,\rho}(-A(n)t^\rho)\right|^2 \leq$$

$$\leq Ct^{-2}\sum_{A(n)<k}(A(n)+1)^{2\tau}|\varphi_n|^2 \leq Ct^{-2}C_\varphi. \tag{49}$$

Here we used the inequality $(A(n)+1)t^\rho(1+A(n)t^\rho)^{-1} < C$.

Taking into account (49), one can rewrite the estimate (48) as

$$||D^\alpha S_k^1||^2_{C(\mathbb{T}^N)} \leq Ct^{-2}C_\varphi.$$

This implies uniform convergence on $x \in \mathbb{T}^N$ of the differentiated sum (46) with respect to variables $x_j,\ j = 1, \ldots, N$, for each $t \in (0, T]$. On the other hand, the sum (47) converges for any permutation of its members, as well, since these terms are mutually orthogonal. This implies the absolute convergence of the differentiated sum (46) on the same interval $t \in (0, T]$.
Further, it is not hard to see that

$$\partial_t^\rho \sum_{A(n)<k} T_n(t) e^{inx} = \sum_{A(n)<k} \partial_t^\rho T_n(t) e^{inx} = - \sum_{A(n)<k} A(n) T_n(t) e^{inx}.$$

Absolute and uniform convergence of the latter series can be proved as above.
Obviously, the function in (38) is 2π-periodic in each argument x_j. Considering the initial condition as (see, (16))

$$\lim_{t \to 0} t^{1-\rho} u(x,t) = \frac{\varphi(x)}{\Gamma(\rho)}, \tag{50}$$

it is not hard to verify, that this condition is also satisfied.
Hence, Theorem 9 is completely proved.

Since Theorem 9 is proved using the Fourier method, it is valid for more general initial-boundary value problems. Namely, let $A(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ be an arbitrary positive formally selfadjoint (symmetric) elliptic differential operator of order $m = 2l$ with sufficiently smooth coefficients $a_\alpha(x)$ in a N-dimensional domain Ω with a sufficiently smooth boundary $\partial\Omega$ and $\rho \in (0, 1]$.The proposed method is applicable to the problem:

$$\partial_t^\rho u(x,t) + A(x, D)u(x,t) = 0, \quad x \in \Omega, \quad t > 0; \tag{51}$$

with initial

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \Omega; \tag{52}$$

and boundary

$$B_j u(x,t) = \sum_{|\alpha|\leq m_j} b_{\alpha,j}(x) D^\alpha u(x,t) = 0, \quad 0 \leq m_j \leq m-1, \quad j = 1, 2, ..., l; \quad x \in \partial\Omega; \tag{53}$$

conditions, where $\varphi(x)$ and coefficients $b_{\alpha,j}(x)$ are given smooth functions.

Using Theorem 5 Theorem 9 can be reformulated as:

Teorema 10

Let $a > \frac{N}{2}$ and $\varphi \in L_2^a(\mathbb{T}^N)$. Then, there exists a solution of initial-boundary value problem (36) - (37) and it has the form

$$u(x,t) = \sum_{n \in Z^N} \varphi_n t^{\rho-1} E_{\rho,\rho}(-A(n)t^\rho)\, e^{inx}, \tag{54}$$

which converges absolutely and uniformly on $x \in \mathbb{T}^N$ and for each $t \in (0,T]$. Moreover, the series obtained after applying term-wise the operators D_t^ρ and A also converge absolutely and uniformly on $x \in \mathbb{T}^N$ and for each $t \in (0,T]$.

Remark 11

Note, when $a > \frac{N}{2}$, according to the Sobolev embedding theorem (see Theorem 6, Chapter 1), all functions in $L_2^a(\mathbb{T}^N)$ are 2π-periodic continuous functions. The fulfillment of the inverse inequality $a \leq \frac{N}{2}$, admits the existence of unbounded functions in $L_2^a(\mathbb{T}^N)$ (see, for example, [12]). Therefore, condition $a > \frac{N}{2}$ for function f of this theorem is not only sufficient for the statement to be hold, but it is also necessary.

