Subject: [External] Message to students
Date: Tuesday, July 26, 2022 at 2:29:27 PM Uzbekistan Standard Time
From: Disconzi, Marcelo M
To: Ibragimov, Zair
CC: Disconzi, Marcelo M
Attachments: Disconzi-Speck2019_Article_TheRelativisticEulerEquationsR.pdf, Disconzi2022_Article_TheRelativisticEulerEquationsW.pdf, PhysRevX.12.021044.pdf, Disconzi2022_Article_RoughSoundWavesIn3DCompressibl.pdf, PhysRevLett.126.222301-with-supplemental.pdf, CPAA.pdf

External Email Use Caution and Confirm Sender

Hi Zair,

Can you please send the email below to students on my behalf? Note that there are six attachments, please do not forget to attach them to the message.

Thanks!

Subject: materials for Marcelo's lecturers

Dear all,

I want to give you some materials to help you with my lectures next week. I should've sent this a month ago, but only now was I able to put together the notes I mention below. I hope this will still be useful to you, despite being last minute.

1. Here:

   http://www.disconzi.net/Research/Uzbekistan2022/Uzbekistan-ASI.pdf

   are some notes I wrote for the lectures. Unfortunately, I didn't have time to Latex them (I plan to do it at some point), so these are handwritten. I think they will be useful to you, although admittedly my handwriting is not the nicest one. Because they are exported from my iPad, the file is large, so it may take a minute for you to open them in your browser.

   This will free you from taking detailed class notes since the notes I wrote contain enough details, so that you can better focus in the lectures.

I'm also attaching some papers to this message:

2. The paper "The Relativistic Euler Equations: Remarkable Null Structures and Regularity Properties" covers parts of section "New formulation of the relativistic Euler equations," roughly pages 46-62 and 98-110 of the notes.

3. The paper "Rough sound waves in 3D compressible Euler flow with vorticity" covers section "Low regularity solutions," which is roughly pages 46-98 of the notes.

4. The paper “The relativistic Euler equations with a physical vacuum boundary: Hadamard local well-posedness, rough solutions, and continuation criterion,” covers section “The relativistic Euler equations with a physical vacuum boundary,” which is pages 111-166 of the notes.

5. The papers “Nonlinear Constraints on Relativistic Fluids Far from Equilibrium” and “First-Order General-Relativistic Viscous Fluid Dynamics” cover section “Relativistic fluids with viscosity,” which is the remaining part of the notes.

(The above papers can all be found on arXiv, but the published versions that I’m sending here are more polished than the arXiv versions.)

Regarding background, I’ll be assuming background in (Lorentzian) geometry and PDEs.

6. If you don’t have a background in geometry or need to brush up some concepts, a quick introduction can be found in chapter 3 of these notes:

   http://www.disconzi.net/Research/USCSummer2019/USC_notes.pdf

   A more complete introduction is available on these notes by S. Aretakis:

   https://www.math.toronto.edu/aretakis/General%20Relativity-Aretakis.pdf

7. If you need a quick introduction to some of the main PDE tools I’ll be assuming, I recommend these notes by J. Luk:

   https://web.stanford.edu/~jluk/NWnotes.pdf

8. Finally, a quick review of the concept of characteristics for general PDEs, which will be used in the lectures, can be found in appendix A of the attached paper “On the existence of solutions and causality for relativistic viscous conformal fluids.”

Naturally, I don’t expect you to go over all this material in the short time before the lectures. But I thought that having these references can be helpful. Of course, do not hesitate to reach out if you have any questions.

Best,

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ON THE EXISTENCE OF SOLUTIONS AND CAUSALITY FOR RELATIVISTIC VISCOUS CONFORMAL FLUIDS

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(Communicated by Igor Kukavica)

Abstract. We consider a stress-energy tensor describing a pure radiation viscous fluid with conformal symmetry introduced in [3]. We show that the corresponding equations of motions are causal in Minkowski background and also when coupled to Einstein's equations, and solve the associated initial-value problem.

1. **Introduction.** Consider the following stress-energy tensor for a relativistic fluid with viscosity:

$$\begin{aligned} T_{\alpha\beta} =& \frac{4}{3}u_\alpha u_\beta \epsilon + \frac{1}{3}g_{\alpha\beta}\epsilon - \eta \pi_\alpha^\mu \pi_\beta^\nu (\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} g_{\mu\nu}\nabla_\lambda u^\lambda) \\ &+ \lambda(u_\alpha u^\mu \nabla_\mu u_\beta + u_\beta u^\mu \nabla_\mu u_\alpha) + \frac{1}{3}\chi \pi_{\alpha\beta}\nabla_\mu u^\mu + \chi u_\alpha u_\beta \nabla_\mu u^\mu \\ &+ \frac{\lambda}{4\epsilon}(u_\alpha \pi_\beta^\mu \nabla_\mu \epsilon + u_\beta \pi_\alpha^\mu \nabla_\mu \epsilon) + \frac{3\chi}{4\epsilon} u_\alpha u_\beta u^\mu \nabla_\mu \epsilon + \frac{\chi}{4\epsilon}\pi_{\alpha\beta} u^\mu \nabla_\mu \epsilon. \end{aligned} \tag{1}$$

Here, u is the four-velocity of fluid particles, normalized so that

$$u^\alpha u_\alpha = -1, \tag{2}$$

ϵ is the energy density of the fluid, g is a (Lorentzian) metric, ∇ is the Levi-Civita connection associated with g, $\pi_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$, and η, λ, and χ are viscous transport coefficients — so that $\eta = \lambda = \chi = 0$ corresponds to an ideal fluid. The transport coefficients are non-negative functions of ϵ. Coefficient η is the usual coefficient of shear viscosity, whereas λ and χ are related to relaxation times. More precisely, while λ and χ, differently than η, have no analogue in more familiar theories such as classical, non-relativistic Navier-Stokes, their physical meaning can be understood from the derivation of (1) from kinetic theory given in [3]. In that case, one may interpret $\lambda/(s\theta)$ and $\chi/(s\theta)$, where s is the entropy density and θ the temperature, as relaxation times that restore causality (since intuitively causality says that the system needs some time to relax back to equilibrium after a perturbation). See [3] for details.

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Key words and phrases. Relativistic viscous fluids, conformal fluids, Einstein's equations, causality, well-posedness.

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We are interested in the case of pure radiation, when the fluid's pressure is given by $p = \frac{1}{3}\epsilon$, and, therefore, p has already been eliminated from $T_{\alpha\beta}$.

Above and throughout, we adopt the following:

Convention 1. We work in units where $8\pi G = c = 1$, where G is Newton's constant and c is the speed of light in vacuum. Our signature for the metric is $-+++$. Greek indices run from 0 to 3 and Latin indices from 1 to 3.

