Recents developments in relativistic fluids

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July 25 - Aug 4, 2022

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Notation and conventions

Unless stated otherwise, we adopt:

- Greek indices run from 0 to 3, Latin indices from 1 to 3, and repeated indices are summed over their range.
- $\{x^\alpha\}_{\alpha=0}^{3}$ denotes coordinates in spacetime, with $x^0 = t$ denoting a time coordinate and $\{x^i\}_{i=1}^{3}$ denoting spatial coordinates. We write $\left\{\frac{\partial}{\partial x^\alpha}\right\}_{\alpha=0}^{3}$ or simply $\{\partial_\alpha\}_{\alpha=0}^{3}$ for the corresponding basis of coordinate vectors.
- Signature convention for Lorentzian metrics is $-+++$.
- Indices are raised and lowered with the spacetime metric.
- We use units where $c_\ell = 8\pi G = 1$, where c_ℓ is the speed of light (in vacuum) and G is Newton's gravitational constant.

- ∇ is the covariant derivative associated with the spacetime metric.

- H^μ denotes the Sobolev space with norm $\|\cdot\|_\mu$.

- Def = definition, Theo = theorem, Prop = proposition, EX = example.

- We will assume familiarity with Lorentzian geometry and Einstein's equations. Unless stated otherwise, we will always assume given a differentiable four-dimensional manifold M equipped with a Lorentzian metric g (so (M,g) will be a spacetime), where our objects (tensors etc.) will be defined.

Introduction

The field of relativistic fluid dynamics is concerned with the study of fluids in situations when effects pertaining to the theory of relativity cannot be neglected. It is an essential tool in high-energy nuclear physics, cosmology, and astrophysics [RZ, DR, RR, We]. Relativistic effects are manifest in models of relativistic fluids through the geometry of spacetime. This can be done in two ways: (a) by letting the fluid interact with a fixed spacetime geometry that is determined by a solution to vacuum Einstein's equations, or (b) by considering the fluid equations coupled to Einstein's equations. In (a), we are neglecting the effects of the fluid's matter and energy on the curvature of spacetime, while in (b) such effects are taken into account. We will discuss both situations.

A crucial aspect of relativistic fluid dynamics is that the mathematical structures present in the equations of motion are substantially different than those present in classical (meaning non-relativistic) fluids (e.g., the fluid velocity satisfies a constraint in the relativistic case, something with no analog in classical fluids). Thus, results for relativistic fluids cannot be obtained as a simple extension of techniques used for classical fluids.

The relativistic Euler equations

The dynamics of a perfect (i.e., no viscous) relativistic fluid is described by the relativistic Euler equations to be introduced below.

Def. The energy-momentum tensor of a relativistic perfect isotropic fluid is the symmetric two-tensor

$$T_{\alpha\beta} = (p+\varrho) u_\alpha u_\beta + p g_{\alpha\beta},$$

where g is a Lorentzian metric, p and ϱ are real-valued functions representing the pressure and energy density of the fluid, u is a vector field representing the velocity of the fluid and normalized by

$$|u|_g^2 = g_{\alpha\beta} u^\alpha u^\beta = u^\alpha u_\alpha = -1.$$

(So, u is time-like.)

Remark. u is often referred to as the fluid's four-velocity, emphasising that it is a vector field in spacetime. We will refer to it simply as velocity unless the terminology is ambiguous or we want to emphasize its four-dimensional character. Similarly for other "four-" quantities, e.g., four-acceleration etc.

Remark. Often perfect fluids are also called ideal fluids and both terms are used interchangeably, although some authors (e.g., [RZ]) reserve the terminology ideal for fluids that obey the equation of state of an ideal gas.

The assumption $|u|_g^2 = -1$ can be understood as follows. Recall that in relativity, observers are defined by their (timelike) world-line up to reparametrizations. More precisely, the norm of a tangent vector to the world-line has no physical meaning if the parameter is not specified. Thus, we can choose to normalize the observer's velocity to -1. In the case of a fluid, we can identify the flow lines of u with the world-line of observers traveling with the fluid particles.

$|u|_g^2 = -1$ also says that u is timelike, so fluid particles do not travel faster than or at the speed of light. This normalization has yet another physical interpretation.

The energy density ϱ entering in T is the energy measured by an observer traveling with the fluid (i.e., at rest with respect to the fluid). It is possible to show, using kinetic theory, that the energy density measured by an observer with velocity v will be $v^\alpha v^\beta T_{\alpha\beta}$. Thus, for the fluid velocity itself we need to have $\varrho = u^\alpha u^\beta T_{\alpha\beta}$, thus $u^\alpha u_\alpha = -1$. Let us make another remark about kinetic theory: it also gives the above expression for T as a "continuum limit" when viscosity is ignored (and under certain natural assumptions) [GLW]. While kinetic theory provides what is probably the best justification for defining T by the above formula, it is also possible to postulate T motivated by physical considerations [We].

The normalization $|u|_g^2 = -1$ also implies that the fluid's acceleration $a^\alpha = u^\beta \nabla_\beta u^\alpha$ is orthogonal to u (hence spacelike), since $u^\alpha \nabla_\beta u_\alpha = 0$.

Finally, the velocity normalization allows us to define a fluid's local rest frame (LRF), which is an orthonormal frame $\{e_\alpha\}_{\alpha=0}^3$ such that $e_0 = u$, The fluid is called isotropic as we are assuming that if one is at rest with respect to the fluid then the stresses in all directions of the fluid are the same. This means that in a LRF, $T_{ii} = p$. It is possible to construct fluid models without this assumption [RZ] (so, e.g., $T_{11} \neq T_{22}$ in a LRF). We will not deal with non-isotropic perfect fluids. For fluids with viscosity, to be introduced later, isotropy does not hold.

Def. The baryon density current of a relativistic perfect fluid is defined by

$$J^\alpha = n u^\alpha,$$

where n is a real valued function representing the baryon number density of the fluid and u^α is the fluid's velocity as above.

Physically, the baryon number density gives the density of matter of the fluid: the rest mass density (measured by an observer at rest w.r.t. the fluid) is given by nm, where m is the mass of the baryonic particles that constitute the fluid (these are notions from kinetic theory [RZ]).

Physically, the quantities p, ϱ, and n are not all independent and are related by a relation known as an equation of state (whose choice depends on the nature of the fluid). Under "normal circumstances" (e.g., absent phase transitions) this relation is invertible: knowledge of any two quantities, e.g., ϱ and n, determines the third, e.g., p. In this case, we can choose any two out of the three quantities to be the fundamental/primitive variables/unknowns. We will choose here ϱ and n, assuming that p is given as a function of these quantities, i.e., $p = p(\varrho, n)$. It is also possible to use thermodynamic relations (see below) to introduce other scalar quantities of physical interest, such as temperature or entropy, and use them instead as primary variables.

Def. The relativistic Euler equations are defined by the equations:

$$\nabla_\alpha T^\alpha_\beta = 0, \quad \text{(conservation of energy-momentum)}$$

$$\nabla_\alpha J^\alpha = 0, \quad \text{(conservation of baryonic charge)}$$

$$g_{\alpha\beta} u^\alpha u^\beta = -1, \quad \text{(velocity normalization)}$$

$$p = p(\varrho, n), \quad \text{(equation of state)}$$

where T and J are as above, $p(\varrho, n)$ is a given equation of state, ∇ is the covariant derivative of the metric g figuring in T.

Remark. On physical grounds we want $\varrho \geq 0$, $n \geq 0$ and, in most models, $p \geq 0$. From the point of view of the Cauchy problem, these should be assumed for the initial data and showed to propagate.

Remark. As said in the introduction, we can consider a relativistic fluid on a fixed background or couple to Einstein's equations. In the first case, which will be treated in this section, we assume g given, but we keep track of derivatives of g for future application to Einstein's eq.

We introduce the tensor symmetric two-tensor

$$\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta,$$

wich corresponds to projection onto the space orthogonal to u, i.e.,

$\Pi_{\alpha\beta} u^\beta = u_\alpha + u_\alpha \underbrace{u_\beta u^\beta}_{=-1} = 0$, and if v is orthogonal to u we have

$$\Pi_{\alpha\beta} v^\beta = v_\alpha + u_\alpha \underbrace{u_\beta v^\beta}_{=0} = v_\alpha.$$

It is convenient to decompose $\nabla_\alpha T^\alpha_\beta$ in the directions parallel and orthogonal to u.

$$\nabla_\alpha T^\alpha_\beta = \nabla_\alpha \left((p+\rho) u^\alpha u_\beta + p g_{\alpha\beta} \right)$$

$$= u^\alpha \nabla_\alpha (p+\rho) u_\beta + (p+\rho) \nabla_\alpha u^\alpha u_\beta + (p+\rho) u^\alpha \nabla_\alpha u_\beta + \nabla_\beta p, \text{ thus}$$

$$u^\beta \nabla_\alpha T^\alpha_\beta = - u^\alpha \nabla_\alpha (p+\rho) - (p+\rho) \nabla_\alpha u^\alpha + (p+\rho) u^\alpha \underbrace{u^\beta \nabla_\alpha u_\beta}_{=0} + u^\beta \nabla_\beta p$$

$$= - u^\alpha \nabla_\alpha \rho - (p+\rho) \nabla_\alpha u^\alpha .$$

$$\Pi^{\gamma\beta} \nabla_\alpha T^\alpha_\beta = u^\alpha \nabla_\alpha (p+\rho) \underbrace{\Pi^{\gamma\beta} u_\beta}_{=0} + (p+\rho) \nabla_\alpha u^\alpha \underbrace{\Pi^{\gamma\beta} u_\beta}_{=0} + (p+\rho) \Pi^{\gamma\beta} u^\alpha \nabla_\alpha u_\beta$$

$$+ \Pi^{\gamma\beta} \nabla_\beta p = (p+\rho) u^\alpha (\underbrace{g^{\gamma\beta} \nabla_\alpha u_\beta}_{= \nabla_\alpha u^\gamma} + u^\gamma \underbrace{u^\beta \nabla_\alpha u_\beta}_{=0}) + \Pi^{\gamma\beta} \nabla_\beta p$$

$$= (p+\rho) u^\alpha \nabla_\alpha u^\gamma + \Pi^{\gamma\beta} \nabla_\beta p .$$

Writing $\nabla_\alpha J^\alpha$ explicitly: $\nabla_\alpha J^\alpha = \nabla_\alpha (n u^\alpha) = u^\alpha \nabla_\alpha n + n \nabla_\alpha u^\alpha$.

Therefore we can rewrite the relativistic Euler equations as:

$$u^\alpha \nabla_\alpha \rho + (p+\rho) \nabla_\alpha u^\alpha = 0,$$

$$(p+\rho) u^\alpha \nabla_\alpha u^\beta + \Pi^{\beta\alpha} \nabla_\alpha p = 0,$$

$$u^\alpha \nabla_\alpha n + n \nabla_\alpha u^\alpha = 0,$$

$$g_{\alpha\beta} u^\alpha u^\beta = -1.$$

The first equation is the conservation of energy, the second equation is the conservation of momentum, and the third equation, a.k.a. the continuity equation, is the conservation of baryon density. These equations reduce to the non-relativistic Euler equations in the non-relativistic limit [RZ].

Observing that without assuming $u^\alpha u_\alpha = -1$ but still taking u timelike, so that the projection onto the orthogonal to u is

$$\Pi_{\alpha\beta} = g_{\alpha\beta} - \frac{u_\alpha u_\beta}{u_\lambda u^\lambda},$$

contracting the momentum equation with u gives

$$(p+\varrho)\, u^\alpha \nabla_\alpha (u_\lambda u^\lambda) = 0.$$

Thus, for $p+\varrho > 0$, $u_\lambda u^\lambda = -1$ provided it holds initially, i.e., the constraint $u_\lambda u^\lambda = -1$ is propagated by the flow.

Remark. Henceforth, we will always assume that one of the equations of motion is the constraint $g_{\alpha\beta} u^\alpha u^\beta = 0$. This will be the case including for the viscous theories we discuss later. Thus, $g_{\alpha\beta} u^\alpha u^\beta = -1$ will often be omitted.

While it is not difficult to obtain local existence and uniqueness by writing the above equations as a first order symmetric hyperbolic system (see, e.g., [An, CB]), we will use a different approach due to Lichnerowicz [Li] (generalizing earlier work of Choquet-Bruhat [FB]) that makes the role of the characteristics manifest and connects with what we will discuss later. In fact, as we will see, but also as expected physically, there are two types of propagation in the fluid: sound waves and transport of vorticity. These correspond to different characteristics and thus should be treated differently. The first order symmetric hyperbolic system, however, treats both at the same level.

Before continuing, we will need a few more notions.

Thermodynamic properties of relativistic fluids

We begin introducing the following quantities:

- The internal energy density ϵ of the fluid:

$$\varrho = n(1+\epsilon)$$

(strictly speaking the factor n should be the rest mass density nm, see above, but there is no harm in setting $m=1$ here). Thus, the energy density of the fluid takes into account the energy coming from the fluid's rest mass.

- The specific enthalpy h of the fluid

$$h = \frac{p + \varrho}{n}, \quad \text{assuming } n > 0.$$

- We assume the existence of functions s and θ, called the entropy density, a.k.a specific entropy, and temperature of the fluid, such that the first law of thermodynamics holds:

$$dp = n\, dh - n\, \theta\, ds,$$

which can also be written

$$d\varrho = h\, dn + n\, \theta\, ds,$$

$$dE = -p\, d\left(\frac{1}{n}\right) + \theta\, ds.$$

(The specific entropy and temperature can be introduced in a more systematic way, see [LL, RZ].) We will often drop "specific" and refer simply to the entropy, enthalpy, etc.

As before, we can choose which two functions a these thermodynamic quantities are independent, with the remaining one being functions of those two. Different choices will be more appropriate for different questions.

With these definitions, we can write

$$T_{\alpha\beta} = (p+\rho) u_\alpha u_\beta + p g_{\alpha\beta} = n h u_\alpha u_\beta + p g_{\alpha\beta},$$ then

$$\nabla_\alpha T^\alpha{}_\beta = \nabla_\alpha (n h u^\alpha) u_\beta + n h u^\alpha \nabla_\alpha u_\beta + \nabla_\beta p,$$ so

$$u^\beta \nabla_\alpha T^\alpha{}_\beta = -\nabla_\alpha (n h u^\alpha) + u^\beta \nabla_\beta p$$

$$= -h \underbrace{\nabla_\alpha (n u^\alpha)}_{=0} \underbrace{- n u^\alpha \nabla_\alpha h + u^\beta \nabla_\beta p}_{= u^\alpha \left(\underbrace{-n \nabla_\alpha h + \nabla_\alpha p}_{= -n \theta \nabla_\alpha s}\right)}$$

Under the physically natural assumption $\theta > 0$, which we will hereafter assume, we conclude:

$$u^\alpha \nabla_\alpha s = 0.$$

Physical interpretation: the fluid motion is locally adiabatic, i.e., entropy is constant along the flow lines of the fluid.

The characteristics of the Euler system

Using ϱ and s as primary variables, the relativistic Euler system can be written as

$$(p+\varrho)u^\alpha \nabla_\alpha u^\beta + \frac{\partial p}{\partial \varrho}\Pi^{\alpha\beta}\nabla_\alpha \varrho + \frac{\partial p}{\partial s}\Pi^{\alpha\beta}\nabla_\alpha s = 0$$

$$u^\alpha \nabla_\alpha \varrho + (p+\varrho)\nabla_\beta u^\beta = 0$$

$$u^\alpha \nabla_\alpha s = 0$$

or equivalently $A^\alpha \nabla_\alpha \Phi = 0$ where $\Phi = (u^\lambda, \varrho, s)$ and

$$A^\alpha = \begin{bmatrix} \underbrace{(p+\varrho)u^\alpha \delta^\beta_\lambda}_{4\times 4} & \underbrace{\Pi^{\beta\alpha}\frac{\partial p}{\partial \varrho}}_{4\times 1} & \underbrace{\Pi^{\beta\alpha}\frac{\partial p}{\partial s}}_{4\times 1} \\ \underbrace{(p+\varrho)\delta^\alpha_\lambda}_{1\times 4} & \underbrace{u^\alpha}_{1\times 1} & \underbrace{0}_{1\times 1} \\ \underbrace{0}_{1\times 4} & \underbrace{0}_{1\times 1} & \underbrace{u^\alpha}_{1\times 1} \end{bmatrix}$$

Thus

$$\det(A^\alpha \xi_\alpha) = u^\alpha \xi_\alpha \det \begin{bmatrix} (p+\varrho)u^\alpha \xi_\alpha \delta^\beta_\lambda & \Pi^{\beta\alpha}\xi_\alpha \frac{\partial p}{\partial \varrho} \\ (p+\varrho)\xi_\lambda & u^\alpha \xi_\alpha \end{bmatrix}$$

In the matrix, if we multiply the first four rows by ξ_ρ and subtract from it the fifth row times $u^\alpha \xi_\alpha$

$$= \det \begin{bmatrix} (p+\varrho)\, u^\alpha \xi_\alpha\, \delta^\rho_\lambda & \Pi^{\alpha\rho} \xi_\alpha \frac{\partial p}{\partial \varrho} \\ 0 & (u^\alpha \xi_\alpha)^2 - \Pi^{\alpha\rho} \xi_\alpha \xi_\rho \frac{\partial p}{\partial \varrho} \end{bmatrix}$$

$$= (p+\varrho)^4 (u^\alpha \xi_\alpha)^4 \left[(u^\mu \xi_\mu)^2 - \frac{\partial p}{\partial \varrho} \Pi^{\mu\nu} \xi_\mu \xi_\nu \right]$$

One set of characteristics is thus given by $u^\alpha \xi_\alpha = 0$, i.e, the flow lines. For the term in brackets, the invariance of the characteristics allows us to introduce a convenient frame $\{e_A\}_{A=0}^3$ with $e_0 = u$ and $\{e_1, e_2, e_3\}$ orthonormal and orthogonal to u. We also introduce the dual frame $\{e^A\}_{A=0}^3$ given by $(e^A)_\alpha := m^{AB}(e_B)_\alpha$ (where m is g expressed in this frame which then takes the form of the Minkowski matrix), so that $e^A(e_B) = \delta^A_B$. Decomposing ξ with respect to the dual frame $\xi_A = e^\mu_A \xi_\mu$ we have $\xi_{A=0} = -\xi^{A=0} = u^\mu \xi_\mu$ and $\xi_{A=i} = \sigma_{A=i}$, where $\sigma^\mu = \Pi^{\mu\alpha} \xi_\alpha$ and $\sigma_A = e^\mu_A \xi_\mu$.

Therefore, the remaining characteristics are determined by

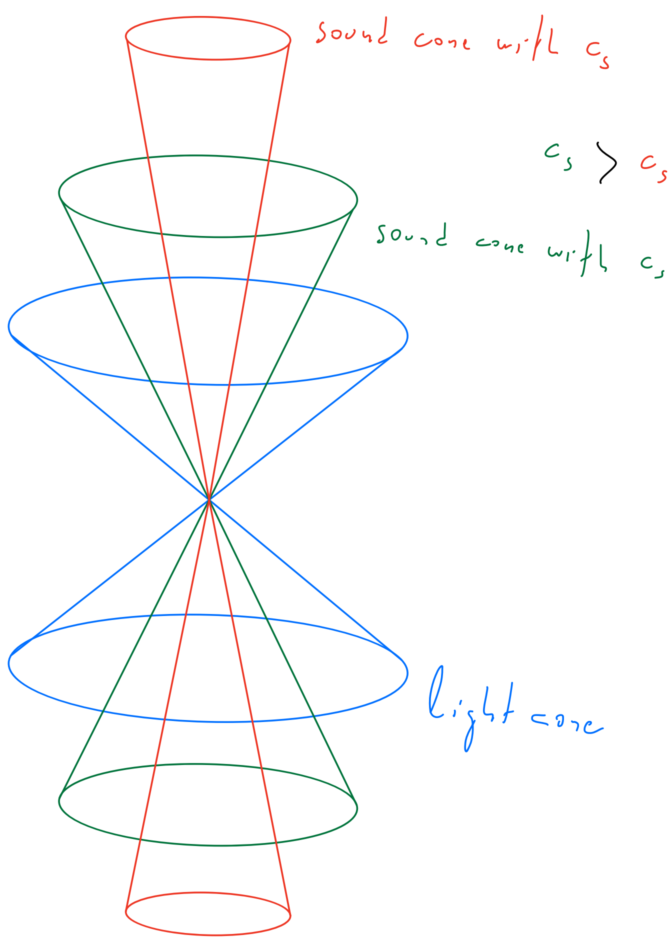
$$\xi_{A=0}^2 - \frac{\partial p}{\partial \varrho} \sum_{i=1}^{3} \xi_{A=i}^2 = 0 .$$

If $\frac{\partial p}{\partial \varrho} < 0$, there are no real solutions so the equations will not be hyperbolic. If $\frac{\partial p}{\partial \varrho} > 1$ then ξ must be timelike, so the corresponding characteristic speeds will be greater than the speed of light (see also remark below). Both cases lead to an evolution incompatible with relativity so we henceforth restrict our attention to systems for which $0 \leq \frac{\partial p}{\partial \varrho} \leq 1$. The case when $\frac{\partial p}{\partial \varrho} = 0$ is allowed has to be treated with some additional care as it corresponds to some sort of degeneracy (which will in fact be present in the case of a free boundary fluid studied later), so we consider for now only $0 < \frac{\partial p}{\partial \varrho} \leq 1$. In this case, the corresponding characteristics have the structure of two opposite cones with opening given by $\sqrt{\frac{\partial p}{\partial \varrho}}$ (this can be seen, e.g., from the above expression for $\xi_{A=0}$). This cone structure is interpreted as corresponding to the propagation of sound waves (see below). It makes sense to call these cones *sound cones* or *acoustic cones* and to define the fluid's sound speed as

$$c_s^2 = \frac{\partial p}{\partial \rho}\Big|_s$$

where we write $|_s$ to emphasize that $\frac{\partial p}{\partial \rho}$ is taken at constant s, i.e., when $p = p(\rho, s)$. (One can check that c_s has units of speed.)

The corresponding picture in tangent space is



To see that the sound cones indeed correspond to the propagation of sound waves, we take a u-derivative of the conservation of energy equation:

$$
\begin{aligned}
0 &= u^\mu \nabla_\mu \left(u^\lambda \nabla_\lambda \varrho + (p+\varrho) \nabla_\lambda u^\lambda \right) \\
&= u^\mu u^\lambda \nabla_\mu \nabla_\lambda \varrho + (p+\varrho) \nabla_\lambda \underbrace{(u^\mu \nabla_\mu u^\lambda)}_{\substack{\| \text{ by the momentum equation} \\ -\frac{c_s^2}{p+\varrho} \Pi^{\lambda\mu} \nabla_\mu \varrho}} + \overbrace{L.O.T}^{\text{includes curvature terms}} \\
&= u^\mu u^\lambda \nabla_\mu \nabla_\lambda \varrho - c_s^2 \Pi^{\lambda\mu} \nabla_\lambda \nabla_\mu \varrho + L.O.T.
\end{aligned}
$$

which is a wave operator for ϱ whose characteristics are the sound cones and which corresponds to the physical intuition of sound waves propagating as disturbances (expansion and rarefaction) of density.

The above discussion motivates the following:

Def. The acoustical metrical is the Lorentzian metric given by

$$G_{\alpha\beta} = c_s^{-2} g_{\alpha\beta} + (c_s^{-2} - 1) u_\alpha u_\beta$$

whose inverse is

$$(G^{-1})^{\alpha\beta} = c_s^2 \Pi^{\alpha\beta} - u^\alpha u^\beta$$

$$= c_s^2 g^{\alpha\beta} + (c_s^2 - 1) u^\alpha u^\beta .$$

Note that $(G^{-1})^{\alpha\beta} \xi_\alpha \xi_\beta = 0$ are the sound cones. The assumptions $0 < c_s \leq 1$ and $|u|_g^2 = -1$ ensures that G is indeed a Lorentzian metric. Note also that $G_{\alpha\beta} u^\alpha u^\beta = -1$.

The existence of the acoustical metric and its relation to the sound cones is indicative of the following big idea to be exploited later: the relevant geometry for the study of a perfect fluid is the acoustic geometry, i.e., the characteristic geometry of the acoustical metric — and not the spacetime geometry. The acoustic geometry will not be flat even if the spacetime is Minkowski. When coupling to Einstein's equations is considered, then the spacetime and acoustic geometry interact with each other, giving rise to a very rich dynamics. We can see now how the case $c_s = 0$ is special, as we no longer obtain a Lorentzian metric in this case.

In sum, the characteristics of the Euler system are the sound cones corresponding to the propagation of sound and the flow lines (i.e., the integral curves of u) which, as we will see next, corresponding to the transport of vorticity in the fluid.

Remark. Above, we excluded $\frac{\partial p}{\partial \varrho} > 1$ based on the physical requirement that no information propagates faster than the speed of light (often called the principle of *causality*; we will have more to say about causality when we study viscous fluids). One can ask, however, if we could study fluids with $\frac{\partial p}{\partial \varrho} > 1$ from a purely mathematical point of view. Computing

$$\det A^0 = (p+\varrho)^4 (u^0)^4 \left(1 + \left(1 - \frac{\partial p}{\partial \varrho}\right) u^i u_i \right)$$

(where we chose normal coordinates at a point for simplicity).

We see that while A^0 is invertible for any u if $\frac{\partial p}{\partial \varrho} \leq 1$, the invertibility of A^0 can fail otherwise (so, e.g., u cannot be prescribed arbitrarily). Since invertibility of A^0 is needed for use of many basic PDE tools (e.g., the Cauchy-Kowalevskaya theorem in the simplest case of analytic data; alternatively, we can say that if $\frac{\partial p}{\partial \varrho} > 1$ then there are choices of u that make the "initial surface" $\{t=0\}$ characteristic), we see that the assumption $\frac{\partial p}{\partial \varrho} \leq 1$ is also justified mathematically.

Relativistic vorticity

A very important quantity in fluids is the vorticity. For classical fluids, it is the curl of the velocity (although one often works with the specific vorticity, i.e., the vorticity divided by the density). Since the curl in 3d can be identified (using Hodge duality) with the exterior derivative of the velocity (thought of as a one-form) or a suitable multiple of it in the compressible case, it seem natural to define the vorticity of a relativistic fluid (where we are in four dimensions) as the exterior derivative of the four-velocity u, with an important distinction that we discuss below, this is what we will do.

Def. The enthalpy current w is defined as

$$w^\alpha = h u^\alpha.$$

The vorticity Ω is defined as the two-form dw.