Section 2.4. Inverse problem.

Let $0 < \rho < 1$ be an unknown number to be determined. Consider the initial-boundary value problem

$$\partial_t^\rho u(x,t) - A(D)u(x,t) = 0, \quad x \in \mathbb{T}^N, \ 0 < t \le T, \tag{55}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in \mathbb{T}^N. \tag{56}$$

As it was proved in Theorem 10, if initial function $\varphi \in L_2^a(\mathbb{T}^N)$, then the solution of this problem exists and is unique. This solution obviously depends on ρ. The purpose of this section is to determine the order $\rho \in (0,1)$ of the time derivative. To do this one needs an extra condition. Different types of such conditions were considered by a number of authors. Our extra condition has the form:

$$f(\rho; t_0) \equiv (2\pi)^{-\frac{N}{2}} \int\limits_{\mathbb{T}^N} u(x,t_0)dx = d_0 \neq 0. \tag{57}$$

The quantity $f(\rho; t_0)$ is, in fact, the projection of the solution $u(x,t_0)$ onto the first eigenfunction.

The initial-boundary value problem (55) - (56) together with extra condition (57), is called an inverse problem.

Definition 12

A pair $\{u(x,t), \rho\}$ of the function $u(x,t)$ and the parameter ρ with the properties

1. $\rho \in (0,1)$,
2. $\partial_t^\rho u(x,t), A(D)u(x,t) \in C(\mathbb{T}^N \times (0,\infty))$,
3. $\partial_t^{\rho-1} u(x,t) \in C(\mathbb{T}^N \times [0,\infty))$

and satisfying all the conditions of problem (55) - (57) in the classical sense is called a classical solution of inverse problem (55) - (57).

We will also call the classical solution simply the solution to the inverse problem.

Now we formulate the main result for the inverse problem.

Teorema 13

Let $a > \frac{N}{2}$ and $\varphi \in L_2^a(\mathbb{T}^N)$. Moreover, let the Fourier coefficient φ_0 of $\varphi(x)$ be non-zero: $\varphi_0 = (2\pi)^{-N} \int_{\mathbb{T}^N} \varphi(x)dx \neq 0$. Then inverse problem (55)- (57) has a unique solution $\{u(x,t), \rho\}$ if and only if

$$0 < \frac{d_0}{\varphi_0} < (2\pi)^{\frac{N}{2}}. \tag{58}$$

Remark 14

Theorem defines the unique ρ from (57). Hence, if we define the integral (57) at another time instant t_1 and get a new ρ_1, i.e. $f(\rho_1; t_1) = d_1$, then from the equality $f(\rho_1; t_0) = d_0$, by virtue of the theorem, we obtain $\rho_1 = \rho$.

Proof of Theorem 13 is based on the following auxiliary lemma:

Lemma 15

Let the Fourier coefficient φ_0 of $\varphi(x)$ be non-zero and $t_0 \geq 1$. Then $f(\rho; t_0)$ as a function of $\rho \in (0, 1)$ is strictly monotone: if $\varphi_0 > 0$, then $f(\rho; t_0)$ increases, and if $\varphi_0 < 0$, then $f(\rho; t_0)$ decreases. Moreover

$$\lim_{\rho \to +0} f(\rho; t_0) = 0, \quad f(1; t_0) = (2\pi)^{\frac{N}{2}} \varphi_0. \tag{59}$$

Proof of Lemma 15. Since the system of eigenfunctions $\{\gamma_N e^{inx}\}$ are orthonormal and $A(0) = 0$, then integrating (54) over $\mathbb{T}^N$ one has

$$f(\rho; t_0) = \varphi_0 t_0^{\rho-1} E_{\rho,\rho}(0) \, (2\pi)^{\frac{N}{2}} = \frac{\varphi_0 t_0^{\rho-1}}{\Gamma(\rho)} \, (2\pi)^{\frac{N}{2}}. \tag{60}$$