We shall couple (1) to Einstein's equations:

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta}, \tag{3}$$

where $R_{\alpha\beta}$ and R are, respectively, the Ricci and scalar curvature of the metric g, and Λ is a constant (the cosmological constant). We recall that in light of the Bianchi identities, a necessary condition for (3) to hold is that

$$\nabla_\alpha T^\alpha_\beta = 0. \tag{4}$$

Naturally, equations (3)-(4) are defined in a four-dimensional differentiable manifold, the space-time.

We shall establish the following.

Main result. *(see Theorems 2.2 and 2.3 for precise statements) Under appropriate conditions on the initial data and the transport coefficients, the system of Einstein's equations coupled to (1) is causal and admits a unique solution. Causality and uniqueness are here understood in the usual sense of general relativity. Existence, uniqueness, and causality remain true if we consider solely (4) in Minkowski spacetime.*

The tensor (1) was introduced[1] in [3]. As discussed there, (1) is the first example in the literature of a stress-energy tensor for relativistic viscous fluids satisfying the following list of physical requirements: in Minkowski background, equations (4) are (i) linearly stable with respect to perturbations around homogeneous thermodynamic equilibrium, (ii) well-posed, and (iii) causal; (iv) Einstein's equations coupled to (1) are well-posed and causal; (v) equations (4) reduce to the standard Navier-Stokes equations in the non-relativistic limit; (vi) an out-of-equilibrium entropy can be defined so that solutions to (4) satisfy the (out of equilibrium) second law of thermodynamics; and (vii) $T_{\alpha\beta}$ can be derived from microscopic kinetic theory.

One reason for seeking a stress-energy tensor satisfying the above properties is that the traditional forms of the relativistic Navier-Stokes equations fail to be causal and stable [23, 35], and attempts to construct a relativistic viscous theory satisfying (i)-(vi) have been limited so far[2]. See [12, 15, 16, 37] for a discussion. In [3] it is also shown that $T_{\alpha\beta}$ yields a well-defined temperature in the test-case of the Gubser flow, in contrast to the traditional relativistic Navier-Stokes' equations that yield a negative temperature, and that a hydrodynamic attractor exists for the dynamics of the Bjorken flow.

Tensor (1) describes a conformal fluid. Loosely speaking, this means that (1) is well-behaved under conformal changes of the metric. More precisely, consider

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[1]In [3], (1) is written in a different form, using the so-called Weyl derivative (whose definition is given in [3]; see [33] for more details) instead of the covariant derivative. Both expressions agree once the Weyl derivative is expanded in terms of the covariant derivative.

[2]It is interesting to note that the seemingly easier task of generalizing the non-relativistic Navier-Stokes to Riemannian manifolds is not without problems either, see [5].

a conformal transformation $g'_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta}$, and the transformed quantities $u'_\alpha = e^{-\phi} u_\alpha$, $\epsilon' = e^{4\phi}\epsilon$. Then the fluid is called conformal if $T_{\alpha\beta}$ is traceless and the corresponding transformed $T'_{\alpha\beta}$ satisfies

$$T'_{\alpha\beta} = e^{2\phi} T_{\alpha\beta}.$$

One can show [2, 4] that under these conditions

$$\nabla'_\alpha (T')^\alpha_\beta = e^{4\phi} \nabla_\alpha T^\alpha_\beta,$$

so in particular solutions are preserved by the above transformations. There exists a large literature on conformal fluids and their applications in physics, to which the reader is referred for a discussion (see, e.g., [11, 20] and references therein; for the mathematical background for these references, see [19]). We restrict ourselves to mentioning that conformal fluids are of importance in the study of the quark-gluon plasma that forms in high-energy collisions of heavy-ions; the quark-gluon plasma at very high temperatures is the prototypical example of a relativistic viscous fluid with an equation of state of pure radiation.

The definition of conformal fluid, stated above, will play no direct role in this work per se. Rather, we shall use one of its main consequences, namely, that for such fluids we have

$$\chi = a_1 \eta, \lambda = a_2 \eta, \tag{5}$$

where a_1 and a_2 are constants. Therefore all transport coefficients are determined once we are given $\eta = \eta(\epsilon)$.

Our main result has previously appeared in [3], but the letter format of that manuscript and the fact that it was addressed primarily to a physical audience prevented us from presenting several details of the proof. In particular, the argument in [3] may not be entirely satisfactory for a mathematical audience.

Definition 1.1. For the rest of the the paper, we shall refer to the system of equations (3), with $T_{\alpha\beta}$ given by (1) and u satisfying (2), as the viscous Einstein-conformal fluid (VECF) system.

2. **Statement of the results.** We now turn to the precise formulation of the Main Result. We begin by discussing the initial data for the VECF system.

Definition 2.1. An initial data set for the VECF system consists of a three-dimensional smooth manifold Σ, a Riemannian metric g_0 on Σ, a symmetric two-tensor κ on Σ, two real-valued functions ϵ_0 and ϵ_1 defined on Σ, and two vector fields v_0 and v_1 on Σ, such that the Einstein constraint equations are satisfied.

We recall that the constraint equations are given by the following system of equations on Σ:

$$R_{g_0} - |\kappa|^2_{g_0} - (\operatorname{tr}_{g_0} \kappa)^2 = 2\rho$$
$$\nabla_{g_0} \operatorname{tr}_{g_0} \kappa - \operatorname{div}_{g_0} \kappa = j$$

where R_{g_0} is the scalar curvature of g_0, ∇_{g_0}, tr_{g_0}, div_{g_0}, and $|\cdot|_{g_0}$ are the covariant derivative, trace, divergence, and norm with respect to g_0. The quantities ρ and j are given by $\rho = T(n, n)$ and $j = T(n, \cdot)$, where n is the future-pointing unit normal to Σ inside a development of the initial data and T is the stress-energy tensor.

Because $T_{\alpha\beta}$ involves first derivatives of u and ϵ, initial conditions for their time derivatives have to be given, hence the necessity of two functions and two vector fields. Even though u is a four-vector, it suffices to specify vector fields on Σ, with initial conditions for the non-tangential components of u derived from (2) (see section 3.2). It is well-known that initial data for Einstein's equations cannot be prescribed arbitrarily, having to satisfy the associated constraint equations, see, e.g., [21], for details.

We can now state our main result. The definition of spaces G^s and $G^{m,s}$ is recalled in Appendix A.1. We refer the reader to the general relativity literature (e.g., [7, 21, 25, 38, 40]) for the terminology employed in Theorem 2.2.