In components it is given by the equivalent expressions:

$$\Omega_{\alpha\beta} = \partial_\alpha(hu_\beta) - \partial_\beta(hu_\alpha)$$
$$= \nabla_\alpha(hu_\beta) - \nabla_\beta(hu_\alpha).$$

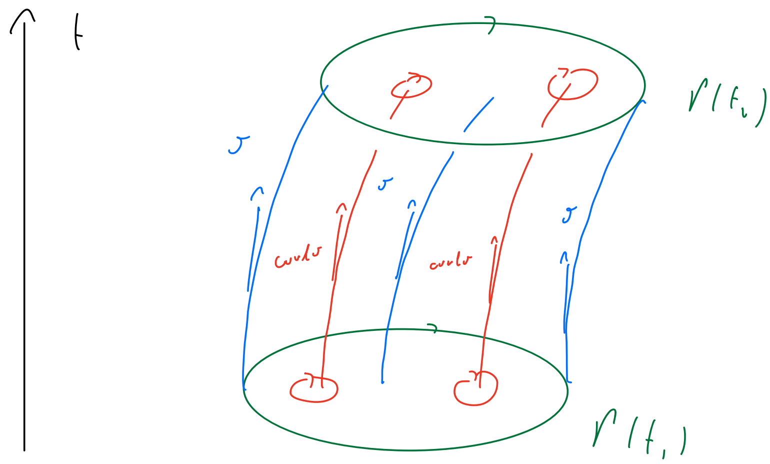
One reason to define the vorticity as above (rather than, say, du) is to have a relativistic version of Kelvin's circulation theorem. For a classical fluid with velocity v, we define its circulation along a closed loop Γ as

$$\mathcal{C}_{cl.} = \oint_\Gamma v \cdot dl .$$

Kelvin's theorem states that this quantity is conserved along fluid lines, i.e.,

$$(\partial_t + v \cdot \nabla)\mathcal{C}_{cl.} = 0$$

The picture below illustrates this situation, with Γ depicted at two different times



This theorem has such a clean physical interpretation as "conservation of vortices" that we expect something similar to hold for relativistic fluids. Indeed it does but the quantity that is conserved now is

$$\mathcal{C} = \oint_\Gamma w_\alpha \, dx^\alpha = \oint_\Gamma h u_\alpha \, dx^\alpha .$$

With this definition:

$$u^\mu \nabla_\mu \mathcal{C} = 0 .$$

The same way that the classical proof goes through using dv, which is the vorticity, the relativistic version involves $d(hu)$, leading to a natural definition of the vorticity as we did. See [RZ] for details.

Next, we derive an important relation between the vorticity and the entropy. Direct computation gives

$$u^\alpha \Omega_{\alpha\beta} = u^\alpha (h \nabla_\alpha u_\beta + \nabla_\alpha h\, u_\beta - h \nabla_\beta u_\alpha - \nabla_\beta h\, u_\alpha)$$

$$= \underbrace{h u^\alpha \nabla_\alpha u_\beta}_{\substack{\parallel \text{ by } \Pi^{\mu\beta} \nabla_\alpha T^\alpha_\beta = 0 \\ -\frac{1}{p+\rho} \Pi^\alpha_\beta \nabla_\alpha p \, = \, -\frac{1}{nh} \Pi^\alpha_\beta \nabla_\alpha p}} + u_\beta u^\alpha \nabla_\alpha h + \nabla_\beta h$$

$$= -\frac{1}{n} \Pi^\alpha_\beta \nabla_\alpha p + u_\beta u^\alpha \nabla_\alpha h + \nabla_\beta h$$

$$= \underbrace{-\frac{1}{n} \nabla_\beta p + \nabla_\beta h}_{= \theta \nabla_\beta s} - u_\beta \underbrace{\left(\frac{1}{n} u^\alpha \nabla_\alpha p - u^\alpha \nabla_\alpha h \right)}_{= - u^\alpha \nabla_\alpha s = 0} .$$

Therefore:
$$u^\alpha \Omega_{\alpha\beta} = \theta \nabla_\beta s .$$

This equation is known as the Lichnerowicz equation. It implies that for an irrotational fluid, i.e., a fluid with $\Omega = 0$, the entropy must be constant, a result with no analogue in classical physics.

Local existence and uniqueness

We will rewrite the relativistic Euler equations as a system for w, Ω, h, and s. We assume that p, n, θ, and E are known functions of h and s.

We begin with an evolution equation for the vorticity. We can write the Lichnerowicz equation as (after multiplying by h)

$$i_w \Omega = h \theta \, ds,$$

where i_w is the interior contraction of the two-form Ω with w, given by

$$(i_w \Omega)_\alpha = w^\mu \Omega_{\mu\alpha}.$$

Taking the exterior derivative:

$$d(i_w \Omega) = d(h\theta) \wedge ds,$$

where we used that $d^2 = 0$, and $\wedge$ is the wedge product of forms, which for one-forms is simply

$$\omega \wedge \mu = (\omega_\alpha dx^\alpha) \wedge (\mu_\beta dx^\beta) = \omega_\alpha \mu_\beta \, dx^\alpha \wedge dx^\beta$$

$$= \sum_{\alpha<\beta} (\omega_\alpha \mu_\beta - \mu_\beta \omega_\alpha) \, dx^\alpha \wedge dx^\beta .$$

We now recall the following formula for the Lie derivative of a form in the direction of a vector field X:

$$\mathcal{L}_X \mu = d(i_X \mu) + i_X(d\mu).$$

In our case, $d\Omega = 0$ since $\Omega = dw$, so

$$\mathcal{L}_w \Omega = d(h\theta) \wedge ds .$$

Using the formula for the Lie derivative in terms of covariant derivatives, expanding the RHS and writing everything in components gives:

$$w^\mu \nabla_\mu \Omega_{\alpha\beta} + \nabla_\alpha w^\mu \Omega_{\mu\beta} + \nabla_\beta w^\mu \Omega_{\alpha\mu}$$

$$= \nabla_\alpha (h\theta) \nabla_\beta s - \nabla_\beta (h\theta) \nabla_\alpha s ,$$

which is our evolution equation for the vorticity.

This equation is interesting because of the following. From the momentum equation we have $u^\alpha \nabla_\alpha u \sim \partial p \sim \partial s$. Commuting with h to get w we have $u^\alpha \nabla_\alpha w \sim \partial s, \partial h$. Since $\Omega \sim \partial w$, we would thus naively expect $u^\alpha \nabla_\alpha \Omega \sim \partial^2 s, \partial^2 h$. However, this does not happen: the structure of the Lichnerowicz equation (which in particular casts ∂s as an exact derivative ds) leads to only one derivative on the RHS. This "gain of derivative" will help with existence and uniqueness below.

In particular, we point out how the first law of thermodynamics was used in the derivation of the vorticity equation; we did not simply apply $u^\mu \nabla_\mu$ to Ω and used $\nabla_\alpha T^\alpha_\mu = 0$.

Before continuing, let us consider an application. As seen, a necessary condition for irrotacionality is that $s =$ constant. In fact, we have:

Prop. If $s =$ constant and $\Omega = 0$ on $\{t = 0\}$, then $s =$ constant and $\Omega = 0$ for $t > 0$.

proof: Integrating $u^\alpha \nabla_\alpha s = 0$ along the flow lines of s gives that $s =$ constant on spacetime. Thus, the equation for the vorticity gives

$$\mathcal{L}_w \Omega = 0,$$

which is a homogeneous transport equation for Ω. Since $\Omega|_{t=0} = 0$, uniqueness gives $\Omega = 0$. $\square$

Remark. Of course, when we say $\Omega = 0$ for $t > 0$, we are referring to t belonging to an interval where the solution exists.

Next we derive an evolution equation for w. We start with the Hodge-Laplacian (not really a Laplacian because g is Lorentzian) of w:

$$\square_H w = (dd^* + d^* d) w = dd^* w + d^* \Omega,$$

where d^* is the adjoint of d. Since $d^* w = -\nabla_\alpha w^\alpha$, compute:

$$d^* w = -\nabla_\alpha w^\alpha = -\nabla_\alpha (h u^\alpha) = -u^\alpha \nabla_\alpha h - h \underbrace{\nabla_\alpha u^\alpha}_{= -\frac{u^\alpha \nabla_\alpha n}{n}}$$

$$= -u^\alpha \nabla_\alpha h + \frac{h}{n} u^\alpha \nabla_\alpha n$$

$$= - w^\alpha \left(\frac{\nabla_\alpha h}{h} - \frac{u^\alpha \nabla_\alpha n}{n} \right) = i_w dF,$$

where $F = \log \frac{n}{h}$. Thus

$$dd^* w = d(i_w dF) = \mathcal{L}_w dF.$$

It will be convenient to introduce $\tilde{h} = h^2$ and consider $F = F(\tilde{h}, s)$. Then, since $w^\alpha w_\alpha = -h^2$

$$\nabla_\alpha F = \frac{\partial F}{\partial \tilde{h}} \nabla_\alpha \tilde{h} + \frac{\partial F}{\partial s} \nabla_\alpha s = - \frac{\partial F}{\partial \tilde{h}} \nabla_\alpha (w^\beta w_\beta) + \frac{\partial F}{\partial s} \nabla_\alpha s$$

$$= -2 \frac{\partial F}{\partial \tilde{h}} w^\beta \nabla_\alpha w_\beta + \frac{\partial F}{\partial s} \nabla_\alpha s$$

$$= -2 \frac{\partial F}{\partial \tilde{h}} w^\beta (\Omega_{\alpha\beta} + \nabla_\beta w_\alpha) + \frac{\partial F}{\partial s} \nabla_\alpha s$$

$$= -2 \frac{\partial F}{\partial \tilde{h}} w^\beta \nabla_\beta w_\alpha + 2 \frac{\partial F}{\partial \tilde{h}} \underbrace{w^\beta \Omega_{\beta\alpha}}_{= h\theta \nabla_\alpha s} + \frac{\partial F}{\partial s} \nabla_\alpha s$$

$$= -2 \frac{\partial F}{\partial \tilde{h}} w^\beta \nabla_\beta w_\alpha + \left(2 \frac{\partial F}{\partial \tilde{h}} h\theta + \frac{\partial F}{\partial s} \right) \nabla_\alpha s$$

To simplify the notation, we henceforth adopt:

Notation. We will use B to indicate a generic expression (which can vary from line to line) depending on at most the number of derivatives of its arguments.

Using the formula for the Lie derivative in terms of covariant derivatives:

$$(\mathcal{L}_w dF)_\rho = -2\frac{\partial F}{\partial \tilde{h}} w^\alpha w^\beta \nabla_\alpha \nabla_\beta w_\rho + \left(2\frac{\partial F}{\partial \tilde{h}} h\theta + \frac{\partial F}{\partial s}\right) w^\alpha \nabla_\alpha \nabla_\rho s$$

$$+ B_\rho(\partial g, \partial s, \partial w).$$

But $w^\alpha \nabla_\alpha \nabla_\rho s = w^\alpha \nabla_\rho \nabla_\alpha s = \nabla_\rho (\underbrace{w^\alpha \nabla_\alpha s}_{=0}) - \nabla_\rho w^\alpha \nabla_\alpha s$

$= B_\rho(\partial g, \partial s, \partial w)$, so

$$(\mathcal{L}_w dF)_\rho = -2\frac{\partial F}{\partial \tilde{h}} w^\alpha w^\beta \nabla_\alpha \nabla_\beta w_\rho + B_\rho(\partial g, \partial s, \partial w).$$

On the other hand

$$(\Box_H w)_\rho = -g^{\alpha\beta} \nabla_\alpha \nabla_\beta w_\rho + R_{\rho\alpha} w^\alpha, \text{ so}$$

$$-g^{\alpha\beta} \nabla_\alpha \nabla_\beta w_\rho + R_{\rho\alpha} w^\alpha = -2\frac{\partial F}{\partial \tilde{h}} w^\alpha w^\beta \nabla_\alpha \nabla_\beta w_\rho$$

$$+ (d^* \Omega)_\rho + B_\rho(\partial g, \partial s, \partial w).$$

Compute:

$$2\frac{\partial F}{\partial \tilde{h}} = 2\frac{\partial F}{\partial h}\underbrace{\frac{\partial h}{\partial \tilde{h}}}_{=\frac{1}{2h}} = \frac{1}{h}\frac{\partial}{\partial h}\log\frac{n}{h} = \frac{1}{h}\left(\frac{1}{n}\frac{\partial n}{\partial h} - \frac{1}{h}\right)$$

$$= -\frac{1}{h^2}\left(1 - \frac{h}{n}\frac{\partial n}{\partial h}\right),$$ thus

$$\left(-g^{\alpha\beta} - \left(1 - \frac{h}{n}\frac{\partial n}{\partial h}\right)\frac{w^\alpha w^\beta}{h^2}\right)\nabla_\alpha \nabla_\beta w_\mu$$

$$= -R_{\mu\alpha} w^\alpha + (d^*\Omega)_\mu + B_\mu(\partial g, \partial s, \partial w).$$

Next, we apply $w^\mu \nabla_\mu$ to this equation and compute:

$$w^\mu \nabla_\mu (d^*\Omega)_\mu = w^\mu \nabla_\mu \nabla_\nu \Omega^\nu{}_\mu$$

$$= \underbrace{w^\mu \nabla_\nu \nabla_\mu \Omega^\nu{}_\mu}_{} + R_{\mu\nu} w^\mu \Omega^\nu{}_\mu + R_\mu{}^\mu{}_{\nu\lambda} w^\lambda \Omega_\mu{}^\lambda$$

$$\|$$

$$\nabla_\nu(\underbrace{w^\mu \nabla_\mu \Omega^\nu{}_\mu}_{}) - \nabla_\nu w^\mu \nabla_\mu \Omega^\nu{}_\mu$$

$$= B_\mu(\partial g, \partial w, \partial s, \partial h, \Omega)$$

$$= B_\mu(\partial^3 g, \partial^2 w, \partial^2 s, \partial^2 h, \partial\Omega).$$

Thus $\left[g^{\alpha\beta} + \left(1 - \frac{h}{n}\frac{\partial n}{\partial h}\right)\frac{w^\alpha w^\beta}{h^2}\right] w^\mu \nabla_\mu \nabla_\alpha \nabla_\beta w_\rho$

$$= B_\rho(\partial^2 g, \partial^2 w, \partial^2 s, \partial^2 h, \partial\Omega).$$

We now invoke that the sound speed is also given by (see [RZ])

$$\frac{1}{c_s^2} = \frac{h}{n}\frac{\partial n}{\partial h}\bigg|_s$$

so, after multiplying by c_s^2:

$$\left[c_s^2 g^{\alpha\beta} - (1 - c_s^2)\frac{w^\alpha w^\beta}{h^2}\right] w^\mu \nabla_\mu \nabla_\alpha \nabla_\beta w_\rho = B_\rho(\partial^2 g, \partial^2 w, \partial^2 s, \partial^2 h, \partial\Omega)$$

and we recognize the inverse acoustical metric in brackets:

$$(G^{-1})^{\alpha\beta}(h, w)\, w^\mu \nabla_\mu \nabla_\alpha \nabla_\beta w_\rho = B_\rho(\partial^2 g, \partial^2 w, \partial^2 s, \partial^2 h, \partial\Omega)$$

where we wrote $(G^{-1})^{\alpha\beta}(h, w)$ to emphasize that we view G^{-1} as a function of h and w. The characteristics of the operator on the LHS are the sound cones and the flow lines. From this, we obtain:

Prop. The operator

$$(G^{-1})^{\alpha\beta} u^\mu \partial_\mu \partial_\alpha \partial_\beta$$

is a third-order hyperbolic hyperbolic operator.

We now consider the equations derived for s, Ω, and w. In these equations, we treat h as a function of w by $h = \sqrt{-w^\alpha w_\alpha}$, and expand the covariant derivatives, absorbing the terms in the Christoffel symbols into the B terms on the RHS of the equations. Doing so, we find (we multiplied the equation for s by h):

$$w^\alpha \partial_\alpha s = 0,$$

$$w^\rho \partial_\rho \Omega_{\alpha\rho} = B_{\alpha\rho}(\partial g, \partial w, \partial s, \Omega),$$

$$(G^{-1})^{\alpha\rho} w^\gamma \partial_\alpha \partial_\rho \partial_\gamma w_\delta = B_\delta(\partial^3 g, \partial^2 w, \partial^2 s, \partial\Omega).$$

Next, we note that the order of derivatives appearing on the RHS is compatible with the order of this mixed order system (see [Le]) so that its characteristics are given simply by the characteristics of the operators on the LHS (recall that at this point g is considered given). The characteristics are therefore given by $w^\alpha \xi_\alpha = 0$ (the flow lines) and $G^{\alpha\rho}\xi_\alpha \xi_\rho = 0$ (the sound cones). In particular, our derivation did not introduce spurious characteristics.

Denote by $\|\cdot\|_N$ the H^N-Sobolev norm in $\mathbb{R}^3$.

Invoking standard energy estimates for strictly hyperbolic operators (see, e.g., [Ho3, Le]) we obtain

$$\|s\|_N \lesssim \|s(0)\|_N + \int_0^t B(\|w\|_N, \|s\|_N),$$

$$\|\Omega\|_N \lesssim \|\Omega(0)\|_N + \int_0^t B(\|g\|_{N+1}, \|w\|_{N+1}, \|s\|_{N+1}, \|\Omega\|_N)$$

$$\|w\|_{N+2} \lesssim \|w(0)\|_{N+2} + \int_0^t B(\|g\|_{N+3}, \|w\|_{N+2}, \|s\|_{N+2}, \|\Omega\|_{N+1}),$$

where we use the following abuse of notation: when we estimate a term like $\|\partial^2 s\|_N$, the derivatives could be time derivatives so we have $\|\partial^2 s\|_N \lesssim \|s\|_{N+2} + \|\partial_t s\|_{N+1} + \|\partial_t^2 s\|_N$. But from the point of view of derivative counting all terms contribute the same. Also, on the LHS we should have $\|w\|_{N+2} + \|\partial_t w\|_{N+1} + \|\partial_t^2 w\|_N$, but all terms contribute as $\|w\|_{N+2}$. Switching N to $N+1$ in the estimate for s and $N+2$ to $N+1$ in the estimate for w, and defining

$$\mathcal{N} = \|s\|_{N+1} + \|\Omega\|_N + \|w\|_{N+1}$$

we obtain:

$$\mathcal{N} \lesssim \mathcal{N}(0) + \int_0^t \mathcal{N},$$

which implies the energy bound for small t:

$$\mathcal{N} \lesssim C(\mathcal{N}(0)).$$

This estimate is the main ingredient for a proof of local existence and uniqueness, similarly to the standard argument for non-linear wave equations.

Other elements for the proof are:

Under the above assumptions ($0 < c_s \leq 1$, $n, \theta > 0$, etc.), it is possibly to successively solve for the time derivatives $\partial_t^k u$, $\partial_t^k s$, $\partial_t^k h$ in terms of the data. This implies (a) that we can construct initial data for the s, Ω, w system out of data for the original system, and (b) that we can construct analytic solutions to the original equations of motion. These analytic solutions satisfy the system for s, Ω, w with $\Omega_{\alpha\beta} = \partial_\alpha(hu_\beta) - \partial_\beta(hu_\alpha)$ and $w_\alpha = hu_\alpha$. Given non-analytic data to the original system, we approximate it by analytic data and use the energy bound (that holds to the analytic solutions) to obtain, via a limit, a non-analytic solution to the original equations of motion. In particular, we have a solution to

$$(p+\varrho)\, u^\alpha \nabla_\alpha u_\beta + \Pi_\beta^\alpha \nabla_\alpha p = 0,$$

where Π is, as before, the projection onto the orthogonal space to u, but we do not know yet it to have the form $\Pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ because we have not yet showed that $|u|_g^2 = -1$. However, we saw that this constraint is propagated.

Finally, uniqueness can also be proved with an energy estimate (in a lower norm) for the difference of two solutions.

We remark that N in the above estimates has to satisfy $N > 3/2$, since we need to use Sobolev estimates and product estimates. From $u^\alpha \nabla_\alpha s = 0$ we obtain that s will remain positive if initially positive, and from $\nabla_\alpha J^\alpha = 0$, written as $u^\mu \nabla_\mu \log n = -\nabla_\mu u^\mu$, the same holds for n (provided, say, that the fluid's velocity does not blow up). Depending on the equation of state, from the thermodynamic relations we obtain positivity of θ, p, and E. Putting all together, we conclude:

Theo (Lichnerowicz [Li]) Consider initial data in H^{N+1}, $N > 3/2$, for the relativistic Euler equations with an equation of state such that $s, h, \theta, n, E, p\big|_{t=0} > 0$, and such that $0 < c_s\big|_{t=0} \leq 1$. Assume also that $|u|_g^2 = -1$ at $t=0$. Then, there exists a unique classical solution to the relativistic Euler equations defined for time interval.

Remark. We have written the relativistic Euler equations in a way that made its characteristics explicit and allowed us to prove existence and uniqueness. But the way we wrote them is not yet good for further applications, and we will present another form of writing the equations later on.

Irrotational flows

Consider the case of an irrotational fluid, i.e., $\Omega = dw = 0$. In this case, locally

$$w = d\phi$$

for some function ϕ. Computing the Hodge-Laplacian

$$\Box_H \phi = (dd^* + d^*d)\phi = d^*d\phi = d^*w = i_w dF$$

for $F = \log \frac{n}{h}$, according to our previous calculations.

But we also showed that $\nabla_\alpha F = -2\frac{\partial F}{\partial \tilde{h}} w^\beta \nabla_\alpha w_\beta + \frac{\partial F}{\partial s}\nabla_\alpha s$ $(\tilde{h} = h^2)$ and $2\frac{\partial F}{\partial \tilde{h}} = -\frac{1}{h^2}\left(1 - \frac{h}{n}\frac{\partial n}{\partial h}\right) = -\frac{1}{h^2}\left(1 - \frac{1}{c_s^2}\right)$, thus

$$i_w dF = -\frac{c_s^2 - 1}{c_s^2}\frac{w^\alpha w^\beta}{h^2}\nabla_\alpha w_\beta = -\frac{c_s^2-1}{c_s^2}\frac{w^\alpha w^\beta}{h^2}\nabla_\alpha\nabla_\beta\phi,$$ thus

multiplying $d^*d\phi - i_w dF = 0$ by $-c_s^2$ and using that $d^*d\phi = -\nabla_\alpha\nabla^\alpha\phi = -g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi$, we find

$$\left(c_s^2 g^{\alpha\beta} - (1-c_s^2)\frac{w^\alpha w^\beta}{h^2}\right)\nabla_\alpha\nabla_\beta\phi = (G^{-1})^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0,$$

where $w_\alpha = \nabla_\alpha\phi$.

The Einstein - Euler system

We will now consider the relativistic Euler equations coupled to Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta},$$

where Λ is the cosmological constant. As usual, we write the equations as

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} T_{\mu\nu} g_{\alpha\beta} + \Lambda g_{\alpha\beta},$$

We consider the problem in wave (or harmonic) coordinates and employ the above form of the fluid equations, so the system reads:

$$-\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} = B_{\alpha\beta}(\partial g, w, s)$$

$$w^\alpha \partial_\alpha s = 0,$$

$$w^\mu \partial_\mu \Omega_{\alpha\beta} = B_{\alpha\beta}(\partial g, \partial w, \partial s, \Omega),$$

$$(G^{-1})^{\alpha\beta} w^\mu \partial_\alpha \partial_\beta \partial_\mu w_\delta = B_\delta(\partial^3 g, \partial^2 w, \partial^2 s, \partial \Omega)$$

We can carry out energy estimates as before to get (with the same abuse of notation as before):

$$\|g\|_{N+2} \lesssim \|g(0)\|_{N+2} + \int_0^t B(\|g\|_{N+2}, \|w\|_{N+1}, \|s\|_{N+1}),$$

$$\|s\|_{N+1} \lesssim \|s(0)\|_{N+1} + \int_0^t B(\|w\|_{N+1}, \|s\|_{N+1}),$$

$$\|\Omega\|_{N} \lesssim \|\Omega(0)\|_{N} + \int_0^t B(\|g\|_{N+1}, \|w\|_{N+1}, \|s\|_{N+1}, \|\Omega\|_{N}),$$

$$\|w\|_{N+1} \lesssim \|w(0)\|_{N+1} + \int_0^t B(\|g\|_{N+2}, \|w\|_{N+1}, \|s\|_{N+1}, \|\Omega\|_{N}),$$

and once again we observe that these estimates close, leading to existence of solutions (see [Li]). We leave the formulation of a precise statement of existence (and uniqueness in the geometric sense) as an exercise.

New formulation of the relativistic Euler equations

The equations we derived in order to obtain local existence and uniqueness for the relativistic Euler equations involve operators that make the role of the characteristics manifest. Nevertheless, such equations are not yet good enough for more refined applications, such as the study of shock formation or the study of low regularity solutions. Here, we will present yet another way of writing the relativistic Euler equations. As we will explain, this new formulation of the equations exhibit several remarkable features, making it ameneable to certain applications in a way that other formulations are not.

Auxiliary quantities

We continue to use the same notation as before for the relativistic Euler equations, and here we introduce several new quantities that will be useful in what follows. Throughout, we denote by $\epsilon^{\alpha\beta\gamma\delta}$ the totally antysymmetric symbol normalized by $\epsilon^{0123} = 1$.

Assumption. For simplicity, in our new formulation of the relativistic Euler equations we will assume that the spacetime metric is the Minkowski metric. The coordinates $\{x^\alpha\}_{\alpha=0}$ will be standard rectangular coordinates.

Def. We introduce:

- the (dimensionless) log-enthalpy:

$$\hat{h} = \log(h/\bar{h}),$$

where $\bar{h}$ is some fixed reference constant value.

- The u-orthogonal vorticity of a one-form V:

$$\mathrm{vort}^\alpha(V) = -\epsilon^{\alpha\beta\gamma\delta} u_\beta \partial_\gamma V_\delta .$$

- The u-orthogonal vorticity vectorfield

$$\varpi^\alpha = \mathrm{vort}^\alpha(hu) .$$

- The entropy gradient one-form:

$$S_\alpha = \partial_\alpha s .$$

- The modified vorticity of the vorticity:

$$C^\alpha = \mathrm{vort}^\alpha(\varpi) + c_s^{-2} \epsilon^{\alpha\beta\gamma\delta} u_\beta \partial_\gamma \hat{h}\, \varpi_\delta$$

$$+ \left(\theta - \frac{\partial\theta}{\partial\hat{h}}\right) S^\alpha \partial_\lambda u^\lambda + \left(\theta - \frac{\partial\theta}{\partial\hat{h}}\right) u^\alpha S^\lambda \partial_\lambda \hat{h} + \left(\theta - \frac{\partial\theta}{\partial\hat{h}}\right) S^\lambda g^{\alpha\beta} \partial_\lambda u_\beta .$$

· The <u>modified divergence of the entropy gradient</u>:

$$D = \frac{1}{n}\partial_\alpha S^\alpha + \frac{1}{n} S^\alpha \partial_\alpha \hat{h} - \frac{1}{n} c_s^{-2} S^\alpha \partial_\alpha \hat{h}.$$

The modified quantities C^α and D come about because of the following. In the applications we will discuss, we need to estimate $\mathrm{vort}^\alpha(\bar{\omega})$ and $\partial_\alpha S^\alpha$, but a good estimate is not available for these quantities. However, adding the right combination of variables to $\mathrm{vort}^\alpha(\bar{\omega})$ and $\partial_\alpha S^\alpha$, we obtain quantities (C^α and D) that satisfy equations with a good structure for which estimates can be derived.

The u-orthogonal vorticity $\bar{\omega}$ is related to Ω by duality: $\bar{\omega}^\alpha = 2u^\mu ({}^*\Omega)_{\mu\nu}$, where ${}^*\Omega$ is the Hodge dual of Ω, given by $(\Omega^*)_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\Omega^{\mu\nu}$. The role of $\bar{\omega}$ is to provide the vorticity "as a vector" rather than as a two form, as is the classical case.