Let $\Psi(\rho)$ be the logarithmic derivative of the gamma function $\Gamma(\rho)$ (see, Section 1). Then $\Gamma'(\rho) = \Gamma(\rho)\Psi(\rho)$, and for $\rho \in (0, 1)$ we have $\Gamma(\rho) > 0$ and $\Psi(\rho) < 0$. Therefore,

$$\frac{d}{d\rho}\left(\frac{t_0^{\rho-1}}{\Gamma(\rho)}\right) = \frac{t_0^{\rho-1}}{\Gamma(\rho)}\big[\ln t_0 - \Psi(\rho)\big] > 0.$$

for $t_0 > 1$ (the case $t_0 = 1$ is obvious). Thus function $f(\rho; t_0)$ increases or decreases depending on sign of φ_0. It is easy to verify equalities (59).
Lemma is proved.

Proof of Theorem 13. First we show existence of the order of the fractional derivative ρ, which satisfies condition (57). We have

$$f(\rho; t_0) = (2\pi)^{-\frac{N}{2}} \int\limits_{\Omega} u(x, t_0) dx = d_0.$$

It follows from Lemma 15 and representation (60) immediately, that if

$$0 < \frac{d_0}{\varphi_0} < (2\pi)^{\frac{N}{2}},$$

then there exists a unique ρ, which satisfies condition (57).

To prove the uniqueness of a solution of inverse problem (55)-(57) we suppose that there exist two pairs of solutions $\{u_1, \rho_1\}$ and $\{u_2, \rho_2\}$ such, that
$0 < \rho_k < 1, \ k = 1, 2$, and

$$\partial_t^{\rho_k} u_k(x,t) - A(D)u_k(x,t) = 0, \quad k = 1, 2, \quad x \in \mathbb{T}^N, \ 0 < t \le T; \tag{61}$$

$$\lim_{t \to 0} \partial_t^{\rho_k - 1} u_k(x,t) = \varphi(x), \quad k = 1, 2, \quad x \in \mathbb{T}^N. \tag{62}$$

Consider the following functions

$$w_k^n(t) = \int_{\mathbb{T}^N} u_k(x,t) e^{inx} dx, \quad k = 1, 2, \quad n \in \mathbb{Z}^N.$$

Then, for each $n \in \mathbb{Z}^N$, we have

$$\partial_t^{\rho_k} w_k^n(t) + A(n) w_k^n(t) = 0, \quad \lim_{t \to 0} \partial_t^{\rho_k - 1} w_k^n(t) = \varphi_n, \quad k = 1, 2.$$

Therefore (see (44)), for each $n \in \mathbb{Z}^N$,

$$w_k^n(t) = t^{\rho_k - 1} E_{\rho_k,\rho_k}(-A(n)t^{\rho_k})\,\varphi_n, \quad k = 1, 2,$$

and condition (57) implies $w_1^0(t_0) = w_2^0(t_0)$. Since $A(0) = 0$ we obtain

$$\varphi_0 t_0^{\rho_1 - 1} E_{\rho_1,\rho_1}(0) = \varphi_0 t_0^{\rho_2 - 1} E_{\rho_2,\rho_2}(0) = d_0.$$

As we have seen above (see Lemma 15), this equation yields $\rho_1 = \rho_2$. But in this case $w_1^n(t) = w_2^n(t)$ for all t and n. Hence

$$\int\limits_{\mathbb{T}^N} [u_1(x,t) - u_2(x,t)] e^{inx} dx = 0$$

for all n. Since the set of eigenfunctions $\{e^{inx}\}$ is complete in $L_2(\mathbb{T}^N)$, then we finally have $u_1(x,t) = u_2(x,t)$. Thus, "if part" of the theorem is proved.

To prove "only if" part of the theorem assume that condition (58) is not verified. In this case, as it follows evidently from representation (60), equation $f(\rho; t_0) = d_0$ has no solution on the interval $(0, 1)$. Hence, in this case the inverse problem does not have a solution. The proof of Theorem 13 is complete.