Theorem 2.2. *Let $\mathcal{I} = (\Sigma, g_0, \kappa, \epsilon_0, \epsilon_1, v_0, v_1)$ be an initial data set for the VECF system. Assume that Σ is compact with no boundary, and that $\epsilon_0 > 0$. Suppose that χ and λ are given by (5), where $\eta : (0, \infty) \to (0, \infty)$ is analytic, and assume that $a_1 = 4$ and $a_2 \geq 4$. Finally, assume that the initial data is in $G^{(s)}(\Sigma)$ for some $1 < s < \frac{17}{16}$. Then:*

1) There exists a globally hyperbolic development M of $\mathcal{I}$.

2) M is causal, in the following sense. Let (g, ϵ, u) be a solution to the VECF system provided by the globally hyperbolic development M. For any $p \in M$ in the future of Σ, $(g(p), u(p), \epsilon(p))$ depends only on $\mathcal{I}|_{i(\Sigma)\cap J^-(p)}$, where $J^-(p)$ is the causal past of p and $i : \Sigma \to M$ is the embedding associated with the globally hyperbolic development M.

We note that, in the standard PDE language, Theorem 2.2 is local in time. But as usual in general relativity, solutions to Einstein's equations are geometric (a solution to Einstein's equations is a Lorentzian manifold) and, in particular, coordinate independent, whereas a statement like "there exists a $\mathcal{T} > 0$..." (as is usual in local in time results) requires the introduction of coordinates. This is why the theorem is better stated as the existence of a globally hyperbolic development[3]. We assumed that Σ is compact for simplicity, otherwise asymptotic conditions would have to be prescribed. The type of asymptotic conditions one would impose had Σ been non-compact depends on the type of questions one is investigating. For instance, it is customary to require g_0 to be asymptotically flat, but other conditions, such as asymptotically hyperbolic, are often used. As for the matter variables, several choices are possible. One can require v_0 and ϵ_0 to approach zero, a constant, or some other specified profile at infinity. The literature on Einstein's equations with non-compact Σ is vast, and a discussion of asymptotic conditions can be found, e.g., [7, 8] and references therein. The assumption $\epsilon_0 > 0$ in Theorem 2.2 (which implies a uniform bound from below away from zero by the compactness of Σ), however, is crucial. This is apparent from expression (1), but it is worth mentioning that allowing ϵ_0 to vanish leads to severe technical difficulties even in the better studied case of the Einstein-Euler system (see [18, 24, 36] for the known results and [13] for a discussion; in fact, the difficulties with vanishing density are present already in the non-relativistic case, see the discussion in [14, 31]). In particular, if we were dealing with a non-compact Σ and had chosen an asymptotic condition where ϵ_0 approaches zero, the techniques here employed would not directly apply.

[3]We recall that a globally hyperbolic development is, roughly speaking, a Lorentzian manifold where Einstein's equations are satisfied and in which Σ embeds isometrically as a Cauchy surface taking the correct data. We also recall that once a globally hyperbolic development is shown to exist, one can prove the existence of the "largest" possible global hyperbolic development, i.e., the maximal globally hyperbolic development of the initial data, which is (geometrically) unique. See [25, 38] for details.

The assumptions $a_1 = 4$ and $a_2 \geq 4$ are technical[4], but they are consistent with conditions that guarantee the previously mentioned linear stability of (1). Note that while our proof is restricted to the Gevrey class, our result guarantees that causality will be automatically satisfied in any function space where uniqueness can be established. This is relevant in view of the difficulties of constructing causal theories of relativistic viscous fluids.

Next, we consider the case of a Minkowski background.

Theorem 2.3. *Let T be given by (1) with g being the Minkowski metric. Suppose that χ and λ satisfy (5), with $a_1 = 4$, $a_2 \geq 4$, where $\eta : (0,\infty) \to (0,\infty)$ is a given analytic function. Let $\epsilon_0, \epsilon_1 : \mathbb{R}^3 \to \mathbb{R}$ and $v_0, v_2 : \mathbb{R}^3 \to \mathbb{R}^3$ belong to $G^{(s)}(\mathbb{R}^3)$ for some $1 \leq s < \frac{7}{6}$, and assume that $\epsilon_0 \geq C_0 > 0$, where C_0 is a constant.*

Then, there exists a $\mathcal{T} > 0$, a function $\epsilon : [0,\mathcal{T}) \times \mathbb{R}^3 \to (0,\infty)$, and a vector field $u : [0,\mathcal{T}) \times \mathbb{R}^3 \to \mathbb{R}^4$, such that (ϵ, u) satisfies equations (2) and (4) in $[0,\mathcal{T}) \times \mathbb{R}^3$, $\epsilon(0,\cdot) = \epsilon_0$, $\partial_0\epsilon(0,\cdot) = \epsilon_1$, $u(0,\cdot) = u_0$, and $\partial_0 u(0,\cdot) = u_1$, where ∂_0 is the derivative with respect to the first coordinate in $[0,\mathcal{T}) \times \mathbb{R}^3$. This solution belongs to $G^{2,(s)}([0,\mathcal{T}) \times \mathbb{R}^3)$ and is unique in this class. Finally, the solution is causal, in the following sense. For any $p \in [0,T) \times \mathbb{R}^3$, $(\epsilon(p), u(p))$ depends only on $(\epsilon_0, \epsilon_1, v_0, v_1)|_{\{x^0=0\}\cap J^-(p)}$, where $J^-(p)$ is the causal past of p (with respect to the Minkowski metric).

While formally Theorem 2.3 can not be derived as a corollary of Theorem 2.2, its validity should come as no surprise once we know the latter to be true. In fact, the proof of Theorem 2.3 will be essentially contained in that of Theorem 2.2, as we shall see. It is nonetheless useful to state Theorem 2.3 given the importance of viscous fluids in Minkowski background for applications.

Remark 1. The difference between $s > 1$ in Theorem 2.2 and $s \geq 1$ in Theorem 2.3 comes from the fact that in the proof of Theorem 2.2 we work in local coordinates and employ bump functions, which cannot be analytic (case $s = 1$). In Minkowski space, however, we can use global coordinates and analyticity is not prevented.

3. Proof of Theorem 2.2.

In this section we prove Theorem 2.2, thus we henceforth assume its hypotheses. We will always denote by s a number in $(1, \frac{17}{16})$, as in the statement of the theorem. The proof will be split in several parts. Some of the arguments parallel well-known constructions in general relativity in the smooth setting, but we present them because some additional steps are required in the Gevrey class.

3.1. The equations of motion.

Here we write the VECF in coordinates and in a more explicit form. At this point, we are only interested in writing the equations in a suitable form, thus we assume the validity of (2) and (3) (and consequently (4)), and derive relations of interest.