<u>Assumption</u>. In the previous definition, as well as in the ensuing discussion of the new formulation of the relativistic Euler equations, it is assumed that $\hat{h}$ and s are the fundamental thermodynamic variables, with h, n, θ, ϱ, E, and p being functions of $\hat{h}$ and s. We also assume our constructions to be such that $0 < c_s = c_s(\hat{h}, s) \leq 1$.

Def. The null-forms relative to G are the following quadratic forms:

$$Q^{(G)}(\varphi,\psi) = (G^{-1})^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\psi,$$

$$Q_{\alpha\beta}(\varphi,\psi) = \partial_\alpha\varphi\,\partial_\beta\psi - \partial_\beta\varphi\,\partial_\alpha\psi.$$

The use of null-forms has a long history in hyperbolic PDEs and we will highlight their properties below.

The new formulation

We can now state the new formulation of the relativistic Euler equations. As the actual statement of the new formulation is quite long, we will give only a schematic statement. We will use $\simeq$ to denote "up to harmless terms," where harmless here means from the point of view of the applications we discuss further below.

Theo (D - Speck, [DS]). Assume that $(\hat{h}, s, u)$ is a C^3 solution to the relativistic Euler equations. Then, $(\hat{h}, s, u)$ also verify the following system of equations:

Wave equations:

$$\square_G \hat{h} \simeq D + Q(\partial\hat{h}, \partial u) + L(\partial\hat{h}),$$

$$\square_G u^\alpha \simeq C^\alpha + Q(\partial \hat{h}, \partial u) + L(\partial \hat{h}, \partial u)$$

$$\square_G s \simeq D + L(\partial \hat{h}),$$

<u>Transport equations:</u>

$$u^\lambda \partial_\lambda s = 0,$$

$$u^\lambda \partial_\lambda S^\alpha \simeq L(\partial u),$$

$$u^\lambda \partial_\lambda \bar{w}^\alpha \simeq L(\partial \hat{h}, \partial u).$$

<u>Transport-div-curl equations:</u>

$$u^\lambda \partial_\lambda D \simeq C + Q(\partial s, \partial \hat{h}, \partial u) + L(\partial h, \partial u),$$

$$\mathrm{vort}^\alpha(S) = 0,$$

$$\partial_\lambda \bar{w}^\lambda \simeq L(\partial \hat{h}),$$

$$u^\lambda \partial_\lambda C^\alpha \simeq C + D + Q(\partial s, \partial \bar{w}, \partial \hat{h}, \partial u) + L(\partial s, \partial \bar{w}, \partial \hat{h}, \partial u).$$

Above, $L(\partial f_1, \dots, \partial f_m)$ denotes linear combinations of terms that are at most linear in ∂f_i, whereas $Q(\partial f_1, \dots, \partial f_m)$ denotes linear combinations of the null forms relative to G. $\square_G$ is the wave operator w.r.t. G, and in $\square_G u^\alpha$ the wave operator acts on u^α treated as a scalar function.

proof: The proof is quite long and we refer to [DS] for details. The core idea is to differentiate a first-order formulation of the equations with several geometric differential operators and observe remarkable cancellations.

In order to illustrate the type of cancellations we are referring to, let us derive the wave equation for $\hat{h}$.

Simple computations give that

$$\det G = -c_s^{-6},$$

$$|\det G|^{1/2} (G^{-1})^{\alpha\beta} = c_s^{-1} g^{\alpha\beta} + (c_s^{-1} - c_s^{-3}) u^\alpha u^\beta.$$

From this, direct computation gives

$$\Box_G \hat{h} = \frac{1}{|\det G|^{1/2}} \partial_\alpha \left(|\det G|^{1/2} G^{\alpha\beta} \partial_\beta \hat{h} \right)$$

$$= c_s^3 \partial_\alpha \left(-(c_s^{-3} - c_s^{-1}) u^\alpha u^\beta \partial_\beta \hat{h} + c_s^{-1} g^{\alpha\beta} \partial_\beta \hat{h} \right)$$

$$= \underline{-(1-c_s^2) u^\alpha \partial_\alpha (u^\beta \partial_\beta \hat{h}) - (1-c_s^2) \partial_\alpha u^\alpha u^\beta \partial_\beta \hat{h}}$$

$$+ (3c_s^{-1} - c_s) u^\alpha \partial_\alpha c_s u^\beta \partial_\beta \hat{h} - c_s g^{\alpha\beta} \partial_\alpha c_s \partial_\beta \hat{h}$$

$$+ \underline{c_s^2 g^{\alpha\beta} \partial_\alpha \partial_\beta \hat{h}}$$

$$= (c_s^2-1)\, u^\alpha \partial_\alpha (u^\beta \partial_\beta \hat h) + c_s^2 g^{\alpha\beta} \partial_\alpha \partial_\beta \hat h$$

$$+ (c_s^2-1)\, \partial_\alpha u^\alpha\, u^\beta \partial_\beta \hat h + (3c_s^{-1} - c_s) \frac{\partial c_s}{\partial \hat h} u^\alpha \partial_\alpha \hat h\, u^\beta \partial_\beta \hat h$$

$$+ (3c_s^{-1} - c_s) \frac{\partial c_s}{\partial s} \underbrace{u^\alpha \partial_\alpha s}_{=0}\, u^\beta \partial_\beta \hat h - c_s g^{\alpha\beta} \frac{\partial c_s}{\partial \hat h} \partial_\alpha \hat h\, \partial_\beta \hat h$$

$$- c_s g^{\alpha\beta} \frac{\partial c_s}{\partial s} \underbrace{\partial_\alpha s}_{= S_\alpha} \partial_\beta \hat h$$

$$= -\frac{\partial c_s}{\partial \hat h} c_s^{-1} \Big(\overbrace{c_s^2 g^{\alpha\beta} \partial_\alpha \hat h\, \partial_\beta \hat h - u^\alpha \partial_\alpha \hat h\, u^\beta \partial_\beta \hat h + c_s^2 u^\alpha \partial_\alpha \hat h\, u^\beta \partial_\beta \hat h}^{= (G^{-1})^{\alpha\beta} \partial_\alpha \hat h\, \partial_\beta \hat h}$$

$$- 2 u^\alpha \partial_\alpha \hat h\, u^\beta \partial_\beta \hat h \Big)$$

So

$$\square_G \hat h =$$

$$(c_s^2-1)\, u^\alpha \partial_\alpha (u^\beta \partial_\beta \hat h) + c_s^2 g^{\alpha\beta} \partial_\alpha \partial_\beta \hat h + (c_s^2-1)\, \partial_\alpha u^\alpha\, u^\beta \partial_\beta \hat h$$

$$+ 2 \frac{\partial c_s}{\partial \hat h} c_s^{-1} u^\alpha \partial_\alpha \hat h\, u^\beta \partial_\beta \hat h - c_s^{-1} \frac{\partial c_s}{\partial \hat h} (G^{-1})^{\alpha\beta} \partial_\alpha \hat h\, \partial_\beta \hat h - c_s \frac{\partial c_s}{\partial s} S^\alpha \partial_\alpha h .$$

In terms of our variables, the momentum equation reads

$$u^\alpha \partial_\alpha u_\beta + \partial_\beta \hat h + u_\beta u^\alpha \partial_\alpha \hat h - q S_\beta = 0$$

where $q = \theta/h$, and the energy equation as

$$u^\alpha \partial_\alpha \hat{h} + c_s^2 \partial_\alpha u^\alpha = 0.$$

Contracting $c_s^2 g^{\beta\alpha} \partial_\alpha$ with the momentum equation,

$$c_s^2 g^{\alpha\beta} \partial_\alpha \partial_\beta \hat{h} = \underline{- c_s^2 u^\alpha \partial_\alpha \overbrace{\partial_\beta u^\beta}^{= -c_s^{-2} u^\beta \partial_\beta \hat{h} \text{ by energy eq.}}} - c_s^2 \partial_\beta u^\alpha \partial_\alpha u^\beta$$

$$- c_s^2 u^\beta \partial_\beta (u^\alpha \partial_\alpha \hat{h}) - c_s^2 \partial_\beta u^\beta u^\alpha \partial_\alpha \hat{h} + c_s^2 q \partial_\beta S^\beta$$

$$+ c_s^2 \frac{\partial q}{\partial \hat{h}} S^\beta \partial_\beta \hat{h} + c_s^2 \frac{\partial q}{\partial s} S^\beta s_\beta$$

$$= - c_s^2 u^\alpha \partial_\alpha \left(- c_s^{-2} u^\beta \partial_\beta \hat{h} \right) = u^\alpha \partial_\alpha (u^\beta \partial_\beta \hat{h}) - 2 c_s^{-1} \frac{\partial c_s}{\partial \hat{h}} u^\alpha \partial_\alpha \hat{h} \, u^\beta \partial_\beta \hat{h}$$

$$- 2 c_s^{-1} \frac{\partial c_s}{\partial s} \underbrace{u^\alpha \partial_\alpha s}_{= 0} u^\beta \partial_\beta \hat{h}$$

So,

$$c_s^2 g^{\alpha\beta} \partial_\alpha \partial_\beta \hat{h} = u^\alpha \partial_\alpha (u^\beta \partial_\beta \hat{h}) - 2 c_s^{-1} \frac{\partial c_s}{\partial \hat{h}} u^\alpha \partial_\alpha \hat{h} \, u^\beta \partial_\beta \hat{h}$$

$$- c_s^2 \partial_\beta u^\alpha \partial_\alpha u^\beta - c_s^2 u^\beta \partial_\beta (u^\alpha \partial_\alpha \hat{h}) - c_s^2 \partial_\beta u^\beta u^\alpha \partial_\alpha \hat{h}$$

$$+ c_s^2 q \partial_\beta S^\beta + c_s^2 \frac{\partial q}{\partial \hat{h}} S^\beta \partial_\beta \hat{h} + c_s^2 \frac{\partial q}{\partial s} S^\beta s_\beta$$

We use this expression to substitute for the term $c_s^2 g^{\alpha\beta}\partial_\alpha\partial_\beta \hat{h}$ on the RHS of $\Box_G \hat{h}$:

$$\Box_G \hat{h} = \underbrace{u^\alpha \partial_\alpha (u^\beta \partial_\beta \hat{h}) - 2 c_s^{-1} \frac{\partial c_s}{\partial \hat{h}} u^\alpha \partial_\alpha \hat{h}\, u^\beta \partial_\beta \hat{h} - c_s^2 \partial_\beta u^\alpha \partial_\alpha u^\beta - c_s^2 u^\beta \partial_\beta (u^\alpha \partial_\alpha \hat{h}) - c_s^2 \partial_\beta u^\beta\, u^\alpha \partial_\alpha \hat{h} + c_s^2 q\, \partial_\beta S^\beta + c_s^2 \frac{\partial q}{\partial \hat{h}} S^\beta \partial_\beta \hat{h} + c_s^2 \frac{\partial q}{\partial s} S^\beta s_\beta}_{c_s^2 g^{\alpha\beta}\partial_\alpha\partial_\beta \hat{h}}$$

$$+ (c_s^2 - 1) u^\alpha \partial_\alpha (u^\beta \partial_\beta \hat{h}) + (c_s^2 - 1) \partial_\alpha u^\alpha\, u^\beta \partial_\beta \hat{h}$$

$$+ 2 \frac{\partial c_s}{\partial \hat{h}} u^\alpha \partial_\alpha \hat{h}\, u^\beta \partial_\beta \hat{h} - c_s^{-1} \frac{\partial c_s}{\partial \hat{h}} (G^{-1})^{\alpha\beta} \partial_\alpha \hat{h}\, \partial_\beta \hat{h} - c_s \frac{\partial c_s}{\partial s} S^\alpha \partial_\alpha \hat{h}.$$

$$= - c_s^2 \partial_\beta u^\alpha \partial_\alpha u^\beta - \overbrace{\partial_\alpha u^\alpha}^{= - c_s^2 \partial_\beta u^\beta \text{ by energy eq.}} u^\beta \partial_\beta \hat{h} - c_s^{-1} \frac{\partial c_s}{\partial \hat{h}} (G^{-1})^{\alpha\beta} \partial_\alpha \hat{h}\, \partial_\beta \hat{h}$$

$$- c_s \frac{\partial c_s}{\partial s} S^\alpha \partial_\alpha \hat{h} + c_s^2 q\, \partial_\beta S^\beta + c_s^2 \frac{\partial q}{\partial \hat{h}} S^\beta \partial_\beta \hat{h}$$

$$+ c_s^2 \frac{\partial q}{\partial s} S^\beta s_\beta .$$

$$= \underline{- c_s^2 \partial_\beta u^\alpha \partial_\alpha u^\beta + c_s^2 \partial_\alpha u^\alpha \partial_\beta u^\beta} - c_s^{-1} \frac{\partial c_s}{\partial \hat h} (G^{-1})^{\alpha\beta} \partial_\alpha \hat h \partial_\beta \hat h$$

$$- c_s \frac{\partial c_s}{\partial s} S^\alpha \partial_\alpha \hat h + \underbrace{c_s^2 g \partial_\beta S^\beta}_{\substack{= n c_s^2 g D - c_s^2 g S^\beta \partial_\beta \hat h + g S^\beta \partial_\beta \hat h \\ = n c_s^2 g D + (1 - c_s^2) g S^\beta \partial_\beta \hat h}} + c_s^2 \frac{\partial g}{\partial \hat h} S^\beta \partial_\beta \hat h$$

$$+ c_s^2 \frac{\partial g}{\partial s} S^\beta S_\beta .$$

$$= n c_s^2 g D - c_s^{-1} \frac{\partial c_s}{\partial \hat h} (G^{-1})^{\alpha\beta} \partial_\alpha \hat h \partial_\beta \hat h$$

$$+ \underline{c_s^2 \left(\partial_\alpha u^\alpha \partial_\beta u^\beta - \partial_\beta u^\alpha \partial_\alpha u^\beta \right)} + (1 - c_s^2) g S^\beta \partial_\beta \hat h$$

$$- c_s \frac{\partial c_s}{\partial s} S^\alpha \partial_\alpha \hat h + c_s^2 \frac{\partial g}{\partial \hat h} S^\beta \partial_\beta \hat h + c_s^2 \frac{\partial g}{\partial s} S^\beta S_\beta .$$

We claim that this is now the desired expression. The first term is the desired expression linear in D, the second term is a null-form of type $Q^{(G)}(\varphi, \psi) = G^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \psi$ with $\varphi = \psi = \hat h$, the third term is a null form of type $Q_{\alpha\beta}(\varphi, \psi) = \partial_\alpha \varphi \partial_\beta \psi - \partial_\beta \varphi \partial_\alpha \psi$

with $\varphi = u^\alpha$ and $\psi = u^\beta$, the next three terms are linear in $\partial \hat{h}$ and the last term involves no derivatives (recall that we treat S as a variable).

$\square$

When the fluid is irrotational, our new formulation reduces to the equations found by Christodoulou in his landmark work on shock formation [Ch]. In this case, the equations are the equation for the potential ϕ derived earlier and the above equation for h; the latter simplifies considerably when $\Omega = 0$ because then s = constant, so all terms in S vanish (in particular, $D = 0$). Our new formulation generalizes to the relativistic setting a similar new formulation of the classical (non-relativistic) compressible Euler equations found by Luk and Speck [LS1, LS2, LS3, Sp].

It important to stress that our new formulation of the relativistic Euler equations should not be taken for granted, i.e., as a simple addition on the top of the formulations found in the simpler settings of irrotational or classical flows. This is because the structures uncovered by Christodoulou and Luk-Speck are unstable under perturbation, in the following sense: as illustrated in our derivation of the equation for h, the smallest change in a numerical factor or coefficient would prevent the exact cancellation needed for the formulation of the equations.

We will next discuss three applications of the new formulation presented above: improved regularity for the entropy and vorticity, existence of low regularity solutions, and the study of shock formation. None of these applications seem attainable using standard formulations of the equations. The latter observation, in particular, highlights the following: despite looking a monstrossity, the new formulation is very nice, i.e., the equations have good structure, whereas the original first-order formulation, despite looking simple, is bad because no good structure is present.

When discussing these applications, especially the last two, the following big picture idea should be kept in mind. The new formulation allows for the use of geometric techniques from mathematical relativity and the theory of nonlinear waves for the study of relativistic perfect fluids. This is because the new formulation casts the equations as a perturbation of nonlinear wave equations of the form

$$\Box_{g(\psi)} \psi = 0.$$

There is, however, a crucial new aspect (as compared to nonlinear wave equations), namely, one has to account for the interaction of sound waves with transport phenomena, which is a manifestation of the fact that the Euler system is a system with multiple characteristics, the sound cones and the flow lines. (Note that this is not the case for an irrotational fluid, where the only characteristics are the sound cones; in particular, this illustrates how the irrotational and rotational case are fundamentally different.) Therefore, the precise nonlinear structure of the "perturbation terms" matters — hence the emphasis, in particular, on quadratic terms and null forms.

Improved regularity

One new result we can prove using the new formulation is that the entropy and u-orthogonal vorticity can be proven to one degree more regular than what is given by standard theory:

Theo (D-Speck [DS]). The relativistic Euler equations are locally well-posed (i.e., existence, uniqueness, and continuous dependence on the data) with

$$(h, s, u, \bar{\omega}) \in H^N \times H^{N+1} \times H^N \times H^N,$$

$N > \frac{3}{2} + 1.$

In other words, if

$$(h, s, u, \bar{\omega})\Big|_{t=0} \in H^N \times H^{N+1} \times H^N \times H^N,$$

then this regularity is propagated by the flow. The crucial observation is that standard theory (e.g., symmetric hyperbolic systems or the mixed order formulation we derived earlier) gives only $(h, s, u, \bar{\omega}) \in H^N \times H^N \times H^N \times H^{N-1}$

even if $(s, \bar{\omega}) \in H^{N+1} \times H^N$ at $t = 0$.

proof: We again refer to [DS] for details. Here we simply highlight the main ingredient.

First, it is not difficult to see that direct energy estimates on the evolution equations of the new formulation lose derivatives. For example, we want to control u in H^N and $\bar{\omega}$ in H^N. The evolution for $\bar{\omega}$ gives (writing schematically and ignoring the data)

$$u^\alpha \partial_\alpha \bar{\omega} \sim \partial u \Rightarrow \|\bar{\omega}\|_N \leq \int_0^t \|\partial u\|_N$$

(which is consistent with the definition of $\bar{\omega}$). Then, since $C \sim \partial \bar{\omega}$, the evolution for u gives

$$\Box_G u \sim C \sim \partial \bar{\omega}$$

$$\Rightarrow \|u\|_N \lesssim \int_0^t \|\partial \bar{\omega}\|_{N-1} \underset{\text{by the estimate for } \|\omega\|_N}{\lesssim} \int_0^t \|\partial u\|_N \lesssim \int_0^t \|u\|_{N+1}$$

So there is a lost of derivatives. The way around this is to use the fact that $\bar{\omega}$ satisfies not only a transport equation but (taking also into account the evolution for $C \sim \text{curl}\, \bar{\omega}$) a div-curl-transport system. Thus we can use elliptic regularity through the div-curl part to gain derivatives.

It is not, however, so simple. The div and curl operators in the new formulation are spacetime div and curl operators. We need to extract regularity across $\{t = \text{constant}\}$ surfaces and for this we need spatial div-curl operators. To do so, we use the constraint

$$u_\alpha \bar{\omega}^\alpha = 0 \Rightarrow u_\alpha \partial \bar{\omega}^\alpha = -\partial u^\alpha \omega_\alpha$$

which ultimately allows us to independently control the "timelike part" of $\nabla \bar{\omega}$. We can then remove this timelike part of the div-curl system (treating it as a source) obtaining a purely spatial div-curl system. (Similar remarks apply to S and the corresponding div-curl).

$\square$

Remark. The above procedure of excising the timelike part of $\nabla \bar{\omega}$ can be done while preserving the null structure of the equations. While the null structure is not important per se for this improved regularity result, it is important for the study of shocks discussed further below, and in the shock problem we need to rely on the extra differentiability of s and $\bar{\omega}$.

Remark. Improved regularity for the vorticity and entropy had been proven in the classical case by Luk-Speck using the corresponding new formulation of the classical

Euler flow. A key difference is that in the classical setting the div and curl are honest, spatial operators, unlike the relativistic case, where we have to deal with spacetime operators as mentioned above.

Low regularity solutions

The standard existence theory for the relativistic Euler equations gives local well-posedness in H^N for $N > \frac{3}{2} + 1$. (Taking, say, (h, s, u) as primary variables, but the threshold is the same if other pair of thermodynamic scalars are adopted.) A natural question that drives a lot research in PDEs is the of the minimum value of N such that a given PDE or system of PDEs is locally well-posed in H^N. A less ambitious but related question is whether we can establish local well-posedness in H^r for r below the threshold given by standard theory (where what is considered "standard" naturally depends on the equation). Questions of this type are commonly referred to as low regularity questions/problems.

In the irrotational case, the relativistic Euler equations can be written as a system of the form

$$G^{\alpha\beta}(\Psi)\,\partial_\alpha\partial_\beta\Psi = \mathcal{N}(\Psi,\partial\Psi)$$

where $\mathcal{N}$ is a quadratic nonlinearity. (To obtain the equation in this form we in fact differentiate the equation for the potential ϕ and put $\Psi = (h,\partial\phi)$.) The study of low regularity solutions of equations of this form has a long history. Some key results, which we state here in terms of their translation to the irrotational relativistic Euler system are the following. The irrotational relativistic Euler equations are locally well-posed for

$$(h, u = d\phi) \in H^{\nu}$$

with

- $\nu > \frac{9}{4} = 2.25$ (Bahouri–Chemin [BC])
- $\nu > \frac{13}{6} = 2.1666\ldots$ (Tataru [Ta1])
- $\nu > 2 + \frac{2-\sqrt{3}}{2} = 2.13\ldots$ (Klainerman–Rodnianski [KR])
- $\nu > 2$ (Smith–Tataru [ST2]; alternative proof by Wang, 2017 [Wa1]).

We remark the following:

- Within the context of "linear theory," i.e., assuming a pre-specified regularity for the coefficients but no further assumption on them (so one cannot use that $\Box_g G$ satisfies an equation), Tataru's 13/6 result is optimal [ST1].

- Smith–Tataru's $\nu > 2$ is optimal under the stated assumptions, as Lindblad [Lin] proved ill-posedness in H^2

(The breakdown mechanism is the instantaneous formation of shocks.)

We can now ask whether similar low regularity results to hold in the case $\Omega \neq 0$. As said, the rotational and irrotational cases are qualitatively different with the transport part deeply coupled to the wave part (more on this below), a manifestation of the alredy alluded fact that for $\Omega \neq 0$ the relativistic Euler flow is a system with multiple characteristic speeds. Therefore, one would expect that new ideas are needed in this case in comparison to the irrotational case.

Before stating what is known for the relativistic Euler equations, we first turn our attention to the classical compressible Euler system, as its simpler form will allow a clearer discussion. In order to keep the connection with the relativistic setting, however, in the theorem below, which is for the classical compressible system, we make the following notational conventions:

- $\hat{h}$ is the logarithmic density, $h = \log \frac{\rho}{\bar{\rho}}$, $\bar{\rho} > 0$ a fixed background density
- u is the classical velocity (so $u = (u^1, u^2, u^3)$)
- $B = \partial_t + u^i \partial_i$ is the material derivative (the classical analogue of $u^\mu \partial_\mu$).
- Ω is the specific vorticity,

$$\Omega = \frac{\operatorname{curl} u}{\rho / \bar{\rho}} = \frac{\operatorname{curl} u}{e^h}$$

- S is the spatial entropy gradient,

$$S = \nabla s$$

- G is the acoustical metric, which can also be defined for a classical fluid and whose characteristic sets are sound cones, given by

$$G = -dt \otimes dt + c_s^{-2} \sum_{a=1}^{3} (dx^a - u^a dt) \otimes (dx^a - u^a dt)$$

(note that $G(B,B) = -1$) with inverse

$$G^{-1} = -B \otimes B + c_s^2 \sum_{a=1}^{3} \partial_a \otimes \partial_a ,$$

where c_s^2 is the fluid's sound speed $c_s^2 = \frac{\partial p}{\partial \varrho}\Big|_s$,
where we assume $p = p(\varrho, s) = p(h, s)$.

· Variables C and D can be defined similarly to the new formulation of the relativistic Euler equations, with

$$C \sim \operatorname{curl} \Omega \, , \quad D \sim \operatorname{div} S$$

· We introduce $\Psi = (\hat{h}, u, s)$ and call them the wave variables because they satisfy wave equations $\Box_G \Psi = \cdots$, while the variables $\{\Omega, S, C, D\}$ are called transport variables as they satisfy transport equations $B(\Omega, S, C, D) = \cdots$ (in both cases we are referring to the classical analogue of the new formulation previously discussed, see below).

In order to state the theorem, we introduce the notation $\Sigma_0 = \{t = 0\}$ and denote by $C^{0,\alpha}$ the standard Hölder spaces. Also, for later use, $\Sigma_t = \{t = \text{constant}\}$.

Theo (D-Luo-Mazzone-Speck [DLMS]).

Consider a smooth solution to the compressible Euler equations whose initial data obey the following assumptions for some real numbers $\nu := 2+\varepsilon$, a small $\alpha > 0$, $0 < D_{\varepsilon,\alpha} < \infty$, $0 < c_1 < c_2 < \infty$, $0 < c_3$:

1. $\|(h, u, \mathrm{curl}\, u)\|_{H^{2+\varepsilon}(\Sigma_0)} + \|s\|_{H^{3+\varepsilon}(\Sigma_0)} \leq D_{\varepsilon,\alpha}$.

2. $\|(C, D)\|_{C^{0,\alpha}(\Sigma_0)} \leq D_{\varepsilon,\alpha}$.

3. Along Σ_0, the data are contained in the interior of a compact subset K of state space in which $\rho \geq c_3$ and $c_1 \leq c_s \leq c_3$.

Then, the solution's time of classical existence T depends only on $D_{\varepsilon,\alpha}$ and K, $T = T(D_{\varepsilon,\alpha}, K)$, and the Sobolev and Hölder regularity of the data are propagated by the flow (i.e., the norms we can control are uniformly bounded functions of $(D_{\varepsilon,\alpha}, K)$ for $t \in [0,T]$).

Remarks.

- The proof of this result involves several ideas of independent interest: sharp estimates for the characteristic (acoustic) geometry; Strichartz estimates for waves couple to vorticity; Schauder estimates for the div-curl part.

- The main challenge is that the system now has multiple characteristic speeds. Low regularity techniques for quasilinear systems are based on Strichartz estimates, which are well-adapted to the wave part of the system (they are based on dispersion). There are no Strichartz estimates for transport equations (no dispersion). In addition, one has to handle the interaction of the wave and transport parts (transport variables enter as source terms in the estimates for the acoustic geometry, see below). This highlights the fact that the rotational and irrotational problem are qualitatively different; even the tiniest amount of vorticity is a game changer (recall the big idea).