Remark 16

The method used in solving this inverse problem can also be applied to more general subdiffusion equations, for example, to the initial-boundary value problem (51)-(53). The only condition is that the first eigenvalue of the elliptic part of the equation must equal zero.

As an example of application of Theorem 13 consider the following initial-boundary value problem for one-dimensional diffusion equation

$$\partial_t^\rho u(x,t) - u_{xx}(x,t) = 0, \quad x \in (0,\pi), \quad t > 0; \tag{63}$$

with the initial condition

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad x \in [0,\pi]; \tag{64}$$

and the boundary condition

$$u_x(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t \geq 0. \tag{65}$$

where $0 < \rho < 1$. In this case the corresponding spectral problem has the set of eigenfunctions $\{\cos kx\}$ complete in $L_2(0,\pi)$, and eigenvalues k^2, $k = 0, 1, \cdots$. Note that the first eigenvalue in this case is $\lambda_0 = 0$ and the corresponding eigenfunction is $v_0(x) = 1$. Therefore, condition (57) takes the form

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \int_0^\pi u(x,t_0)dx = d_0, \quad t_0 \geq 1. \tag{66}$$

Teorema 17

Let $\varphi \in C^1[0,\pi]$ and $\varphi'(0) = \varphi'(\pi) = 0$. If $\varphi_0 = \frac{1}{\pi}\int\limits_0^\pi \varphi(x)dx \neq 0$ and

$$0 < \frac{d_0}{\varphi_0} < \left(\frac{1}{\pi}\right)^{\frac{1}{2}},$$

then there exists a unique solution $\{u(x,t), \rho\}$ to inverse problem (63) - (66). Moreover, for the solution the representation

$$u(x,t) = \sum_{j=1}^{\infty} \varphi_j t^{\rho-1} E_{\rho,\rho}(-(j-1)^2 t^\rho) \cos(j-1)\, x \quad t > 0,\ x \in [0,\pi],$$

holds, where ρ is the unique root of the algebraic equation

$$\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \frac{t_0^{\rho-1}}{\Gamma(\rho)} = \frac{d_0}{\varphi_0}.$$

The proof of this theorem immediately follows from Theorem 13.

Chapter 3. Generalized solutions of forward problems and the second method for solving inverse problems.

Section 3.1 Introduction. Main results.

Let H be a separable Hilbert space with the scalar product $(\cdot, \cdot)$ and the norm $||\cdot||$ and $A : H \to H$ be an arbitrary positive selfadjoint operator in H. Suppose that A has a complete in H system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of nonnegative eigenvalues λ_k. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e. $0 < \lambda_1 \leq \lambda_2 \cdots$.

Using the definitions of a strong integral and a strong derivative, fractional analogues of integrals and derivatives can be determined for vector-valued functions (or simply functions) $f : \mathbb{R}_+ \to H$, while the well-known formulae and properties are preserved (see, for example, [14]).

[14] C. Lizama, Abstract linear fractional evolution equations, Handbook of Fractional Calculus with Applications V. 2, J.A.T. Marchado Ed. DeGruyter, 465-497 (2019).

Let $\rho \in (0,1)$ be a fixed number and let $C((a,b);H)$ stand for a set of continuous functions $u(t)$ of $t \in (a,b)$ with values in H. Consider the Cauchy type problem:

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = 0, \quad 0 < t \leq T; \\ \lim\limits_{t\to 0} \partial_t^{\rho-1} u(t) = \varphi, \end{cases} \tag{67}$$

where φ is a given vector in H. This problem is called a forward problem.

Definition 18

A function $u(t)$ with the properties $\partial_t^\rho u(t), Au(t) \in C((0,T];H)$, $\partial_t^{\rho-1} u(t) \in C([0,T];H)$ and satisfying conditions (67) is called the generalized solution of the forward problem (67).