As is customary, we shall write (3) in trace-reversed form and in wave coordinates. More precisely, we consider the reduced Einstein equations given by

$$g^{\mu\nu}\partial^2_{\mu\nu} g_{\alpha\beta} = B_{\alpha\beta}(\partial\epsilon, \partial u, \partial g), \tag{6}$$

where above and henceforth we adopt the following:

[4] Other values of a_1 and a_2 are in fact possible as showed in [3], and the proof for these other cases is essentially the same as showed here. The main difference is how one factors the characteristic determinant. This different factorization is carried out in [3]. See Remark 16.

Notation 1. We shall employ the letters B and $\widetilde{B}$, with indices attached when appropriate, to denote a general expression depending on at most the number of derivatives indicated in its argument. For instance, in (6), $B_{\alpha\beta}$ represents an expression depending on at most first derivatives of ϵ, first derivatives of u, and first derivatives of g. As another example, $\widetilde{B}(\epsilon, \partial u, \partial^2 g)$ denotes an expression depending on at most zero derivatives of ϵ, one derivative of u, and two derivatives of g. B and $\widetilde{B}$ can vary from expression to expression. It can be easily verified that B and $\widetilde{B}$ will always be an analytic function (typically involving only products and quotients) of its arguments.

Equations (4) become[5]

$$
\begin{aligned}
&(-\eta g^{\alpha\mu} + (\lambda - \eta) u^\alpha u^\mu)\partial^2_{\alpha\mu} u^\beta + (\lambda+\chi) u^\beta u^\mu \partial^2_{\mu\alpha} u^\alpha + \frac{1}{3}(-\eta+\chi) g^{\beta\mu}\partial^2_{\mu\alpha} u^\alpha \\
&\quad + \frac{1}{3}(-\eta+\chi) u^\beta u^\mu \partial^2_{\mu\alpha} u^\alpha + \frac{1}{4\epsilon} u^\beta (\lambda g^{\alpha\mu} + (\lambda + 3\chi) u^\alpha u^\mu)\partial^2_{\alpha\mu}\epsilon \\
&\quad + \frac{1}{4\epsilon}(\lambda+\chi) u^\alpha g^{\beta\mu}\partial^2_{\alpha\mu}\epsilon + \frac{1}{4\epsilon}(\lambda+\chi) u^\beta u^\alpha u^\mu \partial^2_{\alpha\mu}\epsilon + \widetilde{B}^\beta(\partial u, g)\partial^2 g \\
&= B^\beta(\partial\epsilon, \partial u, \partial g).
\end{aligned}
\tag{7}
$$

The term $\widetilde{B}^\beta(\partial u, g)\partial^2 g$, which is linear in $\partial^2 g$, comes from derivatives of the Christoffel symbols, after expanding the second covariant derivatives of u. This term is of the form $\widetilde{B}^\beta(\partial u, g, \partial^2 g)$ according to Notation 1, but we wrote it as $\widetilde{B}^\beta(\partial u, g)\partial^2 g$ to emphasize that we shall consider it as a second order quasi-linear operator on g. The particular form of this operator will not be needed, but it is important that it be included in the principal part of the system for the derivative counting employed below.

Applying $u^\alpha u^\mu \nabla_\alpha \nabla_\mu$ to (2) produces

$$
u_\lambda u^\alpha u^\mu \partial^2_{\alpha\mu} u^\lambda + \widetilde{B}(\partial u, g)\partial^2 g = B(\partial u, \partial g). \tag{8}
$$

We introduce the vector

$$
U = (u^\beta, \epsilon, g_{\alpha\beta}),
$$

where we adopt the obvious notation with u^β denoting (u^0, u^1, u^2, u^3), etc.; such a notation is used throughout, including in the matrices below. We write equations (6), (7), and (8) in matrix form as

$$
\mathfrak{M}(U, \partial)U = \mathfrak{q}(U), \tag{9}
$$

where

$$
\mathfrak{M}(U, \partial) = \begin{pmatrix} m(U,\partial) & b(U,\partial) \\ 0 & g^{\mu\nu}\partial^2_{\mu\nu} \end{pmatrix} \tag{10}
$$

with

$$
\begin{aligned}
m_{00}(U,\partial) =& (-\eta g^{\alpha\mu} + (\lambda-\eta)u^\alpha u^\mu)\partial^2_{\alpha\mu} + (\lambda+\chi) u^0 u^\alpha \partial^2_{0\alpha} \\
&+ \frac{1}{3}(-\eta+\chi)(g^{0\alpha} + u^0 u^\alpha)\partial^2_{0\alpha},
\end{aligned}
$$

$$
m_{0i}(U,\partial) = (\lambda+\chi) u^0 u^\alpha \partial^2_{\alpha i} + \frac{1}{3}(-\eta+\chi)(g^{0\alpha} + u^0 u^\alpha)\partial^2_{\alpha i},
$$

[5] See Appendix B for a derivation of (6) and (7).

$$m_{i\nu}(U,\partial) = u^i(\lambda+\chi)u^\alpha\partial^2_{\alpha\nu} + \frac{1}{3}(-\eta+\chi)(g^{i\alpha}+u^iu^\alpha)\partial^2_{\alpha\nu},\ \nu\neq i,$$

$$m_{ii}(U,\partial) = (-\eta g^{\alpha\mu} + (\lambda-\eta)u^\alpha u^\mu)\partial^2_{\alpha\mu} + u^i(\lambda+\chi)u^\alpha\partial^2_{\alpha i} + \frac{1}{3}(-\eta+\chi)(g^{i\alpha}+u^iu^\alpha)\partial^2_{\alpha i},$$

with no sum over i,

$$m_{\nu 4}(U,\partial) = \frac{1}{4\epsilon}u^\nu(\lambda g^{\alpha\mu} + (\lambda+3\chi)u^\alpha u^\mu)\partial^2_{\alpha\mu} + \frac{1}{4\epsilon}(\lambda+\chi)(u^\alpha g^{\nu\mu} + u^\nu u^\alpha u^\mu)\partial^2_{\alpha\mu},$$

$$m_{4\nu}(U,\partial) = u_\nu u^\alpha u^\mu\partial^2_{\alpha\mu}.$$

(Recall Convention 1: above we have $1 \leq i \leq 3$.) The matrix $b(U,\partial)$ in (10) corresponds to the matrix with the operators $\widetilde{B}^\beta(\partial u, g)\partial^2$ and $\widetilde{B}(\partial u, g)\partial^2$ that act on g (see (7) and (8)), whose explicit form will not be important here. Finally, $g^{\mu\nu}\partial^2_{\mu\nu}$ in (10) represents the 10×10 identity matrix times the operator $g^{\mu\nu}\partial^2_{\mu\nu}$. The vector $\mathfrak{q}(U)$ corresponds to the right-hand side of equations (6), (7), and (8), i.e.,

$$\mathfrak{q}(U) = (B^\beta(\partial\epsilon,\partial u,\partial g), B(\partial u, g), B_{\alpha\beta}(\partial\epsilon,\partial u,\partial g)).$$

3.2. **Initial data.** We now investigate the appropriate initial conditions for (9). We remind the reader that the geometric data in the assumptions of Theorem 2.2 are intrinsic to Σ, thus they do not determine full data for the system[6]. Hence, we need to complete the given data to a full set of initial data.