- Aside from $(h, u) \in H^{2+\varepsilon}(\Sigma_0)$, we have the "extra" regularity assumptions $\operatorname{curl} u \in H^{2+\varepsilon}(\Sigma_0)$, $s \in H^{3+\varepsilon}(\Sigma_0)$, $(C \sim \operatorname{curl}\operatorname{curl} u, D \sim \partial^2 s) \in C^{0,\alpha}(\Sigma_0)$. However, we are able to propagate the extra regularity of the transport variables, even though they are deeply coupled to the rougher wave-part of the system (again, through source terms in the acoustic geometry, see below). $\operatorname{curl} u \in H^{2+\varepsilon}$ and $s \in H^{3+\varepsilon}$ are like the improved regularity we established before. Ultimately, our regularity assumptions are tied to the regularity of the characteristics.

- Assumption 3 is a type of non-degeneracy.

- In view of Lindblad's result, our result is optimal with respect to the wave component of the system, i.e., $(h, u) \in H^{2+\varepsilon}(\Sigma_0)$.

- Our result was the first low regularity result for a system with multiple characteristics in three spatial dimensions. After it, Wang [Wa2], Zhang [Zh], and Zhang-Andersson [ZA] improved it (removing the Hölder assumptions).

proof: The proof is quite long, so it is not feasible to provide it here. We will discuss the main ingredients at a high level, referring to [DLMS] for details.

Strategy.

1. We will use known techniques from wave equations (energy, Strichartz estimates) to control the wave part. This requires, in particular, controlling the acoustic geometry (the regularity of G-null surfaces, i.e., the sound cones). For this, one needs to derive complementary estimates for several geometrical quantities associated with the sound cones.

2. We need to control the transport variables at a consistent amount of regularity as in 1. Energy estimates for transport equations are not enough and there are no Strichartz estimates for transport equations. We combine the transport-type energy estimates with elliptic estimates.

3. Transport variables appear as source terms in the acoustic geometry; need to handle the interaction (feature of the multi-speed problem).

Energy estimates

For simplicity, let us assume $s =$ constant, so $D = 0$ and $C = e^{-h} \operatorname{curl} \Omega \sim \operatorname{curl} \Omega$. The classical compressible Euler equations can then be written (new formulation in the classical case, [LS]) (Recall $G = G(\Psi)$)

$$\square_G \Psi \simeq \underbrace{\operatorname{curl} \Omega}_{\sim C} + \partial \Psi \cdot \partial \Psi \qquad \text{(a)}$$

$$B \Omega \simeq \partial \Psi \qquad \text{(b)}$$

$$B \underbrace{\operatorname{curl} \Omega}_{\simeq C} \simeq \partial \Psi \cdot \partial \Omega \qquad \text{(c)}$$

$$\operatorname{div} \Omega \simeq \partial \Psi. \qquad \text{(d)}$$

(The $\partial \Psi$ on RHS are spacetime derivatives. In general, $\partial \Psi$ can be ∂_x or ∂_t, which is related to the fact that both are controlled in wave energy estimates. We downplay this distinction for most of our discussion, but at one point below it will be important.)

We make the important observation that (c) is <u>not</u> simply curl (b) (it would give $\partial^2 \Psi$ on RHS): there are some cancellations but this requires working with C instead of curl Ω, but here for simplicity we identify the two. However, the reader should see curl Ω as a placeholder for C^i, as the remarks to be made for curl Ω are strictly speaking applicable for C^i instead.

To control $\|\bar{\Psi}\|_{2+\varepsilon}$, take $\partial^{1+\varepsilon}$ of (a)

$$\square_G \partial^{1+\varepsilon} \bar{\Psi} \simeq \partial^{1+\varepsilon} \operatorname{curl} \Omega .$$

Thus, we need to control $\partial^{1+\varepsilon} \operatorname{curl} \Omega \in L^2$. Cannot use (b) as it gives $B \partial^{1+\varepsilon} \Omega \simeq \partial^{3+\varepsilon} \bar{\Psi}$. But (c) gives

$$B \partial^{1+\varepsilon} \operatorname{curl} \Omega \simeq \partial \bar{\Psi} \cdot \underbrace{\partial^{2+\varepsilon} \Omega}_{\in L^2 ?} + \underbrace{\partial^{2+\varepsilon} \bar{\Psi}}_{\text{good}} \cdot \partial \Omega$$

so we can control $\partial^{1+\varepsilon} \operatorname{curl} \Omega \in L^2$ provided we can also establish $\partial^{2+\varepsilon} \Omega \in L^2$. The latter can be obtained through the Hodge estimate

$$\|\partial \Omega\|_{L^2(\Sigma_t)} \lesssim \|\operatorname{div} \Omega\|_{L^2(\Sigma_t)} + \|\operatorname{curl} \Omega\|_{L^2(\Sigma_t)}$$

so

$$\|\partial^{2+\varepsilon} \Omega\|_{L^2(\Sigma_t)} \lesssim \|\operatorname{div} \partial^{1+\varepsilon} \Omega\|_{L^2(\Sigma_t)} + \|\operatorname{curl} \partial^{1+\varepsilon} \Omega\|_{L^2(\Sigma_t)}$$

combined with the above evolution for $\partial^{1+\varepsilon}$ curl Ω and (d) which gives

$$\partial^{1+\varepsilon} \operatorname{div} \Omega \simeq \partial^{2+\varepsilon} \Psi,$$

provided that we do have $\partial^{2+\varepsilon}\Omega \in L^2$ at $t=0$ (for when we Gronwall), explaining one of our extra regularity assumptions. In the end, we obtain the estimate

$$\|\partial \Psi\|_{1+\varepsilon} + \|\partial \Omega\|_{1+\varepsilon} \lesssim e^{\int_0^t \|\partial \Psi\|_{L^\infty(\Sigma_\tau)} + \|\partial\Omega\|_{L^\infty(\Sigma_\tau)} \, d\tau}.$$

Key element: need to control mixed spacetime norm

We thus see that we can close the estimates and prove the theorem if we can bound

$$\|\partial \Psi\|_{L^1_t L^\infty_x} := \int_0^t \|\partial \Psi\|_{L^\infty(\Sigma_\tau)} \, d\tau \lesssim \text{data}$$

$$\|\partial \Omega\|_{L^1_t L^\infty_x} := \int_0^t \|\partial \Omega\|_{L^\infty(\Sigma_\tau)} \, d\tau \lesssim \text{data}$$

For $\|\partial\Psi\|_{L^1_t L^\infty_x}$ the goal is to use Strichartz estimates (since they are designed to estimate mixed spacetime norms for wave systems; recall that Ψ is a wave variable).

For $\|\partial\Omega\|_{L^1_t L^\infty_x}$ there are no Strichartz estimates, as said. Since Ω satisfies a div-curl-transport system, we would like to estimate $\|\partial\Omega\|_{L^1_t L^\infty_x}$ with elliptic estimes. This does not seem possible though since Calderón-Zygmund operators are not bounded in L^∞. We can, however, $\|\partial\Omega\|_{L^1_t L^\infty_x}$ by the stronger norm $\|\partial\Omega\|_{L^1_t C^{0,\alpha}_x}$, and for $C^{0,\alpha}$ elliptic estimates are available. This explains our Hölder assumption on the data.

Using Cauchy-Schwarz in the time integrals, it suffices to bound $\|\partial\Psi\|_{L^2_t L^\infty_x}$ and $\|\partial\Omega\|_{L^2_t L^\infty_x}$. The proof is established by improving (for small time) the bootstrap assumptions

$$\|\partial\Psi\|_{L^2_t L^\infty_x} + \sum_{\nu\geq 2} \nu^{2\delta_0} \|P_\nu \partial\Psi\|^2_{L^2_t L^\infty_x} \lesssim 1,$$

$$\| \partial \Omega \|_{L^2_t L^\infty_x} + \sum_{\nu \geq 2} \nu^{2\delta_0} \| P_\nu \partial \Omega \|_{L^2_t L^\infty_x} \lesssim 1,$$

where P_ν is the Littlewood-Paley projection onto dyadic frequencies and δ_0 is a small depending on ε. We refer to the first one as a bootstrap assumption on the wave part and to the second as a bootstrap assumption on the transport part.

Remark. Only the bootstrap assumptions on $\| \partial \Psi \|_{L^2_t L^\infty_x}$ and $\| \partial \Psi \|_{L^2_t L^\infty_x}$ are needed for the energy estimates. The bootstrap assumptions involving the sums are needed for control of the acoustic geometry.

This discussion should not cause the impression that the estimates for $\| \partial \Psi \|_{L^1_t L^\infty_x}$ and $\| \partial \Omega \|_{L^1_t L^\infty_x}$ are decoupled; we need to handle the interaction between the wave and transport parts (see below), even if our presentation discusses these estimates more or less separately.

The logic of the argument is as follows, where we highlight some (but not all) of the new (in comparison to the pure wave case) ideas that are needed and that are discussed below.

Bootstrap assumptions

Energy estimates

Control of the acoustic geometry. Wave-transport interaction: L^2 estimates for transport variables along sound cones; Hölder estimates on spheres $S_{t,u}$ (control flow lines of L); modified mass aspect function equation sourced by transport variables.

Boundedness of a conformal energy for linear waves on G background

Decay for linear waves in G background

Linear Strichartz estimates

Relies on previous GR/waves techniques.

Transport part needs to be consistent with rescaling/reductions procedures

Hölder (transport and elliptic) estimates for transport variables. Estimates for transport equations in Hölder spaces (control flow lines of B).

Improvement of bootstrap assumptions for the transport part. Hölder estimates for wave part consistent with improved wave bootstrap

Strichartz estimates for quasilinear problem. Close argument.

Improvement of bootstrap assumptions for the wave part

We will discuss these steps in a "constructive" way, i.e., more or less in a reverse order to the logic above, starting with what we want to establish and identifying what we need to be true for that to hold.

The Strichartz estimate and reductions

In view of the above, we have to establish the Strichartz estimate $\| \partial \Phi \|_{L^1_t L^\infty} \lesssim$ data.

Next, through a series of technical reductions that involve rescaling, energy estimates, and the use of Duhamel's principle, it is possible to show that control of $\| \partial \Phi \|_{L^2_t L^\infty_x}$ follows from the following frequency-localized Strichartz estimate for the linear-in-φ (ultimately, because of Duhamel) equation $\Box_G \varphi = 0$

$$\| P_\lambda \partial \varphi \|_{L^q_t L^\infty_x} \lesssim \lambda^{\frac{3}{2} - \frac{1}{q}} \| \partial \varphi \|_{L^2(\Sigma_0)}$$

where P_λ = Littlewood-Paley projection onto dyadic frequency λ and $q \gtrsim 2$. With a further reduction such estimate, in turn, follows from the following fixed frequency Strichartz estimate

$$\| P \partial \varphi \|_{L^q_t L^\infty_x} \lesssim \| \partial \varphi \|_{L^2(\Sigma_0)}$$

where P = Littlewood-Paley projection onto unit frequencies $\{ 1/2 \leq |\xi| < 2 \}$. Finally, an abstract duality argument, the TT^* argument, can be used to show that the fixed frequency Strichartz estimate follows from a dispersive estimate stated below.

We remark that while this series of reductions are technical, they follow known steps used in the aforementioned series of results on low regularity for quasilinear wave equations. (In particular, Φ in $\Box_{G(\Phi)} \varphi = 0$ for the fixed frequency estimate is a rescaled version obtained from the reductions.)

The dispersive estimate

We have now reduced the estimate $\| \partial \Phi \|_{L^1_t L^\infty} \lesssim$ data to the following dispersive estimate:

$$\|PB\varphi\|_{L^\infty(\Sigma_t)} \lesssim \left(\frac{1}{(1+t)^{2/q}} + d(t)\right)\left(\|\partial\varphi\|_{L^2(\Sigma_0)} + \|\varphi\|_{L^2(\Sigma_0)}\right).$$

where $q \gtrsim 2$ and we recall that φ is a solution to

$$\Box_G \varphi = 0.$$

The function d satisfies $\|d\|_{L^{q/2}_t} \lesssim 1$ (i.e., it has the same integrability as $(1+t)^{-2/q}$). The term d is "quasilinear in nature," i.e., even though we seek an estimate for a solution to the linear wave equation $\Box_G \varphi = 0$, the coefficients of $\Box_G$ depend on the solution since $G = G(\Psi)$, and hence need to be suitably controlled. This control then leads to the existence and integrability of d.

We observe that we have reduced a Strichartz estimate for $P\partial\varphi$ to a decay estimate not for $P\partial\varphi$ but for $PB\varphi$. That is because in the duality TT^* argument spatial derivatives can be handled with an integration by parts. We are left with a time derivative (see [Wang 17]). The argument is geometric in nature so we are left with a time direction that is the true normal (i.e., w.r.t. G) to constant-time hypersurfaces, which in our case is B.

We finally note further reduction; since we want now an estimate at unit frequency, we can (via Bernstein's inequality) replace $\|PB\varphi\|_{L^\infty(\Sigma_t)}$ by $\|PB\varphi\|_{L^2(\Sigma_t)}$ on the RHS. The use of L^2 allows us to rely on energy estimates for wave equations.

Decay properties and the acoustic geometry

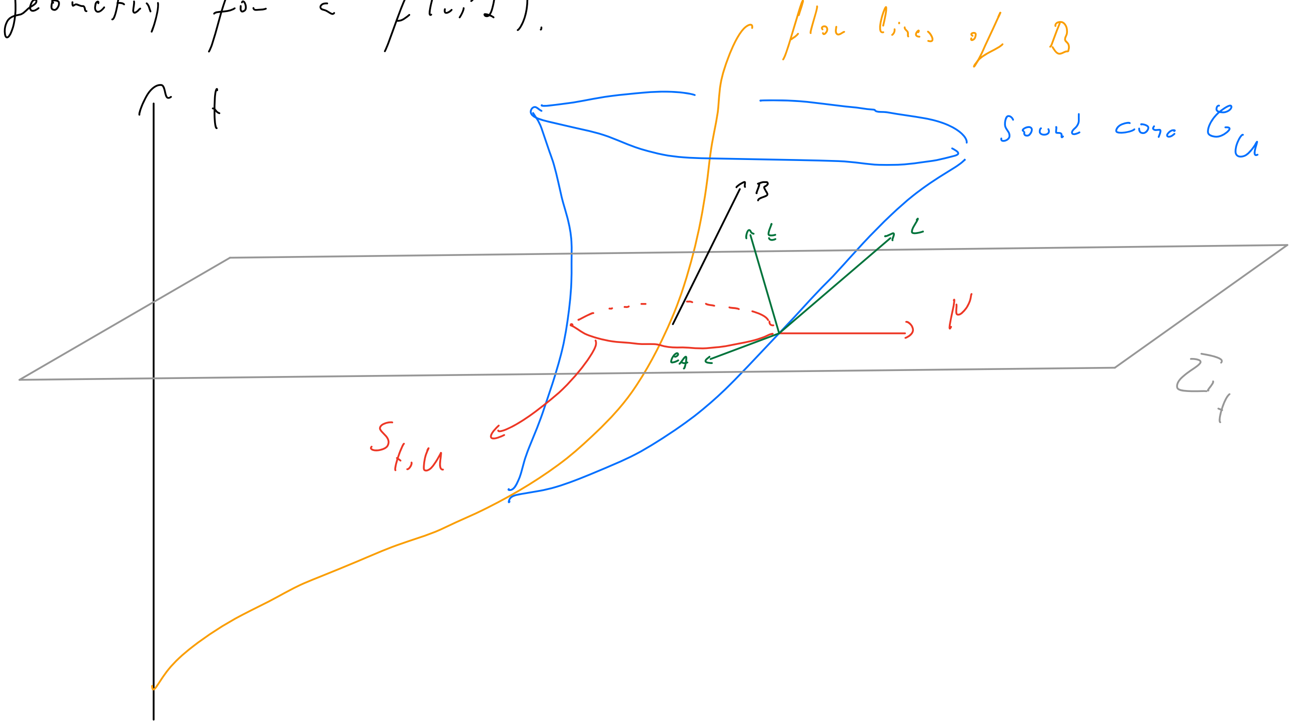
We have now reduced the desired Strichartz estimate for the wave part to a decay estimate for solutions to $\Box_G\varphi = 0$. At this point we can apply the machinary of mathematical GR/wave equations, which we briefly recall.

Decay properties of solutions to $\Box_G \varphi = 0$ are directional dependent, with derivatives of φ in directions tangent to the characteristics decay differently (faster) than derivatives of φ in directions transversal to the characteristics. Thus we need to get a hold on the characterisics of the operator $\Box_G$, which are the sound cones. This is accomplished by introducing an eikonal or optical function, which is a solution to the eikonal equation

$$(G^{-1})^{\alpha\beta} \partial_\alpha U \partial_\beta U = 0$$

with suitable initial conditions. (Note that U depends on $\bar{\Psi}$ since $G = G(\Psi)$, so in particular the regularity of U is tied to that of $\bar{\Psi}$.) The sound cones are then the level sets $\mathcal{C}_u$ of u.

We next introduce a null (w.r.t. G) frame $\{e_1, e_2, \underline{L}, L\}$ adapted to u, $L := B + N$, $\underline{L} := B - N$, where N is the unit outer normal to the spheres $S_{t,u} = \{t = \text{constant}\} \cap \{u = \text{constant}\}$, and $\{e_A\}_{A=1}^{2}$ is an orthonormal frame on $S_{t,u}$. It follows that $G(L, L) = G(\underline{L}, \underline{L}) = G(L, e_A) = G(\underline{L}, e_A) = 0$ and $G(L, \underline{L}) = -2$. This is of course very much like a similar construction in GR, but using the acoustical metric (recall our big idea about the acoustic geometry being the relevant geometry for a fluid).



To prove decay, we follow the usual approach of constructing a weighted energy (called a conformal energy because the method also involves a conformal rescaling, see below) and using certain multipliers with suitable weighted vectorfields. It turns out that we need to use two different vectorfields: one whose weights are good in an "interior" region but become weak in the "exterior" part and one whose weights behave the opposite way. For the interior we take $f(\tilde{r})N$ for suitable f, and for the exterior $\tilde{r}^m L$ for suitable m. Here,

$$\tilde{r} := t - u$$

should be thought of as the quasilinear analogue of the radial coordinate in $\mathbb{R}^3$. (The interior estimate is like a Morawetz estimate adapted to the acoustic geometry and produces integrated energy-decay estimates; the exterior estimate is related to the r^p-method of Dafermos-Rodnianski [DR]).

After testing the equation $\Box_G \varphi = 0$ with multipliers $f(\bar{r}) N\varphi$ and $\bar{r}^m L\varphi$ and integrate by parts we are left with with error terms involving ∂N and ∂L. Since N and L depend on $\mathcal{U}$ which depends on Φ. That is what we meant by saying that the coefficients of $\Box_G$ need to be controled: the quasilinear nature of the problem is still with us, even after all the reductions that led us to the linearising problem $\Box_G \varphi = 0$.

The weighted energy construct in the above procedure is called a conformal energy because, for reasons that we discuss below, in the end we consider not $\Box_G \varphi = 0$ but $\Box_{\tilde{G}} \varphi = 0$ where $\tilde{G}$ is a metric conformal to G.

Control of the acoustic geometry

To estimate ∂N and ∂L we decompose them relative to the null frame obtaining connection coefficients of the null frame, which we are then tasked with estimate. Ultimately this is done by studying a delicate evolution-elliptic system satisfied by the connection coefficients, the null-structure equations. Thus, the desired decay estimate can only be obtained in conjunction with appropriate estimates for the connection coefficients. It is beyond our goal to discuss these estimates here. We will restrict to a few remarks that illustrate what is different in our case in comparison to the case without transport, i.e., GR/nonlinear waves.

One key connection coefficient that plays an important role in the argument is the null mean curvature of the sound cones $\mathcal{C}_u$,

$$\mathrm{tr}_{\not{g}} \chi = G(D_{e_A} L, e_A),$$

where $\not{g}$ = metric induced on $S_{t,u}$ by G, D = covariant derivative of G. Analytically, $\mathrm{tr}_{\not{g}} \chi$ is a special combination of up to second order derivatives of u with coefficients depending on up to first order derivatives of G. $\mathrm{tr}\, \chi$ satisfies the Raychaudhuri equation

$$L\, \mathrm{tr}_{\not{g}} \chi = -R_{LL} + \dots$$

which after a careful decomposition of the Ricci tensor reads

$$L(\mathrm{tr}_{\not{g}} \chi + \Gamma_L) = \frac{1}{2} L^\alpha L^\beta G^{\mu\nu} \partial_\mu \partial_\nu G_{\alpha\beta} + \dots$$

where $\Gamma_L := L^\alpha \Gamma_\alpha$, $\Gamma^\alpha \sim (G^{-1})^\alpha \partial G \sim \partial \Psi$ is a contracted Cartesian Christoffel symbol of G. We group $\mathrm{tr}_{\not{g}} \chi$ with Γ_L because Γ_L does not have enough regularity to be a source. This follows from the delicate structure of estimates, which implies that we would need to control a tangential derivative $\not\partial$ of $\mathrm{tr}_{\not g}\chi$, thus we need to differentiate its evolution equation. If we move Γ_L to the RHS then $L\,\mathrm{tr}_{\not g}\chi = L\Gamma_L + \dots \Rightarrow L\not\partial\,\mathrm{tr}_{\not g}\chi = \not\partial L\Gamma_L \sim \partial^3\Psi$ = too many derivatives. Thus, $\mathrm{tr}_{\not g}\chi + \Gamma_L$ is the good variable to consider. Recalling the equation satisfied by Ψ, $\Box_G\Psi = \mathrm{curl}\,\Omega + \dots$ and using $\Box_G G \simeq \Box_G \Psi$ since $G = G(\Psi)$,

$$L(\mathrm{tr}_{\not g}\chi + \Gamma_L) \simeq \mathrm{curl}\,\Omega$$

Thus to control $\mathrm{tr}_{\not g}\chi + \Gamma_L$ we need to control $\mathrm{curl}\,\Omega$ at consistent regularity level. The presence of $\mathrm{curl}\,\Omega$ on the RHS is an example of the aforementioned interaction between the wave and transport part, i.e., transport variables entering as source terms in the estimates for the acoustic geometry. This is a manifestation of the presence of multiple characteristics.

The arguments used to control $\mathrm{tr}_{\not g}\chi + \Gamma_L$ in the absence of vorticity involve controlling its G_u-tangent derivatives along the sound cones, i.e.,

$\slashed{\partial}(\mathrm{tr}_{\slashed{g}}\chi + \Gamma_L) \in L^2(\mathcal{C}_u)$ (and other spaces along $\mathcal{C}_u$ as well, but we do not discuss them here), where $\slashed{\partial}$ denotes derivatives tangent to $\mathcal{C}_u$. Thus, we need to control $\slashed{\partial}\,\mathrm{curl}\,\Omega$ in $L^2(\mathcal{C}_u)$. At first sight, this seems hopeless because $\mathrm{curl}\,\Omega$ satisfies a transport equation and there is no reason to expect estimates for transport equations to hold along cones. In our case, however, we can estimate $\slashed{\partial}\,\mathrm{curl}\,\Omega \in L^2$ as follows. Energy estimates for transport equations give control of $\slashed{\partial}\,\mathrm{curl}\,w$ in $L^2(\Sigma_t)$. Defining

$$J := |\slashed{\partial}\,\mathrm{curl}\,\Omega|^2 B$$

we find

$$D_\alpha J^\alpha \simeq \slashed{\partial}\,\mathrm{curl}\,\Omega\, B\, \slashed{\partial}\,\mathrm{curl}\,\Omega + \dots$$

We now integrate this in the region interior to $\mathcal{C}_u$ and apply the divergence theorem:

$$\int_0^t \underset{\substack{\Sigma_t \\ (\text{interior to } \mathcal{C}_u)}}{\int} \mathscr{D}\,\mathrm{curl}\,\Omega\, B\, \mathscr{D}\,\mathrm{curl}\,\Omega = \int_0^t \underset{\substack{\Sigma_t \\ (\text{interior to } \mathcal{C}_u)}}{\int} D_\alpha J^\alpha$$

$$= -\int_{\mathcal{C}_u} G(J, V) + \int_{\Sigma_t} G(J, B) + \text{data}$$

where V is a suitably constructed null vector (w.r.t. G) normal to $\mathcal{C}_u$ that allows us to apply the divergence theorem with a null boundary and all integrals are with respect to suitable geometrically induced volume elements. From the construction of V and $G(B,B) = -1$ it comes $G(V,B) = -1$, so

$$G(J, V) = |\mathscr{D}\,\mathrm{curl}\,\Omega|^2 \underbrace{G(B, V)}_{=-1} = -|\mathscr{D}\,\mathrm{curl}\,\Omega|^2. \text{ Thus}$$

$$\int_{\mathcal{C}_u} |\mathscr{D}\,\mathrm{curl}\,\Omega|^2 \lesssim \int_0^t \underset{\substack{\Sigma_t \\ (\text{interior to } \mathcal{C}_u)}}{\int} |\mathscr{D}\,\mathrm{curl}\,\Omega\, B\, \mathscr{D}\,\mathrm{curl}\,\Omega|$$

$$+ \int_{\Sigma_t} |G(J, B)|$$

Using again $G(B,B)=-1$, the second integrand on RHS is simply $|\not{\partial} \operatorname{curl} \Omega|^2$. Using equation (c) we find $B \not{\partial} \operatorname{curl} \Omega \simeq \partial^2 \Psi$. Thus

$$\int_{\mathcal{C}_u} |\not{\partial} \operatorname{curl} \Omega|^2 \lesssim \int_{\Sigma_t} |\not{\partial} \operatorname{curl} \Omega|^2$$

$$+ \int_0^t \int_{\Sigma_t} |\not{\partial} \operatorname{curl} \Omega| \, |\partial^2 \Psi|$$

(interior to $\mathcal{C}_u$)

The first term on RHS is controlled from the energy estimates we derived earlier, as it is the second term on the RHS after applying the Cauchy-Schwarz inequality. Of course, the energy estimates depend on the mixed spacetime norms we are ultimately trying to control, but recall that in the argument everything is organized in a consistent bootstrap. Thus, we obtain the desired control

$$\not{\partial} \operatorname{curl} \Omega \in L^2(\mathcal{C}_u).$$

We make the following two crucial observations.

- The argument relies fundamentally on $G(B,V)=1$, which is only true because B is everywhere transversal to $\mathcal{C}_u$ in view of $G(B,B)=-1$. Absent such a transversality, $G(B,V)$ could change sign or be zero and thus the boundary term $-\int_{\mathcal{C}_u} G(J,V)$ would not correspond to the norm along $\mathcal{C}_u$ we want to control.

- Control of the interior, spacetime integral only works because curl Ω (in reality, C, recall our simplification for exposition purposes) has improved regularity properties as compared to a generic derivative $\partial\Omega$. If we had a generic derivative $\partial\Omega$ instead of curl Ω then we would need to use (b), obtaining $B \cancel{\partial} \partial\Omega \simeq \partial^3 \Psi$, which involves too many derivatives.

In particular, the above highlights that if we had a generic derivative of Ω as a source term in the equation for $\mathrm{tr}_{\cancel{g}} \chi + \Gamma$, the argument would not close.

Generic derivatives of Ω cannot be estimated along cones because they require the Hodge estimates previously employed, which cannot be implemented along cones. This is a feature that repeats throughout the estimates for the acoustic geometry: it's a remarkable feature that the transport variables that appear as source terms for the acoustic geometry estimates appear only in certain especial combination of derivatives for which improved estimates are available. Had generic derivatives be present, the argument would break down.