We first prove the existence and uniqueness of a solution of problem (67).

Teorema 19

For any $\varphi \in H$ problem (67) has a unique solution and this solution has the form

$$u(t) = \sum_{k=1}^{\infty} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^{\rho})(\varphi, v_k) v_k. \tag{68}$$

Obviously solution (68) depends on $\rho \in (0,1)$. Now let us consider the order of fractional derivative ρ as a unknown parameter and consider an inverse problem: can we identify uniquely this parameter ρ, if we have as a additional information the norm

$$W(t,\rho) = ||u(t)||^2 = d_0 \tag{69}$$

at a fixed time instant $t_0 > 0$?

Problem (67) together with extra condition (69) is called the inverse problem.

To solve this inverse problem we fix a number $\rho_0 \in (0,1)$ and consider the problem for $\rho \in [\rho_0, 1)$.

Definition 20

A pair $\{u(t), \rho\}$ of the solution $u(t)$ to the forward problem and the parameter $\rho \in [\rho_0, 1)$ is called the solution of the inverse problem.

Lemma 21

Given ρ_0 from interval $0 < \rho_0 < 1$, there exists a number $T_0 = T_0(\lambda_1, \rho_0)$, such that for all $t_0 \geq T_0$ and for arbitrary $\varphi \in H$ function $W(t_0, \rho)$ is monotonously decreasing with respect to $\rho \in [\rho_0, 1]$.

The main result of this section is the following:

Teorema 22

Let $t_0 \geq T_0$. Then the inverse problem has a unique solution $\{u(t), \rho\}$ if and only if

$$W(t_0, 1) < d_0 \leq W(t_0, \rho_0).$$

Theorem 22 gives a positive answer to the problem posed in review article by Z. Li et al. [16] (p. 440) in the Conclusions and Open Problems section (see Problem 1). Observe, our result shows, that one may recover the order of fractional derivative by the value of $W(t, \rho)$ at a fixed time instant t_0 as "the observation data".

Subsection 3.2. Forward problem. In the present section we prove Theorem 19. To prove the existence of the forward problem's solution we remind the following estimate of the Mittag-Leffler function with a negative argument (see (13))

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t>0. \tag{70}$$

Consider the sum

$$S_j(t) = \sum_{k=1}^{j} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)\,(\varphi, v_k) v_k.$$

Then

$$AS_j(t) = \sum_{k=1}^{j} \lambda_k t^{\rho-1} E_{\rho}(-\lambda_k t^\rho)\,(\varphi, v_k) v_k.$$

Due to the Parseval equality we may write

$$||AS_j(t)||^2 = \sum_{k=1}^{j} |\lambda_k t^{\rho-1} E_\rho(-\lambda_k t^\rho)\,(\varphi, v_k)|^2 \le Ct^{-2}||\varphi||^2,\ t > 0.$$

Here we used estimate (70) and the inequality $\lambda t^\rho (1 + \lambda t^\rho)^{-1} < 1$.

Hence, we obtain $Au(t) \in C((0, T]; H)$.

Further, from equation (67) one has $\partial_t^\rho S_j(t) = -AS_j(t)$. Therefore, from above reasoning, we finally have $\partial_t^\rho u(t) \in C((0, T]; H)$.

It is not hard to verify the fulfillment of equation (67) (see Lemma 2) and the initial condition therein.

Now we prove the uniqueness of the forward problem's solution.
Suppose that problem (67) has two solutions $u_1(t)$ and $u_2(t)$. Our aim is to prove that $u(t) = u_1(t) - u_2(t) \equiv 0$. Since the problem is linear, then we have the following homogenous problem for $u(t)$:

$$\partial_t^\rho u(t) + Au(t) = 0, \quad t > 0; \tag{71}$$

$$\lim_{t \to 0} \partial_t^{\rho-1} u(t) = 0. \tag{72}$$

Set

$$w_k(t) \;=\; (u(t), v_k).$$

It follows from (71) that for any $k \in \mathbb{N}$

$$\partial_t^\rho w_k(t) = (\partial_t^\rho u(t), v_k) = -\,(Au(t), v_k) = -\,(u(t), Av_k) = -\,\lambda_k w_k(t).$$