Assume that $\mathcal{I}$ is given as in the statement of Theorem 2.2. Embed Σ into $\mathbb{R}\times\Sigma$ and consider $p \in \{0\}\times\Sigma$. We shall initially obtain a solution in a neighborhood of p, hence we prescribe initial data locally.

Take coordinates $\{x^\alpha\}_{\alpha=0}^3$ in a neighborhood $\mathcal{U}$ of p such that $\{x^i\}_{i=1}^3$ are coordinates on Σ, which we assume to be normal coordinates for g_0 centered at p. We remark that in these coordinates the initial data will be in $G^{(s)}(\{x^0=0\}\cap\mathcal{U})$. For, by our assumption on $\mathcal{I}$, there exist local coordinates $\{y^i\}_{i=1}^3$ in a neighborhood $\mathcal{Y}\subseteq\Sigma$ of p such that, in these coordinates, the initial data is Gevrey regular. One obtains (short-time) geodesics starting at p by solving the geodesic equation, which will be an ODE with Gevrey data in the $\{y^i\}$ coordinates. Since we can equip Gevrey spaces with a norm, the usual Picard iteration can be applied to solve the geodesic equation, and hence we obtain solutions that are Gevrey regular and vary within the Gevrey class with the initial data. Therefore, the exponential map and, as a consequence, the coordinates $\{x^i\}$ are Gevrey regular in $\mathcal{Y}$ with respect to the $\{y^i\}$ coordinates. Expressing the initial data now in $\{x^i\}$ coordinates, we conclude from standard properties of composition and products of Gevrey maps (see, e.g., [32]) that the initial data is in $G^{(s)}(\{x^0=0\}\cap\mathcal{U})$ in the $\{x^i\}$ coordinates.

We prescribe the following initial conditions for $g_{\alpha\beta}$ on $\{x^0=0\}\cap\mathcal{U}$:

$$g_{ij}(0,\cdot) = (g_0)_{ij},\ g_{00}(0,\cdot) = -1,\ g_{0i}(0,\cdot) = 0,\ \partial_0 g_{ij}(0,\cdot) = \kappa_{ij},$$

[6] For example, g_0 is a metric on Σ which is a three-manifold; thus, g_0 contains only nine (six independent) components locally, whereas there are sixteen (ten independent) components in the full space-time metric. Similarly, κ does not determine all transversal derivatives of g on Σ, and v_0 and v_1 determine only the initial three-velocity and its transversal derivatives, whereas we need the four-velocity u and its tranversal derivatives initially. These mismatches are, as it is well-known, related to the gauge freedom of Einstein's equations. See, e.g., [7] for more discussion.

and $\partial_0 g_{0\alpha}(0,\cdot)$ is chosen such that $\{x^\alpha\}$ are wave coordinates for g at $x^0 = 0$ (which is well-known to always be possible).

For u^β, we prescribe

$$u^i(0,\cdot) = v_0^i,\ u^0(0,\cdot) = \sqrt{1 + (g_0)_{ij} v_0^i v_0^j},\ \partial_0 u^i(0,\cdot) = v_1^i,$$

$$\partial_0 u^0(0,\cdot) = \frac{1}{\sqrt{1+(g_0)_{ij}v_0^i v_0^j}} \Big((g_0)_{ij} v_0^j v_1^i + \frac{1}{2}\kappa_{ij} v_0^i v_0^j + \frac{1}{2}\partial_0 g_{00}(0,\cdot)(1 + (g_0)_{ij} v_0^i v_0^j) + \partial_0 g_{0i}(0,\cdot) v_0^i \sqrt{1 + (g_0)_{ij} v_0^i v_0^j} \Big).$$

(Note that the radicands are non-negative because g_0 is a Riemannian metric.) The initial conditions for u^0 and $\partial_0 u^0$ have been derived from (2) and the above initial conditions for $g_{\alpha\beta}$. Finally,

$$\epsilon(0,\cdot) = \epsilon_0,\ \partial_0\epsilon(0,\cdot) = \epsilon_1.$$

3.3. **Initial conditions for the system in $\mathbb{R}^4$.** Consider the local coordinates introduced in section 3.2. Via these coordinates and identifying p with the origin, we can regard system (9) as defined in an open set $\mathcal{U}$ of $\mathbb{R}^4$ containing the origin, with the initial conditions prescribed on $\{x^0 = 0\} \cap \mathcal{U}$. Note that we can also take (9) as a system of equations on the whole of $\mathbb{R}^4$, and we therefore do so. We seek to extend the initial data to the whole hypersurface $\{x^0 = 0\}$, thus determining initial conditions for the system in $\mathbb{R}^4$.

Let $\mathcal{V}$ be compactly contained in $\{x^0 = 0\} \cap \mathcal{U}$ and $\mathcal{W}$ be compactly contained in $\mathcal{V}$. Let $\varphi : \{x^0 = 0\} \to \mathbb{R}$ be a function in $G^{(s)}(\mathbb{R}^3)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $\mathcal{W}$, and $\varphi = 0$ in the complement of $\mathcal{V}$. Denote by h the Minkowski metric and set, on $\{x^0 = 0\}$,

$$\mathring{g}_{ij} = \varphi (g_0)_{ij} + (1-\varphi) h_{ij},\ \mathring{g}_{00} = -1,\ \mathring{g}_{0i} = 0,\ \partial_0 \mathring{g} = \varphi \kappa_{ij}.$$

These will be initial conditions for $g_{\alpha\beta}$ (for equations (9) in $\mathbb{R}^4$), with an usual abuse of notation to denote the initial conditions involving ∂_0. As our coordinates have been chosen with $\{x^i\}$ normal coordinates for g_0 centered at p, we have that $\mathring{g}_{ij}(0) = h_{ij}$ and the deviations of $\mathring{g}_{ij}$ from the Minkowski metric restricted to $\{x^0 = 0\} \cap \mathcal{U}$ are quadratic on the coordinates away from the origin. Writing

$$\mathring{g}_{ij} = \varphi(g_0)_{ij} + (1-\varphi)h_{ij} = h_{ij} + \varphi((g_0)_{ij} - h_{ij}),$$

we see that, shrinking $\mathcal{U}$ if necessary and taking into account our choice for $\mathring{g}_{0\alpha}$, $\mathring{g}_{\alpha\beta}$ is a perturbation of the Minkowsi metric restricted to $\{x^0 = 0\}$. Therefore, $\mathring{g}_{\alpha\beta}$ defines a Lorentzian metric.