We can now comment on the aforementioned conformal change. When using multipliers in $\Box_G \varphi = 0$ to estimate φ, we obtain a $\mathrm{tr}_{\not G} \chi$ term. It is, however, $\mathrm{tr}_{\not G} \chi + \Sigma_L$ that we can control, as seen. Thus, we conformaly change G to $\tilde{G}$ with the property that $\mathrm{tr}_{\not G} \tilde{\chi} = \mathrm{tr}_{\not G} \chi + \Sigma_L$. But this now requires controlling the conformal factor of the change. This is done with the help of a modified mass aspect function.

Control of the transport part

We next turn to control of the transport variables. We already discussed one important aspect, namely, control along sound cones.

Bounds for $\partial\Omega$ in $C^{0,\alpha}(\Sigma_t)$ can be obtained as follows. First, we establish a div-curl estimate in Hölder spaces

$$\|\partial\Omega\|_{C^{0,\alpha}(\Sigma_t)} \lesssim \|\operatorname{div}\Omega\|_{C^{0,\alpha}(\Sigma_t)} + \|\operatorname{curl}\Omega\|_{C^{0,\alpha}(\Sigma_t)} + \underbrace{L.O.T}_{\text{ignore}}$$

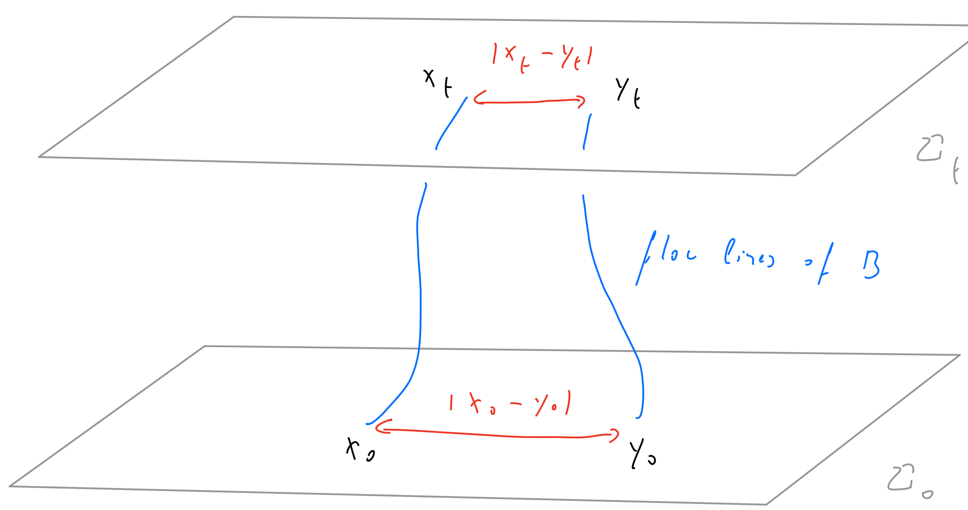
Using (b),

$$\|\partial\Omega\|_{C^{0,\alpha}(\Sigma_t)} \lesssim \|\partial\Psi\|_{C^{0,\alpha}(\Sigma_t)} + \|\operatorname{curl}\Omega\|_{C^{0,\alpha}(\Sigma_t)}$$

For $\operatorname{curl}\Omega$, we use that it satisfies the transport equation (c). To use this equation we derive an energy estimate for transport equations in Hölder spaces that reads

$$\|\operatorname{curl}\Omega\|_{C^{0,\alpha}(\Sigma_t)} \lesssim \|\operatorname{curl}\Omega\|_{C^{0,\alpha}(\Sigma_0)} + \int_0^t \|\partial\Psi\|_{C^{0,\alpha}(\Sigma_\tau)}\, d\tau$$

This estimate is proven by integrating along the characteristics of the transport operator, i.e., the flow lines B and comparing ratios at nearby points. In particular it requires comparing nearby points at time t with their initial positions along the flow, i.e., $|x_t - y_t| \approx |x_0 - y_0|$. This is the case because, with our regularity assumptions, we have control over the flow lines of B.



The estimates will now close provided we can control the Hölder norm of $\partial\Psi$. More precisely, because we need to control only ∂u in $L^2_t C^{0,\alpha}_x$, it suffices to control $\|\partial\Psi\|_{L^2_t C^{0,\alpha}_x}$, which is controlled by the bootstrap assumption.

We also remark that control of the acoustic geometry also involves estimates on the spheres $S_{t,u}$. Because of our functional framework, this eventually leads to Hölder estimates for geometric quantities on the spheres. These are obtained by transport along integral curves of L (which thus need to be controlled), resembling what we did above for the integral curves of B. $\square$

The previous theorem is for the classical compressible Euler system. What about the relativistic case that concerns us here? The equations are significant more complicated. However, Sifan Yu was able to generalize the above techniques to the relativistic setting:

Theo (S. Yu [Yu]). A similar low regularity result as in the previous theorem holds for the relativistic Euler equations.

proof: See [Yu]. We stress that this result should not be taken for granted. Due to the increased complexity of the relativistic equations, there is no reason to believe that results from the classical setting will generalize to the relativistic case. This is especially the case for a result involving many delicate estimates as the one we just presented.

$\square$

Remark. The presence of null forms is not important for these low regularity results, although it is key that they are quadratic. Other special structures of the equations are, as seen, crucial because of the applications of L^2-L^∞ estimates that produce the mixed spacetime norms that can be controlled with Strichartz estimates and our methods.

The study of shock formation

Roughly, a shock wave, or shock for short, is a singularity on solutions to a PDE where the solution remains bounded but one of its derivatives blows up. While it is known that solutions to the relativistic Euler equations can breakdown in finite time [GS] for smooth initial data, we want to understand the nature of the singularity. Thus, we want to discuss the problem of constructive proofs of stable shock formation without symmetric assumptions in more than one spatial dimension, henceforth referred to simply as the problem of shock formation, by which we mean:

- Shocks form for an open set $\mathcal{B}$ of (small) initial data (usually perturbations of constant solutions). (Stability.)
- $\mathcal{B}$ contains "arbitrary" initial data, i.e., not restricted to a symmetry class
- Proofs are constructive, so that we can get a precise description of the shock profile. (Needed for continuing the solution past the shock in a weak sense).

The framework needed to establish proofs of shock formation involves the following ingredients:

Ingredient one: nonlinear geometric optics. This is done by introducing an eikonal function u, which is a solution to the eikonal equation

$$G^{\alpha\beta} \partial_\alpha u \, \partial_\beta u = 0,$$

with appropriate initial condition. The eikonal function plays two crucial roles.

First, the level sets of u are the characteristics associated with the metric G, which are the sound cones. In this regard, we note that u is adapted to the wave part of the system and not to the transport part. This choice is based on the fact that the transport part corresponds to the evolution of the vorticity and entropy, and there are no known blow-up results for these quantities. On the other hand, the only known mechanism of blow-up for relativistic Euler is the intersection of the sound cones. (For classical Euler, other types of singularities have been recently constructed, but their stability is unknown [MRRS]). In particular, this shows the importance, in the context of shock formation, of not treating the transport and sound part of the system together, as it is done in the first order symmetric hyperbolic formalism. The intersection of the sound cones is measured by the inverse foliation density μ defined as

$$\mu = - \frac{1}{G^{\alpha\beta} \partial_\alpha t \, \partial_\beta u},$$

which has the property that $\rho \to 0$ corresponds to the intersection of the characteristics.

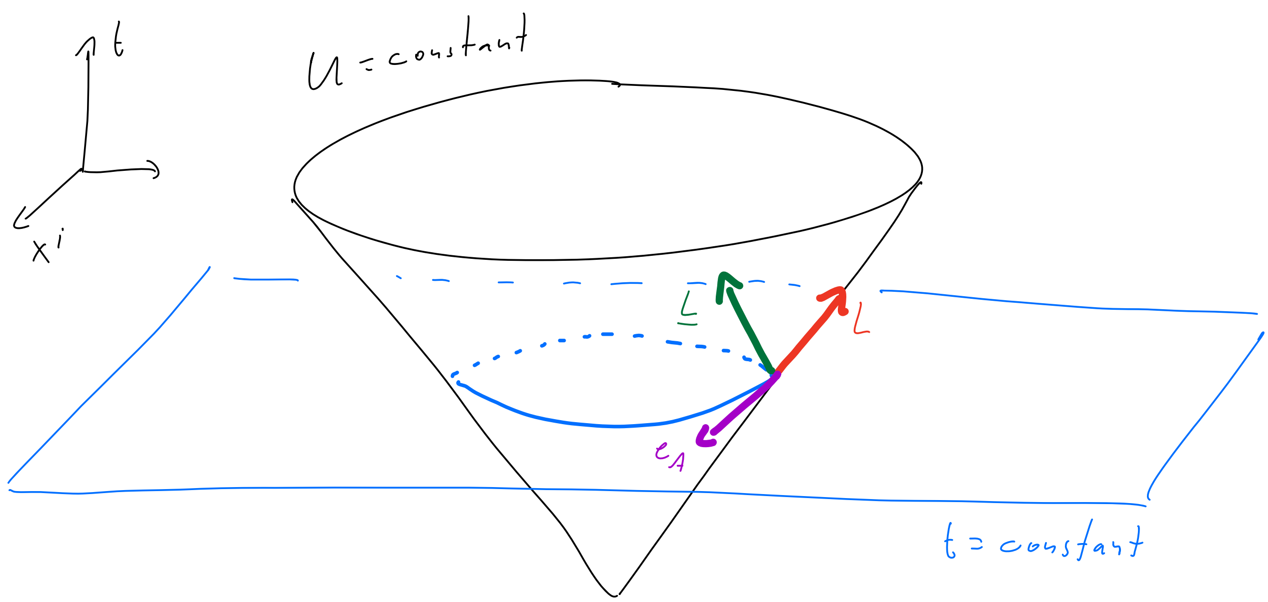
Second, in order to detect the blow up, we need to identify precisely in which direction(s) the solution blows up, and which direction it remains bounded. This is done with the introduction of a null-frame

$$\{ e_1, e_2, \underline{L}, L \}$$

adapted to the sound cones. Here, L and $\underline{L}$ are null vectors with respect to G, satisfying $G(\underline{L}, L) = -2$, and $\{e_1, e_2\}$ is an orthonormal, with respect to G, frame on the (topological) spheres given by the intersection

$$\{ t = \text{constant} \} \cap \{ U = \text{constant} \}.$$

We also have that $G(e_A, L) = 0 = G(e_A, \underline{L})$, $A = 1, 2$.



We can decompose quantities w.r.t. this null frame, and identify that blow-up occurs in the $\underline{L}$ direction, while derivatives of the fluid variables in the other directions remain bounded. To carry out the analysis, we also introduce a geometric system of coordinates adapted to the sound characteristics,

$$\{t, U, \vartheta^1, \vartheta^2\},$$

where ϑ^A, $A=1,2$, are coordinates on the spheres $\{t=\text{constant}\}\cap\{U=\text{constant}\}$ (they are constructed upon solving $G^{\alpha\beta}\partial_\alpha U \partial_\beta \vartheta^A = 0$ with appropriate initial conditions).

Ingredient two: nonlinear null-structure. The basic philosophy for the proof of shock formation is to show that, relative to the geometric coordinates $\{t, U, \vartheta^1, \vartheta^2\}$, the solution remains bounded all way to the shock. In this way we transform the problem of shock formation into a more traditional one, where the goal is to derive long-time estimates for the solution (relative to the geometric coordinates). The blow-up of the solution w.r.t. the original coordinates is recovered by showing that the geometric coordinate system degenerates (in a precise fashion) relative to the original coordinates (since the characteristics are intersecting at the shock, we expect the geometric coordinates to degenerate there).

A crucial aspect of these constructions is that the null-frame and the geometric coordinates depend on the fluid's solution variables, since they (the null frame and the geometric coordinates) are constructed out of $\mathcal{U}$ which depends on G. (in broad philosophical terms, this resembles the approach to Einstein's equations, where the wave coordinates depend on the solution, i.e., on the spacetime metric). Therefore, in order to implement these ideas we have to show that the geometric coordinates remain regular all way up to the shocks. And to do so we need to obtain precise estimates for the fluid variables, showing, in particular, that the derivatives tangent to the sound cone do not produce singularities, the latter coming from derivatives in the $\underline{L}$ direction, as mentioned. One important big idea here is the following. We show that the evolution can be decomposed into a Riccati-type term that drives the blow-up (recall that the Riccati ODE is $\frac{dz}{dt} = z^2$, which blows up in finite time) and error terms that do not significantly alter the high-frequency behavior of the Riccati term. Such terms appear as follows (we will illustrate with $\hat{h}$, similar statements hold for u^α). Expanding the covariant wave operator relative to the null frame we find that the equation for $\hat{h}$ reads, schematically,

$$L(\underline{L}\hat{h}) \simeq -(\underline{L}\hat{h})^2 + Q,$$

where Q denotes linear combinations of null forms relative to G (and we omit harmless terms, e.g., terms linear in derivatives). The equation $L(\underline{L}\hat{h}) \simeq -(\underline{L}\hat{h})^2$ is the Riccati equation for the variable $\underline{L}\hat{h}$, since L is differentiation in the direction of L, thus $L = \frac{d}{d\tau}$ for a suitable parametrization of the flow lines of L. Thus, we need to show that Q is a perturbation that does not significantly alter the Riccati behavior. This is problematic because Riccati terms are generally unstable under perturbations. However, and here is where the role of null-forms is important, Riccati terms are stable upon perturbations by null forms. Relative to the null-frame, we have

$$Q(\partial\varphi, \partial\psi) = \tau(\varphi)\partial\psi + \tau(\psi)\partial\varphi,$$

where τ is differentiation tangent to the sound cones. This implies that even though Q is quadratic, it never involves terms quadratic in the direction the system wants to blow-up. Specifically, in our case, we then have

$$L(\underline{L}\hat{h}) \simeq -(\underline{L}\hat{h})^2 + \tau(\hat{h})\partial h,$$

so that the first term on the RHS is the only term quadratic in $\underline{L}\hat{h}$. If insted of $\tau(\hat{h})$ we had $\partial\hat{h}$ then we would get a $(\partial\hat{h})^2$ term. After decomposing in a null frame, this $(\partial\hat{h})^2$ could produce a $(\underline{L}\hat{h})^2$ that cancels or nearly cancels the $-(\underline{L}\hat{h})^2$ term from the Riccati part, thus working against the blow-up and preventing us from proving that shocks form. The term $\tau(\hat{h})\partial\hat{h}$, on the other hand, is at most linear in $\underline{L}\hat{h}$ so that

$$L(\underline{L}\hat{h}) \simeq -(\underline{L}\hat{h})^2 + \tau(\hat{h})\underline{L}\hat{h}.$$

Since the tangential derivatives remain bounded, the first term on the RHS dominates over the last term, leading to the blow-up of $\underline{L}\hat{h}$.

Remark. A straw man ODE analogy of the above is the following. Consider the two following perturbations of the Riccati ODE $\frac{dz}{dt} = z^2$: $\frac{dz}{dt} = z^2 + \varepsilon z$, $\frac{dz}{dt} = z^2 \pm \varepsilon z^3$, $z(0) > 0$, $\varepsilon > 0$ small. The first equation still blows up and it does it at the same rate as the original one. For the second perturbation, depending on the sign $\pm$ the solution will either exist for all time or it will blow up at an entirely different rate (thus effectively altering the blow-up). The null-forms are the PDE analog of the εy perturbation.

Ingredient three: energy estimates and regularity. The previous arguments assumes that we can in fact close estimates establishing several elements needed in the above discussion (e.g. that tangential derivatives do in fact remain bounded). Thus, we need to derive estimates not only for the fluid variables but also for the eikonal function (since the regularity of the null-frame is tied to that of u).

Energy estimates for the fluid variables are obtained by commuting the equations with derivatives, but in order to avoid generating uncontrollable source terms, we need to commute the equations with certain vector fields that are adapted to the sound characteristics. This leads to vector fields of the form $Z \sim \partial \mathcal{U} \cdot \partial$. Commuting through, e.g., the equation for $\hat{h}$:

$$Z(\Box_g \hat{h}) \sim \Box_g (Z\hat{h}) + (\Box_g \partial \mathcal{U}) \partial \hat{h}$$

$$\sim \Box_g (Z\hat{h}) + \partial^3 \mathcal{U} \cdot \partial \hat{h},$$

so the equation for $\hat{h}$ gives

$$\Box_g (Z\hat{h}) \sim \partial^3 \mathcal{U} \cdot \partial \hat{h} + \dots$$

Since $\mathcal{U}$ solves a (fully non-linear) transport equation, standard regularity theory for transport equations gives that $\mathcal{U}$ is only as regular as the coefficients of the equation, which in this case is G, and since $G = G(\hat{h}, s, u^\alpha)$, we find $\partial^3 \mathcal{U} \sim \partial^3 G \sim \partial^3 \hat{h} + \dots$ On the other hand, standard energy estimates for wave equations give that from $\Box_g (Z\hat{h})$ we obtain control of $\partial(Z\hat{h}) \sim \partial^2 \hat{h}$, so in the end we are trying to control $\partial^2 \hat{h}$ in terms of $\partial^3 \hat{h}$ and thus have a derivative loss.

It turns out that we can overcome the regularity loss by exploiting some delicate tensorial properties of the eikonal equation and of the wave equation relative to geometric coordinates. Together these properties can be used to show that certain geometric tensors constructed out of U enjoy *extra regularity in directions tangent to the sound cones.* Carefully accounting for the precise structure of the aforementioned $\partial^2 U \partial h$ term we can show that it is precisely one of such terms with extra regularity. It turns out that all terms that seem to exhibit loss of regularity are of this form and can thus be controled.

Remark. The special structures mentioned above that are used to prevent loss of regularity of the eikonal function are tied to the geometry of the sound cones. The improved estimates, without regularity loss, for U are not based directectly on the eikonal equation, but rather on evolution equations for geometric quantities, i.e., the null-structure equations we saw before.

To close the estimates we also need to use the extra regularity that we obtained for s and $\bar{w}$ to close the estimates. To see this, let us do a naive derivative counting. From the equation for n^α we have $\Box_g n \sim C$, so we can control $\partial u \lesssim C$. But $C \sim \mathrm{vort}(\bar{w}) \sim \partial w$. From the transport equation for $\bar{w}$, $u^\lambda \partial_\lambda \bar{w} \sim \partial u$, we can control $\bar{w} \sim \partial u$, so in the end we are controlling $\partial u \lesssim \partial^2 u$, which has a loss of a derivative. This loss of regularity can be avoided, however, by using the extra regularity for $\bar{w}$ mentioned earlier. Something similar happens with some estimates involving s.

Finally, we mentioned that the energy estimates that are needed are in fact weighted estimates, where the weight is given by the inverse foliation density μ. Since $\mu \to 0$ at the shock, we end up with energies that are singular at top order. This is a major technical point that involves a complex bootstrap argument to close the estimates.

The above ingredients seem to be needed to establish proofs of shock formation, and are used in all known such proofs (in $n \geq 2$, see below). The crucial point for us here is that all such ingredients are present in the new formulation of the relativistic Euler equations.

Some context for the work on shocks

The ingredients outlined above have not all being introduced in [DS]. They are the culmination of a series of beautiful ideas developed by a series of authors. For the sake of time we will not review this history here, but we refer to the introduction of [DS].

As said, the fluid is irrotational, the new equations reduce significantly and agree with those found by Christodoulou [Ch]. The inclusion of vorticity causes several new difficulties and it is quite remarkable that the vorticity case presents many of the good structures found (and needed) in the irrotational case.

Finally, we mention that in one spatial dimension, the picture is compellingly simpler: in 1d we can rely essentially on the method of characteristics. While this is essentially the same as introducing an eikonal functional, in 1d we can dispense with all the geometric machinary discussed above. Also, we do not need to carry out energy estimates. Instead, one uses estimates in BV (bounded variation) spaces. It is possible to prove that such BV estimates do not generalize to two or more spatial dimensions [Ra].

The relativistic Euler equations with a physical vacuum boundary

Consider fluid within a domain that is not fixed but moves with the fluid motion:

Fluids of this type are called free-boundary fluids. Examples include a liquid drop or, more relevant for the relativistic case, a star.

Denoting by Ω_t the region occupied by fluid at time t, the dynamics of the fluid is defined in the spacetime region

$$\Omega := \bigcup_{0 \leq t < T} \{t\} \times \Omega_t$$

for some $T > 0$, known as the moving domain.

The fluid's free-boundary (a.k.a. moving boundary, free interface) is

$$\Gamma := \bigcup_{0 \leq t < T} \{t\} \times \Gamma_t \quad , \quad \Gamma_t := \partial\Omega_t .$$

Note that Ω has to be determined alongside the fluid motion, i.e., it cannot be freely prescribed a priori.

The free-boundary relativistic Euler equations are the relativistic Euler equations defined on a moving domain Ω. In this case, we have to impose additionally the boundary conditions

$$p|_{\Gamma} = 0, \quad u \in T\Gamma,$$

where $T\Gamma$ is the tangent bundle of Γ. The first condition comes from physics and says that the pressure has to vanish in the fluid-vacuum interface (alternatively we could have $p =$ constant if the moving fluid is immersed in fixed medium, e.g., a liquid drop in air). The second condition says that Γ_t is advected by the fluid, i.e., Γ_t moves with velocity equal to that of the fluid on the boundary.

Let assume from now on that we have a barotropic equation of state, $p = p(\varrho)$. Then:

$$p|_{\Gamma} = 0 \implies \text{condition for } \varrho|_{\Gamma}.$$

There are two distinct cases to consider:

Liquid: $\varrho \geq$ constant > 0 on Γ,

Gas: $\varrho = 0$ on Γ.

(In both cases $p|_{\Gamma} = 0$). The liquid and gas cases, whose names are more or less self-explanatory, are very different problems. A key difference is that the equations degenerate on the boundary in the gas case (since $(p+\varrho)|_\Gamma = 0$) but not in the liquid case (since $(p+\varrho)|_\Gamma > 0$). Here, we will consider the case of a gas, in which case Γ is also known as a <u>vacuum boundary</u>. In the gas case, Ω_t is given by

$$\Omega_t = \{ x \in \mathbb{R}^3 \mid \varrho(t,x) > 0 \}.$$

(Other topologies than $\mathbb{R}^3$ can be considered.) In the gas case, we also impose

$$c_s^2 \big|_\Gamma = 0$$

which is related to the fact that sound waves cannot propagate in vacuum.

<u>Remark.</u> The condition $c_s^2|_\Gamma = 0$ implies that the sound cones degenerate to the flow lines on the boundary. Thus, this problem not only has multiple characteristics; it has repeated characteristics.

A standard equation of state in the study of a gas with a free boundary is

$$p(\varrho) = \varrho^{k+1}, \quad k > 0,$$

which we henceforth adopt.

It turns out that the decay rate of c_s^2 near Γ_t plays a crucial role in this problem. To see it, let us assume that near Γ_t c_s^2 decays a power of the distance to the boundary:

$$c_s^2 \approx d^\beta \quad , \quad d(t,x) = dist(x, \Gamma_t).$$

This assumption is natural because d is a natural scale to consider since away from the boundary we essentially have the standard (non-free boundary) relativistic Euler equations in light of finite propagation speed. Alternatively, we can consider a Taylor expansion near Γ_t with coordinates such that $x^3 = d$. Then, the fluid's acceleration is

$$a_\alpha = u^\mu \nabla_\mu u_\alpha = -\frac{\Pi_\alpha^\mu \partial_\mu p}{p+\varrho} \sim \frac{c_s^2 \partial \varrho}{\varrho + \varrho^{\kappa+1}} \sim \frac{c_s^2 \partial \varrho}{\varrho}$$

$$\sim \frac{d^\beta \, d^{\beta/\kappa - 1} \overbrace{\partial d}^{O(1)}}{d^{\beta/\kappa}} \sim d^{\beta - 1}$$

$$\downarrow$$

$$\varrho^\kappa \sim c_s^2 \sim d^\beta$$

Thus

$$a\big|_{\Gamma} = \begin{cases} 0 & \text{if } \beta > 0 \\ \text{finite} & \text{if } \beta = 1 \\ \infty & \text{if } \beta < 1 \end{cases}$$

The first and third conditions are not physical (zero boundary acceleration would not allow the fluid to rotate, as stars do). We henceforth assume that c_s^2 is comparable to the distance to the boundary, i.e.,

$$c_s^2(t,x) \approx \operatorname{dist}(x, \Gamma_t) \text{ for } x \text{ near } \Gamma_t.$$

a condition known as the physical vacuum boundary condition.

This condition should be viewed as a constraint, i.e., a condition imposed on the initial data that is propagated by the flow. In this setting, the free-boundary relativistic Euler equations are referred to as the relativistic Euler equations with a physical vacuum boundary.

Remark. The physical vacuum boundary condition implies that linear waves with speed c_s reach the boundary in finite time. Thus the motion of the boundary is strongly coupled with the bulk evolution and cannot be viewed as a self-contained evolution at leading order.

Our general strategy to study the problem will be:

- A choice of good nonlinear variables that diagonalize the equations w.r.t. the material derivative

$$D_t := \partial_t + \frac{u^i}{u^0}\partial_i .$$

We want to diagonalize the equations in part because we want to apply an Euler method to obtain solutions, so we want ∂_t LHS $= \partial_x$ RHS. We will see later what D_t is the right vectorfield to consider.

- A choice of good linear variables: the analysis of the linearized equation plays a key role in our approach.

- Derive energy estimates for D_t^μ (good nonlinear) by showing it satisfies the linearized equation with good perturbative terms. Use elliptic estimates to control full derivatives.

- We use a regularization + time discretization to obtain solutions.

Assumption. We will henceforth assume that g is the Minkowski metric. This is not an oversimplification: all features of the problem are already present in Minkowski space (coupling to Einstein, on the other hand, is a much harder problem).

Diagonalization

Let us consider a rescale $\sigma = f(\rho)u$ where f will be chosen. In view of the constraint $\sigma^\alpha \sigma_\alpha = -f^2$, it suffices to consider the evolution of σ^i. Using the relativistic Euler equations, we find

$$\underbrace{\frac{p+\rho}{f} u^\mu \partial_\mu \sigma^i}_{\sim D_t,\ \text{good}} + c_s^2 \underbrace{g^{i\mu}\partial_\mu}_{\text{no } \partial_t,\ \text{good}} \rho + \underbrace{\left(-\frac{f'}{f}(p+\rho) + c_s^2 \right)}_{\text{choose } f \text{ to kill this term}} u^i \underbrace{u^\mu \partial_\mu}_{\text{has } \partial_t,\ \text{bad}} \rho = 0$$

$$\Longrightarrow f(\rho) = \exp \int \frac{c_s^2(\rho)}{p(\rho)+\rho}\, d\rho$$

The resulting function f is not unfamiliar. Recall that we defined the vorticity as

$$\Omega = d(hu), \quad h = \frac{p+\rho}{n}.$$

In the absence of a baryon density n (which we are considering here), we can alternatively define

$$\Omega = d(fu) = dv.$$

Thus, the choice of f that kills the bad term is the same that is used to define the vorticity. One can derive the following evolution

$$v^\gamma \partial_\gamma \Omega_{\alpha\beta} + \partial_\alpha v^\gamma \Omega_{\gamma\alpha} + \partial_\beta v^\gamma \Omega_{\alpha\gamma} = 0,$$

which in particular implies that $\Omega = 0$ if so initially. Because we will only consider the evolution of the spatial part v^i, we also look for an evolution involving Ω_{ij}. The following identy can be verified:

$$v^\gamma \Omega_{\gamma\alpha} = 0$$

We can use it to solve for Ω_{0i} in terms of the spatial components Ω_{ij}:

$$\Omega_{0j} = -\frac{v^i}{v^0}\omega_{ij}.$$

Using this into the above evolution equation:

$$D_t \omega_{ij} + \frac{1}{v^0}\partial_i v^h \omega_{hj} + \frac{1}{v^0}\partial_j v^h \omega_{ih} - \frac{1}{(v^0)^2}\partial_i v^0 v^h \omega_{hj}$$

$$+ \frac{1}{(v^0)^2}\partial_j v^0 v^h \omega_{hj} = 0$$

which is the evolution equation for the vorticity we will employ.