Therefore, we have the following Cauchy problem for $w_k(t)$ (see (72)):

$$\partial_t^\rho w_k(t) + \lambda_k w_k(t) = 0, \quad t > 0; \quad \lim_{t\to 0} \partial_t^{\rho-1} w_k(t) = 0.$$

This problem has the unique solution (see Lemma 2). Therefore, $w_k(t) = 0$ for $t > 0$ and for all $k \geq 1$. Then by the Parseval equation we obtain $u(t) = 0$ for all $t > 0$.

Hence uniqueness of the solution is proved.

Thus the proof of Theorem 19 is complete.

Lemma 23

Let $0 < \rho_0 < 1$. Then there is a number $T_0 = T_0(\rho_0, \lambda_1)$ such that functions $e_\lambda(\rho) = t_0^{\rho-1} E_{\rho,\rho}(-\lambda t_0^\rho)$ are positive and monotonically decrease in $\rho \in [\rho_0, 1]$ for all $t_0 \geq T_0$ and $\lambda \geq \lambda_1$.

Proof. Let $0 < \beta < \pi$ and $\delta(1;\beta)$ stand for a contour oriented by non-decreasing $\arg\zeta$ and consisting of the following parts:
1) the ray $\arg\zeta = -\beta$, $|\zeta| \geq 1$,

2) the arc $-\beta \leq \arg\zeta \leq \beta$, $|\zeta| = 1$,

3) the ray $\arg\zeta = \beta$, $|\zeta| \geq 1$.

It is obvious that in this case the complex ζ-plane is divided into two unbounded parts: $G^{(+)}(1;\beta)$ to the right of $\delta(1;\beta)$ in orientation and $G^{(-)}(1;\beta)$ - to the left of it.

The contour $\delta(1;\beta)$ is called the Hankel path.

If $\frac{\pi}{2}\rho < \beta < \min\{\pi, \pi\rho\}$, then by the definition of this contour $\delta(1;\beta)$, we arrive at (see [17], formula (2.29), p. 135, note $-\lambda t_0^\rho \in G^{(-)}(1;\beta)$)

$$t_0^{\rho-1} E_{\rho,\rho}(-\lambda t_0^\rho) = -\frac{1}{\lambda^2 t_0^{\rho+1}\Gamma(-\rho)} + \frac{1}{2\pi i \rho \lambda^2 t_0^{\rho+1}} \int_{\delta(1;\beta)} \frac{e^{\zeta^{1/\rho}} \zeta^{\frac{1}{\rho}+1}}{\zeta + \lambda t_0^\rho} d\zeta = f_1(\rho) + f_2(\rho). \tag{73}$$

We choose $\beta = \frac{3\pi}{4}\rho$, $\rho \in [\rho_0, 1]$.
To show the validity of the lemma, it suffices to prove that $\frac{d}{d\rho} e_\lambda(\rho) < 0$ for all $\rho \in [\rho_0, 1)$, since the positiveness of $e_\lambda(\rho)$ is a consequence of the inequality $e_\lambda(1) = e^{-\lambda t} > 0$.

[17] M. M. Dzherbashian [=Djrbashian], Integral Transforms and Representation of Functions in the Complex Domain (in Russian), M. NAUKA (1966).