Next, we introduce

$$\mathring{u}^i = \varphi v_0^i,\ \partial_0 \mathring{u}^i = \varphi v_1^i,$$

with the initial conditions for $\mathring{u}^0$ and $\partial_0 \mathring{u}^0$ obtained by the same formulas as in section (3.2), with the appropriate replacements by $\mathring{u}^i$ and $\mathring{g}$ on the right-hand sides. Finally, set

$$\mathring{\epsilon} = \varphi\epsilon_0 + 1 - \varphi,\ \partial_0\mathring{\epsilon} = \varphi\epsilon_1.$$

By the compactness of Σ and the assumption $\epsilon_0 > 0$, it follows that $\epsilon_0 \geq C$ for some constant $C > 0$, thus

$$\mathring{\epsilon} \geq \min\{\frac{1}{2}C, \frac{1}{2}\} \geq C' > 0,$$

for some constant C'.

The initial data for (9) in $\mathbb{R}^4$ described in this section will be denoted by $\mathring{U}$.

3.4. **Solving the system in $\mathbb{R}^4$.** In this section, we solve system (9) with the initial conditions described in section 3.3 (see Proposition 1 below). We shall employ the techniques, terminology, and notation of Leray-Ohya systems reviewed in the appendix.

Lemma 3.1. *Equations (9) form a Leray system.*

Proof. Write U as $U = (U^1, U^2)$, with the understanding that $U^1 = (u^\beta, \epsilon) = (u^0, u^1, u^2, u^3, \epsilon)$ and $U^2 = (g_{\alpha\beta})$. Assign to (9) the following indices:

$$\begin{aligned} m_1 = 2, \quad m_2 = 2,\\ n_1 = 0, \quad n_2 = 0, \end{aligned}$$

where $m_1 = m(U^1) \equiv m(u^\beta, \epsilon)$, $m_2 = m(U^2) \equiv m(g_{\alpha\beta})$,

$$\begin{aligned} n_1 &= n(\text{equation } (7))\\ &= n(\text{equation } (8))\\ &\equiv n(\text{equations corresponding to the first five rows of } (9)), \end{aligned}$$

and

$$\begin{aligned} n_2 &= n(\text{equation } (6))\\ &\equiv n(\text{equations corresponding to the last ten rows of } (9)). \end{aligned}$$

It is understood that we have one index m_I for each unknown of the fifteen unknowns and one index n_J for each one of the fifteen equations in (9). For instance, by $m_1 = m_1(u^\beta, \epsilon) = 2$ we mean $m(u^0) = m(u^1) = m(u^2) = m(u^3) = m(\epsilon) = 2$, and so on.

One readily verifies that with this choice of indices, (9) has the structure of a Leray system. Indeed, we list below for each row J in (9) or, equivalently, for each equation in the system (6), (7), and (8), the value of n_J; the highest derivatives of each unknown entering in the coefficients and on the right-hand side of the equation; and the difference $m_I - n_J$:

$$\text{rows 1-4} \equiv \text{eq. } (7) : n_1 = 0;\ \partial u, \partial\epsilon, \partial g; \begin{cases} m(u) - n_1 \equiv m_1 - n_1 = 2,\\ m(\epsilon) - n_1 \equiv m_1 - n_1 = 2,\\ m(g) - n_1 \equiv m_2 - n_1 = 2, \end{cases}$$

$$\text{row 5} \equiv \text{eq. } (8) : n_1 = 0;\ \partial u, \partial g; \begin{cases} m(u) - n_1 \equiv m_1 - n_1 = 2,\\ m(\epsilon) - n_1 \equiv m_1 - n_1 = 2,\\ m(g) - n_1 \equiv m_2 - n_1 = 2, \end{cases}$$

and

$$\text{rows 6-15} \equiv \text{eq. } (6) : n_2 = 0;\ \partial u, \partial\epsilon, \partial g; \begin{cases} m(u) - n_1 \equiv m_1 - n_2 = 2,\\ m(\epsilon) - n_1 \equiv m_1 - n_2 = 2,\\ m(g) - n_1 \equiv m_2 - n_2 = 2. \end{cases}$$

For example, in equations (7), for which $n_1 = 0$, we have that the left-hand side consists of differential operators of order 2 acting on (u^β, ϵ) $(m(u^\beta, \epsilon) - n_1 = 2)$ and differential operators of order 2 acting on $(g_{\alpha\beta})$ $(m(g_{\alpha\beta}) - n_1 = 2)$, whose coefficients depend on at most first derivatives of the unknowns ($\partial u, \partial\epsilon, \partial g$, i.e., $m(u^\beta, \epsilon) - n_1 - 1$

and $m(g_{\alpha\beta})-n_1-1$); the right-hand side of (7), as the coefficients of the differential operators, depends on at most first derivatives of the unknowns. □

Assumption 1. We henceforth make explicit use of (5), with $a_1 = 4$ and $a_2 \geq 4$, in accordance with the assumptions of Theorem 2.2.

For the proof of the next proposition, the reader is reminded of the Definition A.9 of $\mathcal{A}^s(\Sigma, Y)$, which consists of the space of functions sufficiently near the Cauchy data.

Proposition 1. *There exist a $\mathcal{T} > 0$, a vector field $u : [0,\mathcal{T})\times\mathbb{R}^3 \to \mathbb{R}^4$, a function $\epsilon : [0,\mathcal{T})\times\mathbb{R}^3 \to (0,\infty)$, and a Lorentzian metric g defined on $[0,\mathcal{T})\times\mathbb{R}^3$, such that $U = (u^\beta, \epsilon, g_{\alpha\beta})$ satisfies (9) in $[0,\mathcal{T})\times\mathbb{R}^3$ and takes the initial data $\mathring{U}$ on $\{x^0 = 0\}$. Moreover, $(u,\epsilon,g) \in G^{2,(s)}([0,\mathcal{T})\times\mathbb{R}^3)$ and this solution is unique in this class.*