Remark. Above and throughout, we consider only the spatial components v^i as primary variables for v, so v^0 always means $v^0 = \sqrt{f^2 + v^i v_i}$. In particular, when referring to v we will always mean (v^1, v^2, v^3).

Remark. All the estimates we will discuss need to be complemented by estimates for the vorticity. These estimates are obtained by direct estimates using the above evolution equation. For simplicity, we will omit here such vorticity estimates.

Our choice of f also diagonalizes the energy equation:

$$u^\mu \partial_\mu \varrho + \frac{(p+\varrho)}{a_0}\left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right)\partial_i v_j - \frac{c_s^2}{a_0 (v^0)^2} f v^i \partial_i \varrho = 0,$$

$a_0 = 1 - c_s^2 \frac{v^i v_i}{(v^0)^2}$. The above is for a general equation of state. For $p(\varrho) = \varrho^{h+1}$, we find $f(\varrho) = (1+\varrho^h)^{1+\frac{1}{h}}$

Since c_s^2 is an important quantity, it is convenient to take it as primary variable instead of ϱ. So we define $r := \frac{h+1}{h}\varrho^h$, which is the sound speed up to a constant factor. In terms of r and v_i, the relativistic Euler equations read:

$$D_t r + r (G^{-1})^{ij}\partial_i v_j + r a_1 v^i \partial_i r = 0,$$

$$D_t v_i + a_2 \partial_i r = 0,$$

where G^{-1} is an inverse Riemannian metric given by

$$(G^{-1})^{ij} = \frac{h\left(1+\frac{hr}{h+1}\right)}{a_0 v^0}\left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right) \quad (G \text{ is}$$

related to the acoustical metric; note that $G^{-1ij}\partial_i(\cdot)_j$ is a divergence operator), a_0, a_1, and a_2 are smooth functions of (r, v) that are $O(1)$ near Γ_t and $a_2 > 0$.

Function spaces

Let us denote by s and w the linearized variables associated with r and v, respectively. We will see that the linearized equations admit the following energy:

$$\|(s,w)\|_{\mathcal{H}}^2 := \int_{\Omega_t} r^{\frac{1-h}{h}} \left(s^2 + \frac{1}{a_2} r (G^{-1})^{ij} w_i w_j \right)$$

which can be thought as a weighted L^2 norm. We will see below why such weights are needed, but the reader can expect this to be needed since, as said, the equations are degenerate.

While eventually we want r to be a solution to the equations, for this definition it suffices to take r to be a defining function for Ω_t, i.e., $\Omega_t = \{r > 0\}$, and $r \approx \mathrm{dist}(\cdot, \Gamma_t)$.

Next, we want to define higher order spaces.
A hint of how to do so can be taken from the underlying wave evolution, which at leading order is governed by the wave operator $D_t^2 - r\Delta$. This suggests building higher order spaces based on powers of $r\Delta$ in the underlying weighted L^2 space $\mathcal{H}$. We set

$$\|(s,w)\|^2_{\mathcal{H}^{2l}} := \sum_{\substack{|\alpha|=0 \\ |\alpha|-a\leq h}}^{2l} \sum_{a=0}^{l} \| r^{\frac{1-h}{2h}+a} \partial^\alpha s\|^2_{L^2(\Omega_t)}$$

$$+ \sum_{\substack{|\alpha|=0 \\ |\alpha|-a\leq h}}^{2l} \sum_{a=0}^{l} \| r^{\frac{1-h}{2h}+\frac{1}{2}+a} \partial^\alpha w\|^2_{L^2(\Omega_t)}.$$

To better understand this definition, look at top order:

$$\|(s,w)\|_{\mathcal{H}^{2l}} \sim \| r^{\frac{1-h}{2h}+l} \partial^{2l} s\|_{L^2(\Omega_t)} + \| r^{\frac{1-h}{2h}+l+\frac{1}{2}} \partial^{2l} w\|_{L^2(\Omega_t)}$$

$$\sim \|(r\Delta)^l (s,w)\|_{\mathcal{H}}.$$

This definition can be extended to non-integer $l\geq 0$ by interpolation.

Scaling analysis

Ignoring $\mathcal{O}(1)$ terms, our equations of motion reduce to

$$(\partial_t + v^i \partial_i) r + r \delta^{ij} \partial_i v_j + r v^i \partial_i r = 0$$

$$(\partial_t + v^j \partial_j) v_i + \partial_i r = 0$$

As we will se later, the term $r v^i \partial_i r$ can be treated essentially as a perturbation. This a consequence of the fact that it with the weight r but s requires one less power of r in our energies as compared to w. Thus we drop it for now, obtaining:

$$(\partial_t + v^i \partial_i) r + r \delta^{ij} \partial_i v_j = 0$$

$$(\partial_t + v^j \partial_j) v_i + \partial_i r = 0$$

which heuristically we expect captures the leading order dynamics near the boundary. These equations admit the scaling symmetry:

$$(r(t,x), v(t,x)) \mapsto (\lambda^{-2} r(\lambda t, \lambda^2 x), \lambda^{-1} v(\lambda t, \lambda^2 x)).$$

From this we determine the critical space $\mathcal{H}^{2l_0}$,

$$2 l_0 = 3 + 1 + \frac{1}{h}$$

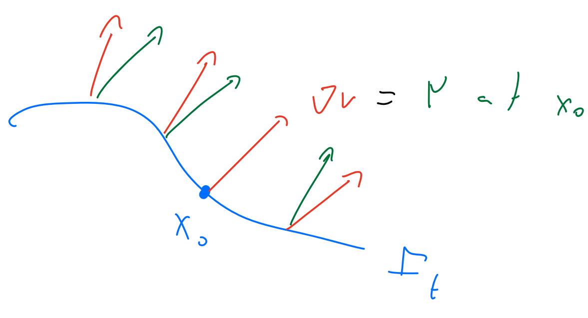
↳ d in d spatial dimensions.

Remark. The full equations do not have a scalling symmetry. Whenever talking about scaling, we mean the scaling symmetry of the above "leading-order" equations.

We next need to define some time dependent control norms that will serve as control norms. Set:

$$A = \|\nabla v - N\|_{L^\infty(\Omega_t)} + \|\sigma\|_{\dot{C}^{1/2}(\Omega_t)}$$

(A is a scale invariant norm) where $\dot{C}^{1/2}$ is the Hölder semi-norm and N is a vectorfield constructed as follows. In each sufficiently small neighborhood of the boundary we can construct N such that $N(x_0) = \nabla v(x_0)$ for some fixed $x_0 \in \Gamma_t$.



The point of introducing N is that we can make A small by localization, whereas $\|\nabla r\|_{L^\infty(\Omega_t)}$ is scale invariant thus cannot be made small by localization or scaling arguments. We also introduce

$$B := A + \|\nabla r\|_{\tilde{C}^{1/2}(\Omega_t)} + \|\nabla v\|_{L^\infty(\Omega_t)}$$

where

$$\|f\|_{\tilde{C}^{1/2}(\Omega_t)} := \sup_{\substack{x,y\in\Omega_t \\ x\neq y}} \frac{|f(x)-f(y)|}{r(x)^{1/2}+r(y)^{1/2}+|x-y|^{1/2}}.$$

We can think of $\|\nabla r\|_{\tilde{C}^{1/2}(\Omega_t)}$ as roughly the $\dot{C}^{3/2}$ Hölder semi-norm, but it is a bit weaker as it uses only one derivative away from the boundary.

The norms A and B are associated with the spaces $\mathcal{H}^{2l_0}$ and $\mathcal{H}^{2l_0+1}$ in view of the embeddings:

$$A \lesssim \|(s,w)\|_{\mathcal{H}^{2l}}, \quad 2l > 2l_0$$

$$B \lesssim \|(s,w)\|_{\mathcal{H}^{2l}}, \quad 2l > 2l_0+1.$$

Local well-posedness and continuation criterion

We can now state our main results.

Theo (D-Ifrim-Tataru, [DIT]) Consider equations

$$D_t r + r (G^{-1})^{ij} \partial_i v_j + a_1 r v^i \partial_i r = 0$$
$$D_t v_i + a_2 \partial_i r = 0 \qquad (*)$$

in Ω, where Ω is as above. Define the state space

$$\mathbb{H}^{2l} := \{ (r,v) \mid (r,v) \in \mathcal{H}^{2l} \}.$$

Then equations $(*)$ are locally well-posed in $\mathbb{H}^{2l}$ for data $(\mathring{r}, \mathring{v}) \in \mathbb{H}^{2l}$ provided that

$$\mathring{r}(x) \approx \operatorname{dist}(x, \Gamma_0), \quad \Omega_0 = \{ \mathring{r} > 0 \}$$

and

$$2l > 2l_0 + 1, \quad 2l_0 = 3 + 1 + \frac{1}{h}.$$

Remarks.

- Local well-posedness above is meant in the usual Hadamard sense: existence and uniqueness of solutions $(r,v) \in C^0([0,T], \mathbb{H}^{2l})$ for some $T > 0$ and continuous dependence of the solution on the initial data in the H^{2l} topology. (We have not defined the relevant topology in H^{2l} and will not do so here, see [DIT] for details.)

– Observe that we obtain local well-posedness for data only half derivative above scaling.

To the best of our knowledge, this is the first local existence and uniqueness result for the relativistic Euler equations with a physical vacuum boundary (in more than one dimension spatial dimension; in one spatial dimension Oliynyk [Ol] established local existence and uniquess. In this setting, however, the boundary is just points and the main difficulties are absent.) A priori estimates had previously been obtained by Hadzic–Shkoller–Speck [HSS] and Jang–LeFloch–Masmoudi [JLM]. (In the case when the boundary cannot accelerate, $c_s^2 \approx d(\cdot, \partial\Omega)$, the problem was treated in [Rn].)

It is possible to transform the moving domain Ω in a fixed domain $[0,T] \times \Omega_0$ via a solution-dependent map $\eta : [0,T] \times \Omega_0 \longrightarrow \Omega$. This has the advantage of fixing domain but introduces new nonlinearities. In this approach, we say that the equations are written in Lagrangian coordinates.

The a priori estimates [JLM, HSS] are done in

Lagrangian coordinates. Our approach, in contrast, deals with the equations in the moving domain Ω, in which case we say that the equations are written in Eulerian coordinates.

We next investigate the question of continuation of solutions:

Theo (D-Ifrim-Tataru, 2020). For each integer $l \geq 0$ there exists an energy functional $E^{2l} = E^{2l}(r,v)$ with the following properties:

a) Coercivity; as long as A remains bounded,

$$E^{2l} \approx \|(r,v)\|^2_{\mathcal{H}^{2l}}.$$

b) Energy estimates hold for solutions to (*):

$$\frac{d}{dt} E^{2l} \lesssim_A B \|(r,v)\|^2_{\mathcal{H}^{2k}}.$$

As a consequence of this theorem, Grönwall's inequality gives

$$\|(r,v)\|^2_{\mathcal{H}^{2l}} \lesssim e^{\int_0^t C(A) B} \|(\mathring{r},\mathring{v})\|^2_{\mathcal{H}^{2l}}$$

Remark. We construct the energies E^{2l} explicitly only for integer $l \geq 0$, but our analysis shows that the last inequality also holds for non-integer $l > 0$.

The previous theorem and the above remark lead to:

Coro (D-Ifrim-Tataru, 2020). The unique solution obtained above can be continued as long as A remains bounded and $B \in L^1_t(\Omega)$.

We will now discuss one important aspect of the proofs, namely, the energy estimates. Readers are referred to [DIT] for full details. In deriving these estimates, we will rely on the following. Due to finite propagation speed, we can localize the problem. Away from the boundary, $r \geq$ constant > 0 and standard estimates are readily available. Therefore, we will implicitly assume throughout that we are working in a neighborhood of the boundary. In particular, we can assume that r is small.

Energy estimates for the linearized equation

Let us consider the linearized equations, which read

$$D_t s + \frac{1}{h}(G^{-1})^{ij}\partial_i r w_j + r(G^{-1})^{ij}\partial_i w_j + r a_1 \omega^i \partial_i s = f,$$

$$D_t w_i + a_2 \partial_i s = h_i$$

where f and h are of the form

$$f = S_1 s + r W_1 w, \quad h = S_2 s + W w$$

where f and h are linear in $\partial(r,v)$ with coefficients that are smooth functions of (r,v). These will be error terms. We make the following observations:

- The linearized system does not require boundary boundary conditions. This is related to the fact that the one-parameter family of solutions used to produce the linearization are not required to have the same domain. Alternatively, we can think that the "boundary conditions" are included in our choice of weights for our function spaces.

- The term $\frac{1}{h}(G^{-1})^{ij}\partial_i r\, w_j$ comes from the linearization of D_t. We obtain precisely a term in G^{-1} when computing the linearization.

- The term $\frac{1}{h}(G^{-1})^{ij}\partial_i r\, w_j$ does not contain derivatives of (s,w), so at first sight it looks like an error term that should be moved to the RHS. We will soon see that this term is not lower order with respect to our energies, as it does not contain the right weight.

To derive control of $\frac{d}{dt}$(energies), we will use the moving domain formula

$$\frac{d}{dt}\int_{\Omega_t} f = \int_{\Omega_t} D_t f + \int_{\Omega_t} f\, \partial_i\left(\frac{v^i}{v^0}\right)$$

which holds true because $\frac{v^i}{v^0} = \frac{u^i}{u^0}$ is the actual physical three-velocity of the fluid particles on the boundary. This is one motivation for this choice of material derivative.

In order to gain intuition, let us consider the case $k = 1$:

$$D_t s + (G^{-1})^{ij} \partial_i r w_j + r (G^{-1})^{ij} \partial_i w_j + r a_1 w^i \partial_i s = f,$$

$$D_t w_i + a_2 \partial_i s = h_i$$

and let us try to bound the "standard" energy

$$E_{st} = \frac{1}{2} \int_{\Omega_t} s^2 + |w|^2$$

Multiplying the first equation by s, the second by $\frac{1}{a_2} w_i$, integrate over Ω_t and use the moving domain formula,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_t} s^2 + \frac{1}{a_2} |w|^2 + \int_{\Omega_t} r (G^{-1})^{ij} s \partial_i w_j + \int_{\Omega_t} a_1 r s w^i \partial_i s$$

$$\simeq \int_{\Omega_t} s^2 + |w|^2 .$$

Above, the term coming from $r a_1 v^i \partial_i s$ was handled with integration by parts

$$\int_{\Omega_t} a_1 r s v^i \partial_i s = \frac{1}{2} \int_{\Omega_t} a_1 r v^i \partial_i s^2 = -\frac{1}{2} \int_{\Omega_t} \partial_i (a_1 r v^i) s^2,$$

where there is no boundary term because $r = 0$ on Γ.

We need the cross-terms in ∂w and ∂s to cancel after integration by parts, but clearly this cannot be case because of the coefficient $r(G^{-1})^{ij}$. This is easily fixed by multiplying the second equation by $r(G^{-1})^{ij} w_j$, but this requires modifying the energy:

$$\frac{1}{2}\frac{d}{dt} \underbrace{\int_{\Omega_t} s^2 + \frac{1}{a_2} r(G^{-1})^{ij} w_i w_j}_{\text{energy}} + \int_{\Omega_t} r(G^{-1})^{ij} s \partial_i w_j + r(G^{-1})^{ij} w_i \partial_i s = \ldots$$

We can combine the last two integrals and then integrate by parts,

$$\int_{\Omega_t} r(G^{-1})^{ij} s \partial_i w_j + r(G^{-1})^{ij} w_i \partial_i s = \int_{\Omega_t} r(G^{-1})^{ij} \partial_i (w_j s)$$

$$= -\int_{\Omega_t} \partial_i r \, (G^{-1})^{ij} w_j s - \int_{\Omega_t} r \, \partial_i (G^{-1})^{ij} w_j s$$

where there is no boundary term because $r = 0$ on the boundary. The second integral is good because Cauchy-Schwarz gives:

$$-\int_{\Omega_t} r \, \partial_i (G^{-1})^{ij} w_j s \lesssim \underbrace{\| r^{1/2} w \|_{L^2(\Omega_t)}}_{\substack{\text{bounded by the } w\text{-part of} \\ \text{the energy since } (G^{-1})^{ij} \approx \delta^{ij}.}} \| s \|_{L^2(\Omega_t)}$$

The first integral, however, is bad because it lacks a weight, i.e., we cannot bound

$$\int_{\Omega_t} \partial_i r \, |w| s \lesssim \int_{\Omega_t} r |w|^2 + \int_{\Omega_t} s^2$$

since $\partial_i r = \mathcal{O}(1)$ on the LHS but $r \to 0$ near Γ on the RHS.

The problem is the term $(G^{-1})^{ij}\partial_i r w_j s$ coming from the linearization of D_t that we prematurely moved to the RHS, a term that itself is not bounded by the energy because it lacks a weight r. If however, we keep this term on the LHS, then

$$\underbrace{\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} s^2 + \frac{1}{a_2} r (G^{-1})^{ij} w_i w_j}_{\text{energy}} + \int_{\Omega_t} (G^{-1})^{ij}\partial_i r\, w_j s$$

$$+ \int_{\Omega_t} r (G^{-1})^{ij} s\, \partial_i w_j + r (G^{-1})^{ij} w_i \partial_i s = \dots$$

We now see that the bad term $-\int_{\Omega_t} \partial_i r (G^{-1})^{ij} w_j s$ coming from the integration by parts exactly cancels with the term coming from the linearization of D_t (in particular such term is not lower order, as said).

Because our energy now has a weight, there are two further things we need to check. First, that the error terms on the RHS written as ... can indeed be bounded by the energy. This is the case because the term f in the first linearized equation is not only linear in s and w but also in s and rw (h in the second equation is linear in s and w only, but the second equation itself gets multiplied by r).

Second, we need to be more careful with the moving domains formula to make sure we do not pick terms where the weight is differentiated, producing terms $\partial r = O(1)$. Going back to the derivation, the relevant term is

$$\int_{\Omega_t} r (G^{-1})^{ij} w_i D_t w_j = \frac{1}{2} \int_{\Omega_t} r (G^{-1})^{ij} D_t (w_i w_j)$$

$$= \frac{1}{2} \int_{\Omega_t} D_t \left(r (G^{-1})^{ij} w_i w_j \right) - \frac{1}{2} \int_{\Omega_t} r D_t (G^{-1})^{ij} w_i w_j - \frac{1}{2} \int_{\Omega_t} D_t r (G^{-1})^{ij} w_i w_j$$

$$= \frac{1}{2} \frac{d}{dt} \int_{\Omega_t} r (G^{-1})^{ij} w_i w_j - \frac{1}{2} \int r (G^{-1})^{ij} w_i w_j \partial_\ell \left(\frac{v^\ell}{v^0} \right)$$

$$-\frac{1}{2}\int_{\Omega_t} r\, D_t (G^{-1})^{ij} w_i w_j \; - \frac{1}{2}\int_{\Omega_t} D_t r\, (G^{-1})^{ij} w_i w_j$$

where in the last step we used the moving formula. The first term is the time derivative of the energy; the second and third terms are good because they have the weight r. The last term looks problematic though. If we had a generic derivative of r in this term it would be $O(1)$ and indeed we would be in trouble, as we would be missing a weight. However, we have a material derivative, thus we can use the equation satisfied by r:

$$D_t r = -r (G^{-1})^{ij} \partial_i v_j - r a_1 \sigma^i \partial_i r \simeq r\, \partial(r, v)$$

to gain back a power of r, so the corresponding integral is good.

We can now go back to the general case $k \neq 1$. The argument is very similar to above. But now the term coming from the linearization of D_t has a $\frac{1}{h}$ factor. So, in order to get an exact cancellation we multiply the equations by $r^{\frac{1-h}{h}} s$ and $\frac{1}{a_2} r^{\frac{1-h}{h}+1} (G^{-1})^{ij} w_i$, yielding:

$$\frac{1}{h} r^{\frac{1-h}{h}} (G^{-1})^{ij} \partial_i r\, w_j s + r^{\frac{1}{h}} (G^{-1})^{ij} \partial_i w_j s + r^{\frac{1}{h}} (G^{-1})^{ij} w_j \partial_i s$$

$$= \partial_i (r^{1/h}) (G^{-1})^{ij} \partial_i r\, w_j s + r^{\frac{1}{h}} (G^{-1})^{ij} \partial_i w_j s + r^{\frac{1}{h}} (G^{-1})^{ij} w_j \partial_i s$$

$$= (G^{-1})^{ij} \partial_i (r^{1/h} w_j s)$$

which can be integrated by parts. We see that in the end we control the energy $\|(s,w)\|^2_{\mathcal{H}}$, as said.

We have one more comment to make about the linearization of D_t. We said it produces the term $\frac{1}{h} (G^{-1})^{ij} \partial_i r\, w_j$. This is true, but only after some intentional algebra. Linearizing the term $\frac{v^i}{v^0} \partial_i r$ and using that $v^0 = \left[\left(1 + \frac{hr}{h+1}\right)^{2+\frac{2}{h}} + |v|^2 \right]^{1/2}$,

$$\delta\left(\frac{v^i}{v^0}\partial_i r\right) = \delta\left(\frac{v^i}{v^0}\right)\partial_i r + \dots$$

$$= \frac{\delta v^i}{v^0}\partial_i r - \frac{v^i}{(v^0)^2}\delta v^0 \partial_i r + \dots = \frac{\delta v^i}{v^0}\partial_i r - \frac{v^i}{(v^0)^2}\delta v^0 \partial_i r$$

But

$$\delta v^0 = \frac{1}{2v^0}\left[\overbrace{\left(2+\frac{2}{h}\right)\left(1+\frac{hr}{h+1}\right)^{1+\frac{2}{h}}\delta r}^{\text{linear in } \delta r = s\text{, absorb into } f} + 2v^j\delta v_j\right]$$

So

$$\delta\left(\frac{v^i}{v^0}\partial_i r\right) = \left(\frac{\delta v^i}{v^0} - \frac{v^i}{(v^0)^3}v^j\delta v_j\right)\partial_i r + \dots$$

$$\sim \frac{1}{v^0}\left(\delta^{ij} - \frac{v^iv^j}{(v^0)^2}\right)\underbrace{\delta v_j}_{= w_j}\partial_i r$$

$$\left.\begin{aligned} &= \frac{1}{h}\overbrace{\frac{h}{a_0v^0}\left(1+\frac{hr}{h+1}\right)\left(\delta^{ij} - \frac{v^iv^j}{(v^0)^2}\right)}^{(G^{-1})^{ij}} w_j\partial_i r \\ &- \frac{1}{h}\frac{h}{a_0v^0}\left(1+\frac{hr}{h+1}\right)\left(\delta^{ij} - \frac{v^iv^j}{(v^0)^2}\right) w_j\partial_i r \end{aligned}\right\} = 0$$

$$+ \frac{1}{v^0}\left(\delta^{ij} - \frac{v^iv^j}{(v^0)^2}\right) w_j\partial_i r$$

$$= \frac{1}{h}(G^{-1})^{ij} u_i \partial_j r + \left[-\frac{1}{a_0}\left(1+\frac{hr}{h+1}\right) + 1 \right] \left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2} \right) \frac{w_j \partial_i r}{v^0}$$

The term in bracket gives, using

$$a_0 = 1 - c_s^2 \frac{|v|^2}{(v^0)^2} = 1 - hr \frac{|v|^2}{(v^0)^2},$$

$$-\frac{1}{a_0}\left(1+\frac{hr}{h+1}\right) + 1 = \frac{1}{a_0}\left[-\left(1+\frac{hr}{h+1}\right) + a_0 \right]$$

$$= \frac{1}{a_0}\left[-1 - \frac{hr}{h+1} + 1 - hr\frac{|v|^2}{(v^0)^2} \right]$$

$$= \frac{1}{a_0}\left[-\frac{h}{h+1} - \frac{h|v|^2}{(v^0)^2} \right] r$$

and therefore, the entire term containing the bracket is linear in rw and can therefore be absorbed into f.

Although the above arguments are simple, they capture the following big idea: it is key to find the right variables to treat the problem. In our case casting the system in terms of (r, v) leads to a linearized with good structure for which we can derive an energy estimate. This good structure is manifest in the cancellation of the cross terms $s\partial v$ and $w\partial s$ to cancel out, a cancellation that happens because the coefficient a_0 has the right form for the algebra to work out, as just seen.

Because of the good structures present on the linearized equation, we build our strategy around it

In addition, the above also points out to the following important idea that will be useful for the derivation of higher order estimates: differentiating the equation with arbitrary derivatives produces $\partial_r \approx O(1)$ terms when the derivatives follow on the weights. As seen, such ∂_r terms tend to destroy the delicate weighted structure of the equation. Differentiating D_t, however, does not create this problem because the equation for r gives $D_t r \sim r\,\partial(v, \sigma)$, i.e., every time that D_t falls on a r we gain it back.

Energy estimates for solutions

The above discussion suggests that in order to derive energy estimates for the equation

$$D_t r + r (G^{-1})^{ij} \partial_i \sigma_j + a_1 r \sigma^i \partial_i r = 0$$

$$D_t \sigma_i + a_2 \partial_i r = 0$$

we could take several material derivatives of the equations, D_t^N, and show that the top order terms $(D_t^N r, D_t^N \sigma)$ satisfy the linearized equations with good perturbative terms. However, this is not the case: the important "cancellation term" for the linearized equation comes from the fact that a regular derivative does not commute with D_t, whereas if we differentiate the equation with D_t, well, D_t commutes with itself.

Our approach is then to introduce the required cancellation term by hand upon defining the following good linear variables:

$$s_0 := r \qquad s_1 := \partial_t r \qquad s_2 := D_t^2 r + \frac{1}{2}\frac{a_0 a_2}{k(1+\frac{kr}{k+1})}(G^{-1})^{ij}\partial_i r \partial_j r$$

$$w_0 := v \qquad w_1 := \partial_t v$$

$$w_N := D_t^N v, \quad N \geq 2$$

$$s_N := D_t^N r - \frac{a_0}{k(1+\frac{kr}{k+1})}(G^{-1})^{ij} D_t^{N-1} v_j \partial_i r$$

(Note that only s_N is modified from D_t^N because only the linearized equation for s needs the cancelation term.)