We first estimate the derivative $f_1'(\rho)$. Let $\Psi(\rho)$ be the logarithmic derivative of the gamma function (see Chapter 1):

$$\Psi(\rho) = \big(\ln\Gamma(\rho)\big)' = \frac{\Gamma'(\rho)}{\Gamma(\rho)}.$$

Then

$$f_1'(\rho) = \frac{\ln t_0 - \Psi(-\rho)}{\lambda^2 t_0^{\rho+1}\Gamma(-\rho)}.$$

Since

$$\frac{1}{\Gamma(-\rho)} = -\frac{\rho}{\Gamma(1-\rho)} = -\frac{\rho(1-\rho)}{\Gamma(2-\rho)}, \quad \Psi(-\rho) = \Psi(1-\rho) + \frac{1}{\rho} = \Psi(2-\rho) + \frac{1}{\rho} - \frac{1}{1-\rho},$$

then

$$f_1'(\rho) = \frac{1}{\lambda^2 t_0^{\rho+1}} \frac{\rho(1-\rho)[\Psi(2-\rho) - \ln t_0] + 1 - 2\rho}{\Gamma(2-\rho)} = -\frac{f_{11}(\rho)}{\lambda^2 t_0^{\rho+1}\Gamma(2-\rho)}. \tag{74}$$

The estimate $\Psi(2-\rho) < 1-\gamma$ (see Chapter 1), where $\gamma \approx 0,57722$ is the Euler-Mascheroni constant, leads to

$$f_{11}(\rho) > \rho(1-\rho)[\ln t_0 - (1-\gamma)]) + 2\rho - 1.$$

For $t_0 = e^{1-\gamma} e^{2/\rho}$ one has $\rho(1-\rho)[\ln t_0 - (1-\gamma)]) + 2\rho - 1 = 1$. Therefore, $f_{11}(\rho) \geq 1$, if $t_0 \geq T_0$ and

$$T_0 = e^{1-\gamma} e^{2/\rho_0}. \tag{75}$$

Thus, by virtue of (74), for all such t_0 we arrive at

$$f_1'(\rho) \leq -\frac{1}{\lambda^2 t_0^{\rho+1}}. \tag{76}$$

We turn to the estimate of derivative $f_2'(\rho)$. Let $F(\zeta,\rho)$ be the integrand in (73):

$$F(\zeta,\rho)=\frac{1}{2\pi i\rho\lambda^2 t_0^{\rho+1}}\cdot\frac{e^{\zeta^{1/\rho}}\zeta^{1/\rho+1}}{\zeta+\lambda t_0^{\rho}}.$$

In order to take into account the dependence of the integration domain $\delta(1;\beta)$ on ρ while differentiating the function $f_2'(\rho)$, we represent integral (73) as

$$f_2(\rho)=f_{2+}(\rho)+f_{2-}(\rho)+f_{21}(\rho),$$

where

$$f_{2\pm}(\rho)=e^{\pm i\beta}\int_1^{\infty}F(s\,e^{\pm i\beta},\rho)\,ds,$$

$$f_{21}(\rho)=i\int_{-\beta}^{\beta}F(e^{iy},\rho)\,e^{iy}dy=i\beta\int_{-1}^{1}F(e^{i\beta s},\rho)\,e^{i\beta s}ds.$$

Let us consider the function $f_{2+}(\rho)$. Since $\beta=\frac{3\pi}{4}\rho$ and $\zeta=s\,e^{i\beta}$, then

$$e^{\zeta^{1/\rho}}=e^{\frac{1}{2}(i-1)\,s^{\frac{1}{\rho}}}.$$

For the derivative of $f_{2+}(\rho)$ one has

$$f'_{2+}(\rho) = I \cdot \int_1^\infty \frac{e^{\frac{i-1}{2}\, s^{1/\rho}} s^{\frac{1}{\rho}+1}\, e^{2ia\rho} \big[\frac{1}{\rho^2}(\frac{1-i}{2} s^{\frac{1}{\rho}} - 1)\ln s + 2ia - \frac{1}{\rho}\ln t_0\big]}{s\, e^{ia\rho} + \lambda t_0^\rho} ds -,$$

$$-I \cdot \int_1^\infty \frac{ias\, e^{ia\rho} + \lambda t_0^\rho \ln t_0}{(s\, e^{ia\rho} + \lambda t_0^\rho)^2} ds,$$

where $I = e^{ia}(2\pi i\rho\lambda^2 t_0^{\rho+1})^{-1}$ and $a = \frac{3\pi}{4}$. By virtue of the inequality $|s\, e^{ia\rho} + \lambda t_0^\rho| \geq \lambda t_0^\rho$ we arrive at

$$|f'_{2+}(\rho)| \leq \frac{C}{\rho\lambda^3 t_0^{2\rho+1}} \int_1^\infty e^{-\frac{1}{2}\, s^{1/\rho}} s^{\frac{1}{\rho}+1} \big[\frac{1}{\rho^2} s^{1/\rho}\ln s + \ln t_0\big] ds.$$