Proof. We fix the initial data $\mathring{U}$ as constructed in section 3.3 and consider $\widehat{U} = (\widehat{u}^\alpha, \widehat{\epsilon}, \widehat{g}_{\alpha\beta}) \in \mathcal{A}^s(\Sigma, Y)$. Shrinking Y if necessary, we can assume that $\widehat{g}_{\alpha\beta}$ is a Lorentzian metric, that $\widehat{\epsilon} > 0$, and that $\widehat{u}$ is time-like for $\widehat{g}_{\alpha\beta}$, since these properties hold for $\mathring{U}$. Because the coefficients of the matrix of differential operators $\mathfrak{M}(U,\partial)$ depend on at most first derivatives of the unknowns, we can evaluate these coefficients on $\widehat{U}$. Denote the corresponding operator by $\mathfrak{M}(\widehat{U},\partial)$. The characteristic determinant $P(\widehat{U},\xi)$ of (9), evaluated at $\widehat{U}$, is

$$P(\widehat{U},\xi) = \det\mathfrak{M}(\widehat{U},\xi) = p_1(\widehat{U},\xi)p_2(\widehat{U},\xi)p_3(\widehat{U},\xi)p_4(\widehat{U},\xi) \tag{11}$$

where[7]

$$p_1(\widehat{U},\xi) \equiv p_1(\xi) = \frac{1}{12\widehat{\epsilon}}\eta^4(\widehat{u}^\mu\xi_\mu)^4, \tag{12}$$

$$\begin{aligned} p_2(\widehat{U},\xi) \equiv p_2(\xi) = \big[&(a_2-1)((\widehat{u}^0)^2\xi_0^2 + (\widehat{u}^1)^2\xi_1^2 + (\widehat{u}^2)^2\xi_2^2 + (\widehat{u}^3)^2\xi_3^2) - \xi^\mu\xi_\mu \\ &+2(a_2-1)(\widehat{u}^1\widehat{u}^2\xi_1\xi_2 + \widehat{u}^1\widehat{u}^3\xi_1\xi_3 + \widehat{u}^2\widehat{u}^3\xi_2\xi_3) \\ &+2(a_2-1)\widehat{u}^0\xi_0\widehat{u}^i\xi_i\big]^2, \end{aligned} \tag{13}$$

$$\begin{aligned} p_3(\widehat{U},\xi) \equiv p_3(\xi) = &-6((a_2+5)a_2 + (a_2^2+7a_2-8)\widehat{u}^\lambda\widehat{u}_\lambda)(\widehat{u}^\mu\xi_\mu)^2 \\ &+6(a_2+2)(1+5\widehat{u}^\lambda\widehat{u}_\lambda)\xi^\mu\xi_\mu, \end{aligned} \tag{14}$$

and

$$p_4(\widehat{U},\xi) \equiv p_4(\xi) = (\xi^\mu\xi_\mu)^{10}, \tag{15}$$

and the contractions in these expressions are done with respect to the metric $\widehat{g}_{\alpha\beta}$. The computation of $P(\widehat{U},\xi)$, and the corresponding factorization in the above polynomials, is done through a lengthy and tedious algebraic calculation, part of which was done with the help of the software Mathematica[8]. Note that the block diagonal form of $\mathfrak{M}(U,\partial)$ allowed us to compute the characteristic determinant without providing the specific form of the operators $\widetilde{B}^\beta(\partial u, g)\partial^2 g$ and $\widetilde{B}(\partial u, g)\partial^2 g$.

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[7] We remark that compared to [3], polynomial $p_3(\widehat{U},\xi)$ looks different. That is because in [3] $\widehat{u}^\lambda\widehat{u}_\lambda$ had been replaced by 1 in view of (2). Strictly speaking, we are not allowed to do that since one has to prove that u remains normalized for positive time, which is done in Lemma 3.3 below, but this was ignored in [3] since there only a sketch of the proof was presented (see the above Introduction).

[8] See Appendix C.

It is easy to see that the polynomials $\widehat{u}^\mu \xi_\mu$ and $\xi^\mu \xi_\mu$ are hyperbolic polynomials as long as $\widehat{g}_{\alpha\beta}$ is a Lorentzian metric and $\widehat{u}$ is time-like with respect to $\widehat{g}_{\alpha\beta}$. Both conditions are satisfied in view of the constructions in section 3.3. Therefore, $p_1(\xi)$ is the product of four hyperbolic polynomials (recall that $\widehat{\epsilon} > 0$ and $\eta(\widehat{\epsilon}) > 0$), and $p_4(\xi)$ is the product of ten hyperbolic polynomials. We now move to analyze $p_2(\xi)$ and $p_3(\xi)$.

Write $p_2(\xi) = (\widetilde{p}_2(\xi))^2$, where $\widetilde{p}_2(\xi)$ is the second-degree polynomial between brackets in the definition of $p_2(\xi)$. We claim that $\widetilde{p}_2(\xi)$ is a hyperbolic polynomial. To show this, we need to investigate the roots $\xi_0 = \xi_0(\xi_1, \xi_2, \xi_3)$ of the equation $\widetilde{p}_2(\xi) = 0$. Consider first the case where $\widetilde{p}_2(\xi)$ is evaluated at the origin, i.e., $\widetilde{p}_2(\xi) = \widetilde{p}_2(\widehat{U}(0), \xi)$, and assume for a moment that $\widehat{g}_{\alpha\beta}(0)$ is the Minkowski metric and that $\widehat{u}^\mu \widehat{u}_\mu = -1$. In this case, the roots are

$$\begin{aligned}\xi_{0,\pm} = -\frac{1}{1 + (a_2 - 1)(1 + \underline{\widehat{u}}^2)} \Bigg(&(a_2 - 1)\underline{\widehat{u}} \cdot \underline{\xi} \sqrt{1 + \underline{\widehat{u}}^2} \\ &\pm \sqrt{(a_2 + (a_2 - 1)\underline{\widehat{u}}^2)\underline{\xi}^2 - (a_2 - 1)(\underline{\widehat{u}} \cdot \underline{\xi})^2} \Bigg),\end{aligned} \tag{16}$$

where $\underline{\widehat{u}} = (\widehat{u}^1, \widehat{u}^2, \widehat{u}^3)$, $\underline{\widehat{u}}^2 = (\widehat{u}^1)^2 + (\widehat{u}^2)^2 + (\widehat{u}^3)^2$, $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$, $\underline{\xi}^2 = \xi_1^1 + \xi_2^2 + \xi_3^2$, and $\cdot$ is the Euclidean inner product. We see that if $\underline{\xi} = 0$, then $\xi_{0,\pm} = 0$, and hence $\xi = 0$. Thus, we can assume $\underline{\xi} \neq 0$. The Cauchy-Schwarz inequality gives $\underline{\widehat{u}}^2 \underline{\xi}^2 - (\underline{\widehat{u}} \cdot \underline{\xi})^2 \geq 0$, hence $\xi_{0,+}$ and $\xi_{0,-}$ are real and distinct for $a_2 \geq 4$. We conclude that $\widetilde{p}_2(\xi)$ is a hyperbolic polynomial at the origin. Since the roots of a polynomial vary continuously with the polynomial coefficient, $\widetilde{p}_2(\xi)$ will have two distinct real roots at any point on $\{x^0 = 0\}$ if $\widehat{g}_{\alpha\beta}$ is sufficiently close to the Minkowski metric and $\widehat{u}^\mu \widehat{u}_\mu$ sufficiently close to -1. We know from section 3.3 that these last conditions are fulfilled upon taking $\mathcal{U}$ and Y sufficiently small (recall that $\mathring{g}_{\alpha\beta}(0)$ equals the Minkowski metric.). Therefore, $\widetilde{p}_2(\xi)$ is a hyperbolic polynomial, and $p_2(\xi)$ is the product of two hyperbolic polynomials.