The reason the definition changes for small N is that our estimates are based on a hierarchy that ultimately needs to connect with estimates for (r, v) themselves. We also remark that the connection term

$$\frac{a_0}{k(1+\frac{kr}{k+1})}(G^{-1})^{ij} D_t^{N-1} v_j \partial_i r$$

could be replaced with $\frac{1}{h}(G^{-1})^{ij}\partial_i r\, D_t^{\nu-1} v_j$ (which

is more alike what we have in the linearized equation) since the difference between both is perturbative as it comes with a good power of r. This is precisely the computation we did above using the explicit form of a_0. Our choice here, however, is more convenient because it is the term $\frac{a_0}{h(1+\frac{hr}{h+1})}$ is what appears

in the commutator $[D_t, \partial]$. This again can be viewed from the above computation for the linearized equation.

To understand our choices, note that

$$D_t s_\nu = D_t^{\nu+1} r - \frac{a_0}{h(1+\frac{hr}{h+1})}(G^{-1})^{ij} D_t^{\nu} v_j \partial_i r + \dots$$

$$= D_t^{\nu}\left(-r(G^{-1})^{ij}\partial_i v_j\right) - \frac{1}{h}(G^{-1})^{ij} D_t^{\nu} v_j \partial_i r + \dots$$

$\hookrightarrow$ using equation for r and above observations

$$= -r(G^{-1})^{ij}\partial_i \underbrace{D_t^{\mu} v_j}_{=(\omega_\mu)_j} - \frac{1}{h}(G^{-1})^{ij}\underbrace{D_t^{\mu} v_j}_{=(\omega_\mu)_j}\partial_i r$$

Thus

$$\underbrace{D_t s_\mu + r(G^{-1})^{ij}\partial_i(\omega_\mu)_j + \frac{1}{h}(G^{-1})^{ij}\partial_i r(\omega_\mu)_j}_{\text{main terms in the linearized equation for } s.} = \dots$$

Indeed, we can show that the good linear variables satisfy the linearized equations with source terms

$$D_t s_{2\mu} + \frac{1}{h}(G^{-1})^{ij}\partial_i r(\omega_{2\mu})_j + r(G^{-1})^{ij}\partial_i(\omega_{2\mu})_j$$

$$+ r a_1 v^i \partial_i s_{2\mu} = f_{2\mu}$$

$$D_t(\omega_{2\mu})_i + a_2 \partial_i s_{2\mu} = (h_{2\mu})_i$$

We construct our hierarchy based on 2μ because we will use the underlying wave evolution which is governed by a second order operator $D_t^2 - r\Delta$, and is ultimately connected with our function spaces $\mathcal{H}^{2k}$ based on an even number of derivatives.

<u>Remark.</u> Although it is not the case that $s_{2\mu} = D_t^{2\mu} v$, to gain intuition it is often helpful to think so and we will do so to construct some heuristics.

Our goal is to show that the source terms $(f_{2\mu}, h_{2\mu})$ are perturbative, i.e., can be bounded by the appropriate energy norms we introduce below and which are the energy for the linearized equations applied to $(s_{2\mu}, \sigma_{2\mu})$.

In order to analyze the source terms, we need an efficient way of analyzing multilinear expressions in $v, \partial^k v, \partial^k \sigma$ (with $O(1)$ coefficients) that arise in these expressions. Based on the scaling identified above,

$$(v(t,x), \sigma(t,x)) \longmapsto (\lambda^{-2} v(\lambda t, \lambda^2 x), \lambda^{-1} \sigma(\lambda t, \lambda^2 x))$$

we introduce the following bookkeeping scheme based on the <u>order</u> of multilinear expressions, defined as follows

- v and σ have order -1 and $-1/2$, respectively (we only count v having order $-1/2$ if it is differentiated. Undifferentiated v has order 0).

- D_t and ∂_i have order $1/2$ and 1, respectively.
- G, a_0, a_1, and a_2 and, more generally, smooth functions of (r, v) not vanishing at $r=0$ have order 0 (the order is defined in terms of the order of the leading term in a Taylor expansion about $r=0$, being order 0 if the term is constant $\neq 0$).
- The order of a multilinear expression is defined as the sum of the order of its factors.

With these conventions, all terms in the v equation have order $-1/2$ except the last one that has order -1 and all terms in the v equation have order $-1/2$. Upon successive differentiations of any multilinear expression w.r.t. D_t or ∂, all terms produce the same (highest) order, unless some these derivatives apply to coefficients, in which case lower order terms are produced.

The basic idea is that terms of high order in our scheme are the "dangerous" ones. This is because such terms are the ones with more derivatives and like in unweighted estimates, the terms with more derivatives are the ones we have to carry about. Unlike unweighted estimates, however, it is not the number of derivatives per se that matters but the delicate balance of derivatives and weights (e.g., a term that is not top order in the number of derivatives but has no weights typically cannot be controlled). More derivatives require more weights, thus powers of r are good and decrease the order of an expression. We also not that a D_t derivative is better than a ∂ derivative because, solving for $D_t^\mu(r, v)$ in the equations gives powers of r back, which is not the case for ∂. r has lower order than v because it requires one less weight than v in $\mathcal{H}^{2\ell}$.

The other ingredient we need to analyze multilinear expressions are some powerful interpolation theorems proven in [IT]:

<u>Lemma.</u> We have:

- $$\| r^{\sigma_j} \partial^j f \|_{L^{p_j}} \lesssim \| r^{\sigma_0} f \|_{L^{p_0}}^{1-\theta_j} \| r^{\sigma_m} \partial^m f \|_{L^{p_m}}^{\theta_j},$$

$1 \le p_j, p_m \le \infty$, $\theta_j = \frac{j}{m}$, $\frac{1}{p_j} = \frac{1-\theta_j}{p_0} + \frac{\theta_j}{p_m}$, $\sigma_j = \sigma_0(1-\theta_j) + \sigma_m \theta_j$,

$m - \sigma_m - d\left(\frac{1}{p_m} - \frac{1}{p_0}\right) > -\sigma_0$, $\sigma_j > -\frac{1}{p_j}$, $0 < j < m$,

$\sigma_0, \sigma_m \in \mathbb{R}$.

- $$\| r^{\sigma_j} \partial^j f \|_{L^{p_j}} \lesssim \| f \|_{L^\infty}^{1-\theta_j} \| r^{\sigma_m} \partial^m f \|_{L^2}^{\theta_j},$$

$\theta_j = \frac{j}{m}$, $\frac{1}{p_j} = \frac{\theta_j}{2}$, $\sigma_j = \sigma_m \theta_j$, $m - \sigma_m - \frac{d}{2} > 0$, $0 < j < m$,

$\sigma_m > -\frac{1}{2}$.

- $$\| r^{\sigma_j} \partial^j f \|_{L^{p_j}} \lesssim \| f \|_{\dot{C}^{1/2}}^{1-\theta_j} \| r^{\sigma_m} \partial^m f \|_{L^2}^{\theta_j},$$

$\theta_j = \frac{2j-1}{2m-1}$, $\frac{1}{p_j} = \frac{\theta_j}{2}$, $\sigma_j = \sigma_m \theta_j$, $m - \frac{1}{2} - \sigma_m - \frac{d}{2} > 0$, $0 < j < m$,

$\sigma_m > -\frac{1}{2}$.

$$- \| r^{\sigma_j} \partial^j f \|_{L^{p_j}} \lesssim \| f \|_{\tilde{C}^{1/2}}^{1-\theta_j} \| r^{\sigma_m} \partial^m f \|_{L^2}^{\theta_j},$$

$\theta_j = \frac{j}{m}$, $\frac{1}{p_j} = \frac{\theta_j}{2}$, $\sigma_j = \sigma_m \theta_j - \frac{1}{2}(1-\theta_j)$, $m - \frac{1}{2} - \sigma_m - \frac{d}{2} \geq 0$,

$0 < j < m$, $\sigma_m > \frac{m-2}{2}$.

$d = 3$ is the space dimension.

<u>proof</u>: [IT].

We are now ready for the energy estimates. Define

$$E^{2l} = E^{2l}(r, v) = \sum_{j=0}^{l} \| (s_{2j}, \omega_{2j} \|_{\mathcal{H}}^2 .$$

We remark that the energy needs an additional term involving an analogue of the good linear variables for the vorticity but, as said, we will not discuss the vorticity estimates.

<u>Theo (coercivity)</u>. $E^{2l} \approx_A \| (r, v) \|_{\mathcal{H}^{2l}}^2$

<u>proof</u>: We begin with the $\lesssim$ part.

Using the equations to successively solve for $D_t(r,v)$, we obtain that (s_{2l}, w_{2l}) is a linear combination of multilinear expressions in $r, \partial r, \partial v$ (with zero order coefficients). It is useful to record here the structure of the linear-in-derivatives top order terms obtained by solving for $D_t^{2l}(r,v)$:

$$D_t^{2l} r \approx r^l \partial^{2l} r + r^{l+1} \partial^{2l} v \approx r^l \partial^{2l} r$$

order: $l-1 \approx (l-1) + (l-\frac{3}{2}) \approx l-1$

$$D_t^{2l} v \approx r^l \partial^{2l} v + r^l \partial^{2l} r \approx r^l \partial^{2l} v$$

order: $l-\frac{1}{2} \approx (l-\frac{1}{2}) + (l-1) \approx l-1$

Incidentally this suggests

$$\| D_t^{2l}(r,v) \|_{\mathcal{H}} \approx \| (r,v) \|_{\mathcal{H}^{2l}}$$

which basically what we want, although, as seen, we cannot work directly with $D_t^{2l}(r,v)$ because they do not solve the linearized equations with good perturbative terms (we have to introduce the good linear variables).

So, again, we consider using the equations to successively solve for $D_t(r,v)$. We begin with the top (w.r.t. our orders) terms, so we ignore the terms coming from $a, v^i \partial_i r$ or from derivatives falling on the zero order coefficients. We also consider first the case that when we commute D_t with ∂, all derivatives fall on v^i and not on r (w.r.t. v^0). Then, the corresponding multilinear expressions $s_{2\ell}$ and $w_{2\ell}$ have the following properties:

- They have orders $\ell - 1$, $\ell - 1/2$, respectively.
- They have exactly 2ℓ derivatives.
- They contain at most $\ell+1$, ℓ factors of r, respectively.

For $s_{2\ell}$, thus, we find multilinear expressions of the form

$$r^a \prod_{j=1}^{J} \partial^{n_j} r \prod_{l=1}^{L} \partial^{m_i} v$$

where $n_j, m_i \geq 1$,

$$\sum n_j + \sum_i m_i = 2\ell$$

$$a + J + \frac{L}{2} = \ell + 1$$

(when $J = 0$ or $L = 0$ the corresponding product is absent.)

With a bit of algebra, we can show that these constraints imply that we can choose b_j and c_i such that:

$$0 \leq b_j \leq (n_j - 1)\frac{\ell}{2\ell - 1}, \quad 0 \leq c_i \leq (m_i - 1)\frac{\ell + 1/2}{\ell - 1/2},$$

$$a = \sum b_j + \sum_i c_i .$$

With these choices, we can verify that the interpolation theorems apply to yield:

$$\| r^{b_j} \partial^{n_j} v \|_{L^{p_j}(r^{\frac{1-h}{h}})} \lesssim (1+A)^{1-\frac{2}{p_j}} \| (v_1, v) \|_{\mathcal{H}^{2\ell}}^{2/p_j},$$

$$\| r^{c_i} \partial^{m_i} v \|_{L^{p_i}(r^{\frac{1-h}{h}})} \lesssim A^{1-\frac{2}{q_i}} \| (v_1, v) \|_{\mathcal{H}^{2\ell}}^{2/q_i},$$

where $\dfrac{1}{p_j} = \dfrac{n_j - 1 - b_j}{2(\ell-1)}$, $\dfrac{1}{q_i} = \dfrac{m_i - 1/2 - c_i}{2(\ell-1)}$,

$$\| f \|_{L^p(h)} = \int |f|^2 h .$$

(Observe that the numerators in $1/p_j$, $\frac{1}{q_i}$, correspond to the orders of the expressions being estimated and add to $\ell-1$ as needed.) This gives the desired estimate for the top order terms considered. The remaining terms in $S_{2\ell}$ are analyzed similarly. In fact, they are easier as they have lower order (i.e., more favorable factors of r). A similar analysis can be done for $w_{2\ell}$. This concludes the $\lesssim$ part.

Now we move to the $\geq$ part. Applying D_t to the equations satisfied by (s_{2j}, w_{2j}) leads to

$$s_{2j} = L_1 s_{2j-2} + F_{2j}$$

$$w_{2j} = L_2 w_{2j-2} + H_{2j}$$

where

$$L_1 s := a_2 (G^{-1})^{ij} \left(r \partial_i \partial_j s + \frac{1}{h} \partial_i r \partial_j s \right),$$

$$(L_2 w)_i := a_2 (G^{-1})^{pq} \left(\partial_i (r \partial_p w_q) + \frac{1}{h} \partial_p r \partial_i w_q \right).$$

To understand the origin and significance of the operators L_1 and L_2, we observe that the wave equations obtained by differentiating the (r, v) equations are

$$D_t^2 r - L_1 r = \dots$$

$$D_t^2 v - L_2 v = \dots$$

(Earlier we wrote $D_t^2 - r\Delta$ for the wave operators, but that is only a crude approximation. The exact expression is with the L_1 and L_2 operators). Taking D_t^{2j}:

$$\underbrace{D_t^{2+2j} v}_{\sim s_{2j+2}} = L_1 \underbrace{D_t^{2j} v}_{\sim s_{2j}} + \dots$$

$$\underbrace{D_t^{2+2j} v}_{w_{2j+2}} = L_2 \underbrace{D_t^{2j} v}_{w_{2j}} + \dots$$

which explains the above relation. We call the operators L_1 and L_2 (second order) transition operators as they relate the variables at level $2j$ with their counterparts at level $2j+2$ in our hierarchy. Therefore, we need to understand the properties of L_1 and L_2. We will show that they satisfy the following elliptic estimates

$$\|s\|_{H^{2,\frac{1}{2k}+\frac{1}{2}}} \lesssim \|L_1 s\|_{H^{0,\frac{1}{2k}-\frac{1}{2}}} + \|s\|_{L^2(r^{\frac{1-k}{k}})},$$

$$\|w\|_{H^{2,\frac{1}{2k}+1}} \lesssim \|L_2 w\|_{H^{0,\frac{1}{2k}}} + \|w\|_{L^2(r^{1/k})},$$

where $H^{s,\sigma}$ is the weighted Sobolev space with norm:

$$\| f \|^2_{H^{s,\sigma}} = \sum_{|\alpha| \leq s} \| r^{\sigma} \partial^{\alpha} f \|^2_{L^2} .$$

Using weighted embeddings, it follows that $\mathcal{H}^{2j}$ is equivalent to $H^{2j, \frac{1-k}{2k} + j} \times H^{2j, \frac{1-k}{2k} + \frac{1}{2} + j}$ (note that their norms are the same at top order), but it is more convenient for the elliptic estimates to work in $H^{s,\sigma}$.

<u>Remark.</u> As stated, the above estimate for L_2 is wrong. Observe that L_2 only controls the divergence part of w, as $(L_2 w)_i \sim (G^{-1})^{pq} \partial_i (r \partial_p w_q) \sim r \partial_i (G^{-1})^{pq} \partial_p w_q$. To bound w we need to also control its curl part, and for that we need the vorticity estimates that we are not discussing.

Let us consider the estimate for s. We first note that integration by parts is the usual elliptic fashion yields the weaker bound

$$\| s \|_{H^{2, \frac{1}{2k} + \frac{1}{2}}} \lesssim \| L_1 s \|_{H^{0, \frac{1}{2k} - \frac{1}{2}}} + \| s \|_{H^{1, \frac{1}{2k} - \frac{1}{2}}} .$$

Thus, it suffices to prove:

$$\|s\|_{H^{1,\frac{1}{2h}-\frac{1}{2}}} \lesssim \|L_1 s\|_{H^{0,\frac{1}{2h}-\frac{1}{2}}} + \|s\|_{L^2(r^{\frac{1-h}{2h}})}$$

Compute

$$\int_{\Omega_t} r^{\frac{1-h}{h}} \partial_3 s \, L_1 s = \int_{\Omega_t} r^{\frac{1-h}{h}} \partial_3 s \,(G^{-1})^{ij} a_2 \Big(\underbrace{r \partial_i \partial_j s}_{\text{by parts}} + \frac{1}{h} \partial_i r \, \partial_j s\Big)$$

$$= - \int_{\Omega_t} r^{\frac{1}{h}} \underbrace{\partial_3 \partial_i s \,(G^{-1})^{ij} \partial_j s}_{= \frac{1}{2} (G^{-1})^{ij} \partial_3 (\partial_i s \, \partial_j s)} a_2 - \int_{\Omega_t} \underbrace{\frac{1}{h} r^{\frac{1-h}{h}}}_{r^{\frac{1-h}{h}} r \text{ hit with } \partial_i} \partial_i r \, \partial_3 s \,(G^{-1})^{ij} a_2 \, \partial_j s$$

cancel

$$- \int_{\Omega_t} r^{1/h} \partial_3 s \, \partial_i (a_2 (G^{-1})^{ij}) \, \partial_j s + \int_{\Omega_t} r^{\frac{1-h}{h}} \partial_3 s \,(G^{-1})^{ij} a_2 \frac{1}{h} \partial_i r \, \partial_j s$$

(there is no boundary term because $r=0$ on the boundary.)

Integrating ∂_3 by parts in the first integral

$$= \int_{\Omega_t} \frac{1}{2h} r^{\frac{1-h}{h}} \partial_3 r \, a_2 (G^{-1})^{ij} \partial_i s \, \partial_j s - \int_{\Omega_t} \frac{1}{2} r^{1/h} \partial_3 (a_2 (G^{-1})^{ij}) \partial_i s \, \partial_j s$$

$$- \int_{\Omega_t} r^{1/h} \partial_3 s \, \partial_i (a_2 (G^{-1})^{ij}) \, \partial_j s .$$

(again, there is no boundary term). Recall now that we can work on a neighborhood of $x_0 \in \partial\Omega_t$ where $\nabla r(x_0) = \nu$, so $|\nabla r - N| \leq A \ll 1$. We can arrange the coordinates such that $\nu = e_3 = (0,0,1)$. In this case $\partial_3 r \geq$ constant > 0. We can further assume that $r \leq \varepsilon$ for small $\varepsilon > 0$, so $r^{1/h} = r^{\frac{1-h}{h}} \mathcal{O}(\varepsilon)$

Then (recall $a_2 > 0$)

$$\int_{\Omega_t} r^{\frac{1-h}{h}} \partial_3 s \, L_1 s \gtrsim \int_{\Omega_t} r^{\frac{1-h}{h}} (G^{-1})^{ij} \partial_i s \partial_j s - \varepsilon \int_{\Omega_t} r^{\frac{1-h}{h}} |\partial s|^2$$

$$\gtrsim \int_{\Omega_t} r^{\frac{1-h}{h}} |\partial s|^2 - \varepsilon \int_{\Omega_t} r^{\frac{1-h}{h}} |\partial s|^2 \gtrsim \int_{\Omega_t} r^{\frac{1-h}{h}} |\partial s|^2$$

where we used the positive definiteness of (G^{-1}). Applying Cauchy-Schwartz-with-ε on the LHS gives the result

The proof for L_2 is similar

To finish the proof of coercivity we need two more elements:

First, we need to show that the terms F_{2j} and H_{2j} are perturbative. This requires a very delicate analysis of such terms, but in the end, with help with our bookkeeping scheme and the above interpolation theorems, we can show that they satisfy the estimate

$$\|(F_{2j}, H_{2j})\|_{\mathcal{H}^{2l-2j}} \lesssim \varepsilon \|(v, w)\|_{\mathcal{H}^{2l}}.$$

(Here, the ε term comes from either terms of $O(A)$, or factors that have an extra power of r that we can use for smallness; the latter comes from the term $a_1 w' \partial_r r$.)

Next, we take the $\mathcal{H}^{2l-2j}$ norm in the equations with the transition operators and using the estimate for (F_{2j}, H_{2j}):

$$\| L_1 s_{2j-2} \|_{\mathcal{H}^{2l-2j}} \lesssim \| s_{2j} \|_{\mathcal{H}^{2l-2j}} + \varepsilon \| (r,v) \|_{\mathcal{H}^{2l}}$$

$$\| L_2 w_{2j-2} \|_{\mathcal{H}^{2l-2j}} \lesssim \| w_{2j} \|_{\mathcal{H}^{2l-2j}} + \varepsilon \| (r,v) \|_{\mathcal{H}^{2l}}$$

At this point we want to prove the elliptic estimates

$$\| L_1 s_{2j-2} \|_{\mathcal{H}^{2l-2j}} \gtrsim \| s_{2j-2} \|_{\mathcal{H}^{2l-2j+2}} ,$$

$$\| L_2 w_{2j-2} \|_{\mathcal{H}^{2l-2j}} \gtrsim \| w_{2j-2} \|_{\mathcal{H}^{2l-2j+2}} .$$

(we can ignore the harmless L^2 term that appears on the LHS.) For $j = l$, this is the elliptic estimate we proved above since

$$\mathcal{H}^2 \simeq H^{2, \frac{1-k}{2k}+1} \times H^{2, \frac{1-k}{2k}+\frac{1}{2}+1}$$

$$\mathcal{H}^0 \simeq H^{0, \frac{1-k}{2k}} \times H^{0, \frac{1-k}{2k}} ,$$

For other values of j, in a typical elliptic fashion we apply the estimate we proved with (s,w) replaced by suitable weighted derivatives of themselves (although we remark that the argument is not straightforward because we need to be careful with the weights), relying again on

$$\mathcal{H}^{2j} \simeq H^{2j, \frac{1-h}{2h}+j} \times H^{2j, \frac{1-h}{2h}+\frac{1}{2}+j}.$$

In the end, we obtain:

$$\|(s_{2j-2}, w_{2j-2})\|_{\mathcal{H}^{2l-2j+2}} \lesssim \|(s_{2j}, w_{2j})\|_{\mathcal{H}^{2l-2j}} + \epsilon \|(r,v)\|_{\mathcal{H}^{2l}}, \quad 1 \leq j \leq l.$$

Concatenating these estimates produces the result. $\square$

Establishing coercivity of the energy is a key ingredient for our main result. Without it, we cannot connect estimates for the linearized variables, which can be obtained because of the good structure of the linearized equation, with estimates for solutions to the nonlinear problem. But it still remains to show that the energy estimates themselves hold:

$$\frac{d}{dt} E^{2\ell}(r,v) \lesssim_A B \, \| (r,v) \|^2_{\mathcal{H}^{2\ell}} .$$

This is proven using ideas similar to those in the proof of coercivity, namely, we use our bookkeeping scheme to keep track of which terms are perturbative, interpolation, and observe some cancelations. Ultimately, these ideas rely on the fact that (s_{2j}, w_{2j}) satisfy the linearized equations with source terms that can be shown to be perturbative. In addition, we need to be careful to ensure that we can interpolate with only factors that are linear in B. We refer to [DIT] for details.

$\square$

Once again, we recall that the proofs have to handle the vorticity as well, which we neglected here.

Remaining arguments

Here we make some brief comments on the remaining arguments that are needed to establish local well-posedness. We construct solutions using a time discretization that involves the following steps:

- Regularization.
- Transport (iteration of the boundary at each time step.)
- Euler's method.

What is interesting is that taken separately each of these steps seems unbounded. When taken together, there is an extra cancellation that comes to rescue. This is a direct analogue of the key cancellation we observed for the linearized equation.

To control the iteration, we need to translate our energy estimates to estimates at fixed time. We do so by reinterpreting the operators D_t^N as operators at fixed time obtained by reiterating the equations.

For uniqueness, we construct a suitable functional that tracks the distance between solutions, in part by measuring the distance between their boundaries (since different solutions are defined in different domains). This functional is, like much in our approach, inspired by the energy for the linearized equation. To show that the functional is propagated by the flow, we rely on ideas of [IT], where a similar functional was constructed for the treatment of the analogous classical problem.

Continuous dependence on the data is established with help of the regularization.

Relativistic fluids with viscosity

So far we discussed only perfect fluids, which have no viscosity and/or dissipation. There are compelling reasons to consider relativistic viscous fluid, including:

- The quark-gluon plasma, which is an exotic state of matter that forms in collisions of heavy ions performed at particle accelerators like the RHIC and LHC. It is well attested that the quark-gluon plasma is a relativistic liquid with viscosity [RR].
- Neutron star mergers. Recent state-of-the-art numerical simulations strongly suggest that viscous and dissipative can affect the gravitational wave signal produced in collisions of neutron stars, and that these effects would be measurable by the next generation of gravitational wave detectors [ADBHRS, MHHZAP].

Because our focus here is on mathematical aspects of relativistic fluid theories, we will not say more about the physical motivation, but we would be remiss not to stress that the above two examples show that two of the most advanced experimental apparatus ever built (LHC and LIGO) are produce/will produce data that requires/may require relativistic fluids with viscosity for its explanation.

Terminology. We will use the terms viscosity and dissipation interchangeably. This is a common practice in the community.

The first difficulty in studying relativistic viscous fluids is to find and appropriate model. Unlike the case of a perfect fluid, there is no Lagrangian for the description of a relativistic viscous fluid (this is already the case for classical fluids). Absent a Lagrangian, there is no canonical way of determining the energy-momentum tensor. A natural thing to do in this case is to modify the perfect fluid energy-momentum tensor and baryon current by adding terms that represent viscous effects:

$$\tilde{T}_{\alpha\beta} = (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \underline{P}) \Pi_{\alpha\beta} + \pi_{\alpha\beta} + \mathcal{Q}_\alpha u_\beta + \mathcal{Q}_\beta u_\alpha,$$

$$J_\alpha = n u_\alpha + \mathcal{J}_\alpha$$

where $\mathcal{R}$, $\underline{P}$, π, $\mathcal{Q}$, and $\mathcal{J}$ are known as viscous fluxes and represent the viscous correction to the energy density, the viscous correction to the pressure, a.k.a. the bulk viscosity, a the viscous shear stress, the heat flux, and the viscous correction to the baryon density, respectively.

Next, one needs to make modeling choices determining the viscous fluxes. The first proposal in this direction was introduced by Eckart in the '40s [Ec], setting

$$R = 0,$$

$$P = -\zeta \nabla_\alpha u^\alpha,$$

$$\pi_{\alpha\beta} = -2\eta\, \Pi_\alpha^{\mu} \Pi_\beta^{\nu} \left(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} \nabla_\lambda u^\lambda g_{\mu\nu} \right),$$

$$Q_\alpha = -k\theta \left(\Pi_\alpha^{\mu} \nabla_\mu \theta + u^\mu \nabla_\mu u_\alpha \right)$$

$$J_\alpha = 0,$$

followed by Landau-Lifshitz [LL], who postulated the same relations except for

$$J_\alpha = n u_\alpha - \frac{Q_\alpha}{h}$$

Above, $\zeta = \zeta(\varrho, n)$ and $\eta = \eta(\varrho, n)$ are the coefficients of bulk and shear viscosity and $k = k(\varrho, n)$ is the heat conductivity.

We will not discuss the physics arguments leading to these choices, other than saying that they are inspired by an attempt to write a covariant (geometric) version of the classical Navier-Stokes equations.