Lemma 24

Let $0 < \rho \leq 1$ and $m \in \mathbb{N}$. Then

$$J(\rho) = \frac{1}{\rho} \int_1^\infty e^{-\frac{1}{2} s^{\frac{1}{\rho}}} s^{\frac{m}{\rho}+1} ds \leq C_m.$$

Proof. Let us make a change of variables:

$$r = s^{\frac{1}{\rho}} \quad s = r^\rho, \quad ds = \rho r^{\rho-1} dr.$$

Then

$$J(\rho) = \int_1^\infty e^{-\frac{1}{2} r} r^{m-1+2\rho} dr \leq \int_1^\infty e^{-\frac{1}{2} r} r^{m+1} dr = C_m.$$

Lemma 24 is proved.

Application of this lemma gives (note, $\frac{1}{\rho}\ln s < s^{\frac{1}{\rho}}$, provided $s \geq 1$)

$$|f'_{2+}(\rho)| \leq \frac{C}{\lambda^3 t_0^{2\rho+1}}\left[\frac{C_3}{\rho} + C_1 \ln t_0\right] \leq \frac{C}{\lambda^3 t_0^{2\rho+1}}\left[\frac{1}{\rho} + \ln t_0\right].$$

One has the same estimate for $f'_{2-}(\rho)$.
Finally consider $f_{21}(\rho)$. We have

$$f'_{21}(\rho) = \frac{a}{2\pi i \lambda^2 t_0^{\rho+1}} \cdot \int_{-1}^{1} \frac{e^{e^{ias}}\, e^{ias}\, e^{2ia\rho s}\left[2ias - \ln t_0 - \frac{iase^{ia\rho s}+\lambda t_0^{\rho}\ln t_0}{e^{ia\rho s}+\lambda t_0^{\rho}}\right]}{e^{ia\rho s} + \lambda t_0^{\rho}} ds.$$

Therefore,

$$|f'_{21}(\rho)| \leq C\,\frac{\ln t_0}{\lambda^3 t_0^{2\rho+1}}.$$

Estimates of $f'_{2\pm}$ and f'_{21}, and estimate (76) leads to

$$\frac{d}{d\rho}e_\lambda(\rho) \le -\frac{1}{\lambda^2 t_0^{\rho+1}} + C\frac{1/\rho + \ln t_0}{\lambda^3 t_0^{2\rho+1}}.$$

Therefore, this derivative is negative provided

$$t_0^{\rho_0} > \frac{C}{\lambda_1}\left(\frac{1}{\rho_0} + \ln t_0\right).$$

Hence, one can specify a number $T_0 = T_0(\rho_0, \lambda_1)$ (see also (75)) such that for all $t_0 \ge T_0$

$$\frac{d}{d\rho}\left[t_0^{\rho-1}E_{\rho,\rho}(-\lambda t_0^\rho)\right] < 0, \quad \lambda \ge \lambda_1, \quad \rho \in [\rho_0, 1].$$

Lemma 23 is proved.

Note

$$W(t,\rho) = ||u(t)||^2 = \sum_{k=1}^{\infty} |(\varphi, v_k)|^2 |t^{\rho-1} E_\rho(-\lambda_k t^\rho)|^2.$$

Therefore, it is easy to see that Lemma 21 is a consequence of Lemma 23. In turn, Theorem 22 follows from Lemma 21.

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Thank you for attention!