Later it became clear that the Eckart and Landau theories do not lead to hyperbolic equations of motion [HL2, Pi], as they include the characteristics

$$\Pi^{\alpha\beta}\xi_\alpha\xi_\beta = 0.$$

In particular, the corresponding equations are acausal, i.e., they admit faster-than-light propagation of information, in clear violation of relativity theory

We will not compute the characteristics here, but simply point out that part of the problem is that the operator $\Pi^{\alpha\beta}\nabla_\alpha\nabla_\beta$ that appears in $\Pi^{\gamma\alpha}\nabla_\beta T^\beta_\alpha = 0$ contributes significantly to the characteristics. This operator is "spatial" and acts like a Laplacian. This is by design in view of the attempt to find a covariant generalization of the classical Navier-Stokes theory. It seems that was precisely the problem: the Navier-Stokes equations are not hyperbolic thus one should be seeking a fully relativistic generalization.

In addition, the Eckart and Landau-Lifshitz theories are unstable. (In)stability here means mode stability of solutions to the equations linearized about thermodynamic equilibrium states characterized by $\varsigma, n, u =$ constant and viscous fluxes $= 0$. Stability should hold for viscous theories is that small perturbations away from equilibrium should decay in time due to dissipation. (More general notions of stability can also be considered.)

It turns out that modeling viscous phenomena in relativity is not a simple task. Seemingly natural modeling choices made over the years kept resulting in acausal and unstable theories [RZ].

We remark that while causality is a statement for a general spacetime, including when there is coupling to Einstein's equations, stability is typically studied in a Minkowski background. In a general spacetime, a stability analysis would have to also account for diffeomorphism invariance.

We will next discuss the mathematical properties of two theories that address the acausality and instability of relativistic viscous models.

The DNMR theory

The Denicol-Niemi-Molnar-Rischke (DNMR) theory is the theory that is primary used in the study of the quark gluon plasma. (For historical reasons, it is also referred to as a Müller-Israel-Stewart theory.) The big idea here is to treat the viscous fluxes as new variables on the same footing as ϱ, n, and u. Since we are now introducing new variables, new equations of motion should be introduced as well. These are obtained from kinetic theory plus extra modeling choices based on physical assumptions. These extra choices are needed because kinetic theory does not uniquely determine the equations in the fluid limit (e.g., Eckart and Landau-Lifshitz can also be obtained from kinetic theory [GLW]).

The new equations for the viscous fluxes and $\nabla_\alpha T^\alpha_\beta = 0$ lead to the DNMR equations [DNMR]

$$u^\alpha \nabla_\alpha \varrho + (\varrho + p + \underline{P}) \nabla_\alpha u^\alpha + \pi^\mu_\alpha \nabla_\mu u^\alpha = 0,$$

$$(\varrho + p + \underline{P})\, u^\mu \nabla_\mu u_\alpha + c_s^2\, \Pi_\alpha^\mu \nabla_\mu \varrho + \Pi_\alpha^\mu \nabla_\mu \underline{P} + \Pi_\alpha^\mu \nabla_\nu \pi^\nu_\mu = 0,$$

$$\tau_{\underline{P}}\, u^\alpha \nabla_\alpha \underline{P} + \underline{P} + \zeta \nabla_\alpha u^\alpha + \delta_{\underline{P}\underline{P}}\, \underline{P} \nabla_\alpha u^\alpha + \lambda_{\underline{P}\pi}\, \pi^{\mu\nu} \sigma_{\mu\nu} = 0,$$

$$\tau_\pi\, \hat{\Pi}^{\mu\nu}_{\alpha\beta}\, u^\lambda \nabla_\lambda \pi^{\alpha\beta} + \pi^{\mu\nu} + 2\eta\, \sigma^{\mu\nu} + \delta_{\pi\pi}\, \pi^{\mu\nu} \nabla_\alpha u^\alpha$$
$$+ \tau_{\pi\pi}\, \pi_\alpha^{\langle\mu} \sigma^{\nu\rangle\alpha} + \lambda_{\pi\underline{P}}\, \underline{P}\, \sigma^{\mu\nu} = 0,$$

subject to the constraints

$$\pi_{\alpha\beta} = \pi_{\beta\alpha}, \quad u^\alpha \pi_{\alpha\beta} = 0, \quad \pi_\alpha^\alpha = 0,$$

in addition to the usual $u_\alpha u^\alpha = -1$.

Above, $\hat{\Pi}^{\mu\nu}_{\alpha\beta} := \frac{1}{2}(\Pi_\alpha^\mu \Pi_\beta^\nu + \Pi_\beta^\mu \Pi_\alpha^\nu) - \frac{1}{3}\Pi^{\mu\nu}\Pi_{\alpha\beta}$

projects a 2-tensor into its u-orthogonal symmetric trace-free part; $A_\lambda^{\langle\mu} B^{\nu\rangle\lambda} := \hat{\Pi}^{\mu\nu}_{\alpha\beta} A^{\alpha\lambda} B^\beta_\lambda$ (A, B symmetric);

$\sigma^{\mu\nu} := \hat{\Pi}^{\mu\nu}_{\alpha\beta} \nabla^\alpha u^\beta$ is the shear tensor; and the coefficients $\{\eta, \zeta, \tau_{\underline{P}}, \tau_\pi, \delta_{\underline{P}\underline{P}}, \lambda_{\underline{P}\pi}, \delta_{\pi\pi}, \tau_{\pi\pi}, \lambda_{\pi\underline{P}}\}$, called *transport coefficients*, are functions of ϱ (in particular, ζ and η are the coefficients of bulk and shear viscosity and $\tau_{\varrho}, \tau_{\pi}$ are known as relaxation times), as it is the pressure $p = p(\varrho)$, with $c_s^2 = p'(\varrho)$.

We remark that above we did not consider the full DNMR equations. We are considering the case where $n = 0$ (so p and the transport coefficients depend only on ϱ) and $Q = 0$, because this is the case we treat in our results. See [DNMR] for the full equations. We also have $R = 0$, but this is always the case for the DNMR theory.

What should become apparent above is the sheer complexity of the equations. With the exception of the linear terms ϱ and π in the last two equations, all terms contribute to the principal part. The system is large, 22×22 (see below). Thus, we have a large system with non-diagonal principal part.

In addition to being successfully used in the study of the quark-gluon plasma, mostly through numerical simulations, the DNMR equations enjoy the following good properties (these properties hold for the full DNMR equations that we did not state):

- Stability holds ([DNMR] based on [HL1, OLs]).

- Causality was established in the following particular cases under reasonable assumptions on the transport coefficients and fluid variables: for the equations linearized about thermodynamic equilibrium (again [DNMR] based on [HL1, DLs]), in 1+1 dimensions [OUKM], and in rotational symmetry [PKR, FG]).

We next turn to the question of causality in 3+1 dimensions without symmetry assumptions and local well-posedness.

<u>Notation.</u> The symmetry and trace-free condition of $\pi^{\alpha\beta}$ allows us to diagonalize it

$$\pi^{\mu}{}_{\nu} e_A^{\nu} = \Lambda_A e_A^{\mu}, \quad A = 0, \dots, 3$$

with $\{e_A\}_{A=0}^{3}$ orthonormal ($g_{\mu\nu} e_A^{\mu} e_B^{\nu} = m_{AB} = \mathrm{diag}(-1,1,1,1)$)

$e_0 = u$, $\Lambda_{A=0} = 0$, $\Lambda_{A=i}$ real, and $\Lambda_1 + \Lambda_2 + \Lambda_3 = 0$.

We can order $\Lambda_1 \le \Lambda_2 \le \Lambda_3$, $\Lambda_1 \le 0 \le \Lambda_3$.

We have the following result.

Theo (Bemfica - D - Noronha - Radosz - Vu [BDHNR])

Consider the DNMR equations. Assume:

(A1) $\tau_\Pi, \tau_\pi > 0$, $\eta, \zeta, \delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \delta_{\pi\pi}, \tau_{\pi\pi}, \lambda_{\pi\Pi} \geq 0$.

(A2) $\varepsilon > 0$, $P \geq 0$, $\varepsilon + P + \Pi > 0$ (note that Π can be negative).

(A3) $\varepsilon + P + \Pi + \Lambda_i > 0$, $i = 1, 2, 3$.

Then, the following are sufficient conditions for causality:

(a) $\varepsilon + P + \Pi - |\Lambda_1| - \dfrac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \dfrac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 \geq 0$,

(b) $2\eta + \lambda_{\pi\Pi}\Pi - \tau_{\pi\pi}|\Lambda_1| > 0$,

(c) $\tau_{\pi\pi} \leq 6\delta_{\pi\pi}$,

(d) $\dfrac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \dfrac{\tau_{\pi\pi}}{12\tau_\pi} \geq 0$,

(e)
$$\frac{1}{3\tau_\pi}\left[4\eta + 2\lambda_{\pi\Pi}\Pi + (3\delta_{\pi\pi} + \tau_{\pi\pi})\Lambda_3\right] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi} + |\Lambda_1| + \Lambda_3 c_s^2$$
$$+ \frac{\dfrac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\dfrac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \dfrac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3 + |\Lambda_1|)^2}{P + \varepsilon + \Pi - |\Lambda_1| - \dfrac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \dfrac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3} \leq (P + \varepsilon + \Pi)(1 - c_s^2),$$

(f) $$\frac{1}{6\tau_\pi}\left[2\eta + \lambda_{\pi\varrho}\varrho + (\tau_{\pi\pi} - 6\delta_{\pi\pi})|\Lambda_1|\right] + \frac{\zeta + \delta_{\varrho\varrho}\varrho - \lambda_{\varrho\pi}|\Lambda_1|}{\tau_\varrho} + (p+\varepsilon+\varrho-|\Lambda_1|)c_s^2 \geq 0$$

(g) $$\frac{\dfrac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\dfrac{\lambda_{\varrho\pi}}{\tau_\varrho} + c_s^2 - \dfrac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3 + |\Lambda_1|)^2}{\left(\dfrac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\varrho}\varrho) - \dfrac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right)^2} \leq 1,$$

(h) $$\frac{1}{3\tau_\pi}\left[4\eta + 2\lambda_{\pi\varrho}\varrho - (2\delta_{\pi\pi} + \tau_{\pi\pi})|\Lambda_1|\right] + \frac{\zeta + \delta_{\varrho\varrho}\varrho - \lambda_{\varrho\pi}|\Lambda_1|}{\tau_\varrho} + (p+\varepsilon+\varrho-|\Lambda_1|)c_s^2$$
$$\geq \frac{(p+\varepsilon+\varrho+\Lambda_2)(p+\varepsilon+\varrho+\Lambda_3)}{3(p+\varepsilon+\varrho-|\Lambda_1|)}\left[1 + \frac{2\left(\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\varrho}\varrho) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right)}{p+\varepsilon+\varrho-|\Lambda_1|}\right].$$

(We abuse notation and denote $c_s^2 = \left.\frac{\partial p}{\partial \varepsilon}\right|_s$ by analogy with the perfect fluid case, but c_s^2 is not the sound speed – found through the characteristics – which now also depends on the viscous fluxes.)

Moreover, the following conditions are necessary for causality:

(a) $$2\eta + \lambda_{\pi\varrho}\varrho - \frac{1}{2}\tau_{\pi\pi}|\Lambda_1| \geq 0,$$

(b) $$\varepsilon + p + \varrho - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\varrho}\varrho) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\Lambda_3 \geq 0,$$

(c) $\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\rho}\,\underline{P}) + \frac{\tau_{\pi\pi}}{4\tau_\pi}(\underline{\Lambda}_a + \underline{\Lambda}_d) \geq 0$, $a,d = 1,2,3$, $a \neq d$,

(d) $p + \varrho + \underline{P} + \underline{\Lambda}_a - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\rho}\,\underline{P}) - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\underline{\Lambda}_a + \underline{\Lambda}_d) \geq 0$,
$a,d = 1,2,3$, $a \neq d$,

(e) $$\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\rho}\,\underline{P}) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\underline{\Lambda}_i + \frac{1}{6\tau_\pi}\left[2\eta + \lambda_{\pi\rho}\,\underline{P} + (6\delta_{\pi\pi} - \tau_{\pi\pi})\underline{\Lambda}_i\right] + \frac{3 + \delta_{\rho\rho}\,\underline{P} + \lambda_{\rho\pi}\,\underline{\Lambda}_i}{\tau_\rho} + (\varrho + p + \underline{P} + \underline{\Lambda}_i)c_s^2 \geq 0,\quad i = 1,2,3,$$

(f) $$p + \varrho + \underline{P} + \underline{\Lambda}_i - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\rho}\,\underline{P}) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\underline{\Lambda}_i - \frac{1}{6\tau_\pi}\left[2\eta + \lambda_{\pi\rho}\,\underline{P} + (6\delta_{\pi\pi} - \tau_{\pi\pi})\underline{\Lambda}_i\right] - \frac{3 + \delta_{\rho\rho}\,\underline{P} + \lambda_{\rho\pi}\,\underline{\Lambda}_i}{\tau_\pi} - (\varrho + p + \underline{P} + \underline{\Lambda}_i)c_s^2 \geq 0,\quad i = 1,2,3.$$

Finally, under the sufficient conditions above, the Cauchy problem admits local existence and uniqueness for data in suitable Gevrey spaces. These results hold with or without coupling to Einstein's equations.

Remarks.

- Both the sufficient and the necessary conditions can be seen to be non-empty. More importantly, they are expected to hold for some reasonable (although not all, see below) physical systems.

- When the only viscous flux is present is $\mathcal{P}$, the equations simplify considerably and in this case it is possible to obtain local existence and uniqueness in Sobolev spaces (with or without coupling to Einstein's equations).

- Recall that Gevrey spaces G^s are the spaces of smooth functions f such that for every compact K there exists a constant $C > 0$ such that $|\partial^\alpha f(x)| \leq C^{|\alpha|+1} (\alpha!)^s$ for every multi-index α and every $x \in K$. This is a generalization of analytic functions since $s=1$ corresponds to analyticity.

proof: Causality boils down to computing the system's characteristics. More precisely, given sub-luminal characteristics we still need to show that the equations satisfy a domain of dependence property, but this can then be done with a Holmgren type of argument. Thus, we need to analyze the roots ξ of $\det(A^\alpha \xi_\alpha) = 0$, where A^α are the 22×22 matrices of the system written as

$$A^\alpha \partial_\alpha \Psi = B(\Psi)$$

where $\Psi = (\varrho, u^\alpha, \mathcal{P}, \pi^{0\mu}, \pi^{1\mu}, \pi^{2\mu}, \pi^{3\mu})$. As it can be seen from the above equations, the calculation of $\det(A^\alpha \xi_\alpha)$ is rather non-trivial. We do it through a series of well-thought-out calculations. After finding $\det(A^\alpha \xi_\alpha)$, we still need to analyze the roots of the corresponding polynomial.

Causality is a stament for every ξ. Thus, if we have a condition, call it S, for which we can find a single ξ that violates the statement needed for all ξ, we have that the negation of S is a necessary condition for causality. In our case we can manage to do this this by taking S to be inequalities whose negation are the ones stated. This is simpler than finding sufficient conditions because it suffices to find one such ξ.

For the sufficient conditions, a very careful analysis of the polynomial $\det(A^\alpha \xi_\alpha)$ needs to done. This is possible if some terms on the polynomial have the right sign, which is the case under the assumptions we make.

Local existence is based on the identity

$$c^T a = \det(a)\, I$$

where c^T is the transpose of the cofactor matrix of the matrix a. In our case, this identity allows us to diagonalize the system, where the (diagonal) principal part will then be the differential operator corresponding to $\det(A^\alpha \xi_\alpha)$. This will be an operator of order 22 which, in view of the causality conditions, will be a product of (strictly) hyperbolic operators. Some of these operators are repeated as $\det(A^\alpha \xi_\alpha) = 0$ has repeated roots (this means that the diagonalized operator is only weakly hyperbolic). Thus, estimates will lose derivatives in Sobolev spaces. But we can still close estimates in Gevrey spaces because of the infinite differentiability and controlled growth of functions on these spaces. Our techniques go back to the seminal work of Leray and Ohya on weakly hyperbolic equations [LO]. See [Di] for an overview of these techniques.

$\square$

The analysis of the characteristics in our proof reveals that the characteristics of the DNMR equations are:

- flow lines, with multiplicity 14.

- sound waves, with single multiplicity (corresponding to two roots, i.e., a cone)

- shear waves, three distinct characteristics of multiplicity one each (two roots for each characteristic, i.e., each is a cone)

More precisely, these are possibly distinct characteristics in that they might coincide for specific values of the fluid variables and transport coefficients, but without such specific fine tuning they will in general be different.

(Note that the number of roots adds to 22).

Our necessary conditions are particularly useful for applications because one can verify at each time step of numerical simulations whether they hold. If they do not, then causality is being violated. This was recently done in [PADNN-H, CS] where the authors checked the causality conditions for numerical simulation of the quark-gluon plasma and found that up to 30% of the initial fluid cells violate causality. This raises questions about the validity of some conclusions about the quark-gluon plasma derived based on these simulations.

The BDNK theory

The Bemfica-Disconzi-Noronha-Kovtun (BDNK) theory is the culmination of a series of works [BDN1, BDN3, BDN4, K, HK2]. The goal is to construct a fully general-relativistic theory of viscous fluids (meaning, a theory that is causal, stable, includes all fluid variables and viscous fluxes, and is locally well-posed in Sobolev spaces, with or without coupling to Einstein's equations) by "fixing" the acausality and instability of the Eckart and Landau-Lifshitz theories.

We will not reproduce here all arguments employed in the construction of the BDNK theory, which are many and rely on ideas of effective field theories, kinetic theory, and thermodynamics, aided by insights from geometry and hyperbolic PDEs. We will only mention that the big idea is to have the fundamental principle of causality determine which terms are allowed in the energy-momentum tensor, rather than (as in Eckart's and Landau-Lifshitz's theories) making possibly unwarranted assumptions and only later investigate causality.

The BDNK theory is defined by the following energy momentum-tensor and baryon current:

$$T_{\alpha\beta} := (\varrho + \mathcal{R}) u_\alpha u_\beta + (p + \mathcal{P})\Pi_{\alpha\beta} + \pi_{\alpha\beta} + Q_\alpha u_\beta + Q_\beta u_\alpha ,$$

$$J^\alpha := n u^\alpha ,$$

with

$$\mathcal{R} := \tau_{\mathcal{R}} (u^\mu \nabla_\mu \varrho + (p+\varrho) \nabla_\mu u^\mu),$$

$$\mathcal{P} := -\zeta \nabla_\mu u^\mu + \tau_{\mathcal{P}} (u^\mu \nabla_\mu \varrho + (p+\varrho) \nabla_\mu u^\mu),$$

$$Q_\alpha := \tau_Q (p+\varrho) u^\mu \nabla_\mu u_\alpha + \beta_Q \Pi^\mu_\alpha \nabla_\mu \varrho + \beta_\varrho \Pi^\mu_\alpha \nabla_\mu \varrho + \beta_n \Pi^\mu_\alpha \nabla_\mu n ,$$

$$\begin{aligned} \pi_{\alpha\beta} &:= -2\eta \sigma_{\alpha\beta} \\ &= -2\eta \Pi^\mu_\alpha \Pi^\nu_\beta \left(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} \nabla_\lambda u^\lambda g_{\alpha\beta} \right), \end{aligned}$$

where the τ's, called relaxation times, are functions of ϱ and n,

$$\beta_\varrho := \tau_Q \left.\frac{\partial p}{\partial \varrho}\right|_n + \kappa \theta n \left.\frac{\partial (\mu/\theta)}{\partial \varrho}\right|_n ,$$

$$\beta_n := \tau_Q \left.\frac{\partial p}{\partial n}\right|_\varrho + \kappa \theta n \left.\frac{\partial (\mu/\theta)}{\partial n}\right|_\varrho .$$

μ is the chemical potential determined by the thermodynamic relation $\frac{dp}{p+\varrho} = \frac{d\theta}{\theta} + \frac{n\theta}{p+\varrho} d\left(\frac{\mu}{T}\right)$. The coefficients of shear and bulk viscosity and the heat conductivity are functions of n and ϱ. Collectively, the relaxation times, β's, ζ, η and κ are called transport coefficients. Observe that all viscous fluxes are present and both ϱ and n are included.

Remark. Because the equations of motion $\nabla_\alpha T^\alpha_\beta = 0$ will be second order in (ϱ, ψ, u), the equation $\nabla_\alpha J^\alpha = 0$ is in fact a constraint. This constraint will be propagated by data such that $\nabla_\alpha J^\alpha \big|_{t=0} = 0$.

Theo (Bemfica-D-Noronha [BDN4]). Assume

$$\varrho + p,\ \tau_\varrho,\ \tau_p,\ \tau_Q > 0,$$

$$\eta,\ \zeta,\ \kappa \geq 0.$$

Then, the system of BDNK equations coupled to Einstein's equations is causal if and only if

$$(\varrho + p)\tau_Q > \eta,$$

$$2(p+\varrho)\tau_\varrho \tau_Q > \tau_\varrho\left((p+\varrho)c_s^2 \tau_Q + \zeta + \frac{4}{3}\eta + \kappa\sigma_s\right) + (p+\varrho)\tau_p \tau_Q$$

$$\geq 0$$

$$\left[\tau_\varrho\left((p+\varrho)c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \kappa\sigma_s\right) + (p+\varrho)\tau_p\tau_Q\right]^2$$

$$\geq 4(p+\varrho)\tau_\varrho\tau_Q\left[\tau_p\left((p+\varrho)c_s^2\tau_Q + \kappa\sigma_s\right) - \tau_\varrho\left(\zeta + \frac{4\eta}{3}\right)\right]$$

$$\geq 0$$

$$(p+\varrho)\tau_\varrho \tau_Q + \kappa \sigma_s \tau_\ell > \tau_\varrho \left((p+\varrho) c_s^2 \tau_Q + \zeta + \frac{4\eta}{3} + \kappa \sigma_s \right) + (p+\varrho) \tau_\ell \tau_Q (1 - c_s^2)$$
$$+ \beta_\varrho \left(\zeta + \frac{4\eta}{3} \right),$$

where

$$\sigma_s := \frac{(p+\varrho)^2 \theta}{n} \left. \frac{\partial(\mu/\theta)}{\partial \varrho} \right|_n + \theta (p+\varrho) \left. \frac{\partial(\mu/\theta)}{\partial n} \right|_\varrho ,$$

$$c_s^2 := \left. \frac{\partial p}{\partial \varrho} \right|_s$$
$$= \left. \frac{\partial p}{\partial \varrho} \right|_n + \frac{n}{p+\varrho} \left. \frac{\partial p}{\partial n} \right|_\varrho .$$

The same result holds in a fixed background.

(We abuse notation and denote $c_s^2 = \left. \frac{\partial p}{\partial \varrho} \right|_s$ by analogy with the perfect fluid case, but c_s^2 is not the sound speed – found through the characteristics – which now also depends on the viscous fluxes.)

proof: Like in the case of the DNMR equations, the proof reduces to an analysis of the characteristics which in this case are given by $\det(A^{\alpha\beta} \xi_\alpha \xi_\beta) = 0$, where $A^{\alpha\beta}$ are the matrices of the principal part of the system. Here, we have differentiated $\nabla_\mu J^\mu = 0$ with $u^\alpha \nabla_\alpha$ to obtain a second-order equation. Also like in the case of the DNMR equations, we need to carry a judicious analysis of the roots.

$\square$

The analysis of the characteristics in our proof reveals that the characteristics of the BDNK equations are:

- flow lines, with multiplicity 2 (2 root for each multiplicity)

- Sound waves, with single multiplicity (corresponding to two roots, i.e., a cone)

- second sound (propagation of temperature perturbations), with a single characteristic (corresponding to two roots, i.e., a cone)

- shear waves, with multiplicity 3 (2 roots for each multiplicity, i.e., a cone)

(Note that the number of roots adds to $12 \Leftarrow 6$ equations of second order. Recall that we differentiated $\nabla_\mu J^\mu = 0$)

Next, we adress local existence and uniqueness

Theo (Bemfica - D - Noronha [BDN4]; Bemfica - D - Rodriguez - Shao [BDRS] Bemfica - D - Graber [BDG]). Let $(\Sigma, \mathring{g}, \mathring{k}, \mathring{\varepsilon}, \hat{\varepsilon}, \mathring{n}, \hat{n}, \mathring{u}, \hat{u})$ be an initial-data set for the BDNK - Einstein system such that $\nabla_\mu J^\mu = 0$ holds for the initial data and $u^\mu u_\mu = -1$. Assume that the assumptions of the previous theorem hold in strict form and that the transport coefficients are analytic functions of their arguments. Finally, assume that the $\circ$ quantities are in H^r and the $\wedge$ quantities in H^{r-1}, $r \geq 5$. Then, there exists a globally hyperbolic development of the initial data, which is unique if it is the maximal development.

proof: The proof is carried out through the following steps.

- We work locally in wave coordinates and decompose all derivatives into their u and u-ortogonal and expand these decompositions in coordinates, obtaining evolutions for $u^\mu \partial_\mu(\cdot)$, $\Pi^{\alpha\mu}\partial_\mu(\cdot)$, which can be turned into a first-order system.

- We show that the matrix of the principal part of the resulting first-order system admits a complete set of eigenvectors. We can then diagonalize the principal part.

- The diagonalization happens at the level of the principal symbol. This needs to be done at the level of the equations. But because of rational functions that are obtained in the eigenvalues and eigenvectors, the resulting equations become pseudo-differential when diagonalized. The pseudo-differential diagonal system admits good energy estimates that can be used to produce solutions.

- Solutions to the original equations are obtained through an approximation by analytic solutions.

$\square$

<u>Remark</u>. Our proof in fact shows that the system, written as a first order system, is strongly hyperbolic.

It remains to show stability. This is accomplished by applying the following theorem to the system of first-order equations derived in the proof of the previous theorem.

<u>Theo (Benfica - D - Noronha [BDN4])</u>. Consider a system of

first order PDEs with constant coefficients whose first-order derivatives can be decomposed in the directions parallel and orthogonal to the unit timelike vectorfield u. If the system is causal, strongly hyperbolic, and stable in the LRF $\mathcal{O}$ then it is stable in any frame connected to $\mathcal{O}$ by a Lorentz transformation.

The proof can be found in [BDNK]. We then show that conditions for stability in the LRF can be found consistent with the previous causality conditions.

The previous theorem was generalized by Gavassino [Ga], who in particular removed the strong hyperbolicity hypothesis.

We conclude with some observations on the physical significance of the BDNK theory.

The BDNK theory reproduces known physics relevant to the study of the quark gluon plasma [BDNK] in some simple settings. The BDNK tensor has been derived from kinetic theory [BDN3, HK2].

Numerical simulations of the BDNK theory have been recently carried out by Pandya-Pretorius [PP], Pandya-Most-Pretorious [PMP], and Bantilan-Bea-Figueras [BBF] for conformal fluids in one in two dimensions. The main conclusion is that for small viscosity (which is the regime viscous theories are expected to be trusted) BDNK and DNMR mostly agree.

These observations in conjuction with the above mathematical results indicate that the BDNK theory posesses all the good features of the DNMR equations plus a good existence and uniqueness theory, and this while incorporating all relevant fluid variables and viscous fluxes.

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