We now investigate the roots $\xi_0 = \xi_0(\xi_1, \xi_2, \xi_3)$ of the equation $p_3(\xi) = 0$. As above, we first consider $p_3(\xi)$ evaluated at the origin and suppose that $\widehat{g}_{\alpha\beta}(0)$ is the Minkowski metric and that $\widehat{u}^\mu \widehat{u}_\mu = -1$, which produces

$$\begin{aligned}\xi_{0,\pm} = \frac{1}{-2(2 + a_2) - (a_2 - 4)(1 + \underline{\widehat{u}}^2)} \Bigg(&(a_2 - 4)\underline{\widehat{u}} \cdot \underline{\xi} \sqrt{1 + \underline{\widehat{u}}^2} \\ &\pm \sqrt{2} \sqrt{(3a_2(2 + a_2) + (a_2^2 - 2a_2 - 8)\underline{\widehat{u}}^2)\underline{\xi}^2 - (a_2^2 - 2a_2 - 8)(\underline{\widehat{u}} \cdot \underline{\xi})^2} \Bigg).\end{aligned}$$

As above, we can assume $\underline{\xi} \neq 0$, and the Cauchy-Schwarz inequality again gives $\underline{\widehat{u}}^2 \underline{\xi}^2 - (\underline{\widehat{u}} \cdot \underline{\xi})^2 \geq 0$. We readily verify that $(a_2^2 - 2a_2 - 8) \geq 0$ and $3a_2(2 + a_2) > 0$ for $a_2 \geq 4$. Therefore, $\xi_{0,+}$ and $\xi_{0,-}$ are real and distinct, and $p_3(\xi)$ is a hyperbolic polynomial at the origin. As above, this implies that $p_3(\xi)$ is a hyperbolic polynomial.

We conclude that $P(\widehat{U}, \xi)$ is the product of four degree one (i.e., $p_1(\xi)$), two degree two (i.e., $p_2(\xi)$), one degree two (i.e., $p_3(\xi)$), and ten degree two (i.e., $p_4(\xi)$) hyperbolic polynomials. The Gevrey index of (9) is thus $\frac{17}{16}$ (see Remark 15). Recall that $1 < s < \frac{17}{16}$ by assumption.

Since $m_I - n_J = 2$ for all I, J, and $\sum_I m_I - \sum_J n_J \geq 2$, we have verified the conditions of Theorem A.14 in the appendix. Hence we obtain the diagonalized system

$$\widetilde{\mathfrak{M}}(U,\partial)U = \widetilde{\mathfrak{q}}(U), \tag{17}$$

where $\widetilde{\mathfrak{M}}(U,\partial)$ is a diagonal matrix whose entries are differential operators of order 30 (the order of the characteristic determinant, see the appendix) whose coefficients depend on at most 29 derivatives of U, and $\widetilde{\mathfrak{q}}(U)$ contains all the lower order terms. We want to invoke Theorem A.10 to solve (17). To do so, we need to provide initial conditions for (17). Since our goal is to obtain a solution to (9) out of a solution to (17), such initial conditions need to be compatible with solutions to (9).

We shall show that all derivatives of U, restricted to $\{x^0 = 0\}$, can be formally computed from (9) and written in terms of the initial data. In particular, initial conditions to (17) compatible with (9) can be determined. As usual in these situations, it suffices to show that we can inductively compute $\partial_0^k U$ on $\{x^0 = 0\}$ as the tangential derivatives ∂_i can always be computed.

From (6), we can determine $\partial_0^2 g_{\alpha\beta}\big|_{\{x^0=0\}}$ in terms of the initial data $\mathring{U}$. Using the result into (7), we can write $\widetilde{B}^\beta(\partial u, g)\partial^2 g$ restricted to $\{x^0 = 0\}$ in terms of $\mathring{U}$. Equations (7) and (8) then give

$$\mathfrak{a}\begin{pmatrix}\partial_0^2 u^\beta \\ \partial_0^2 \epsilon\end{pmatrix} = \mathfrak{b},$$

where $\mathfrak{b}$ can be written in terms of the initial data on $\{x^0 = 0\}$, and the matrix $\mathfrak{a}$ is the matrix of the coefficients of the terms $\partial_0^2 u^\beta$ and $\partial_0^2\epsilon$ in equations (7) and (8). At the origin, where $\mathring{g}_{\alpha\beta}(0)$ equals the Minkowski metric, the determinant of $\mathfrak{a}$ is

$$\frac{\eta^4}{\epsilon_0}(1+\underline{\mathring{u}}^2)^2(3a_2 + (a_2-4)\underline{\mathring{u}}^2)(a_2 + (a_2-1)\underline{\mathring{u}}^2)^2,$$

which is never zero for $a_2 \geq 4$ (recall that $\epsilon_0 > 0$ and $\eta(\epsilon_0) > 0$). Invoking once more the fact that $\mathring{g}_{\alpha\beta}$ is a perturbation of the Minkowski metric, we conclude that $\det(\mathfrak{a})|_{\{x^0=0\}}$ never vanishes. We can thus invert $\mathfrak{a}$ and write $\partial_0^2 u^\beta$ and $\partial_0^2\epsilon$ at $x^0 = 0$ in terms of $\mathring{U}$.

It is clear that we can continue this process: differentiate (6) with respect to ∂_0 to determine $\partial_0^3 g_{\alpha\beta}\big|_{\{x^0=0\}}$; differentiate (7) and (8) with respect to ∂_0, use $\partial_0^3 g_{\alpha\beta}\big|_{\{x^0=0\}}$ to eliminate the resulting terms $\widetilde{B}^\beta(\partial u, g)\partial^3 g$ and $\widetilde{B}(\partial u, g)\partial^3 g$, and then solve for $\partial_0^3 u^\beta$ and $\partial_0^3\epsilon$ at $x^0 = 0$ (notice that the matrix $\mathfrak{a}$ remains unchanged). Inductively, we can determine all derivatives $\partial_0^k U$ on $\{x^0 = 0\}$, $k = 2, 3, \dots$, in terms of $\mathring{U}$. Moreover, $\partial_0^k U\big|_{\{x^0=0\}}$ are analytic expressions of $\mathring{U}$ and, therefore, the initial conditions for (17) determined in this fashion will be in $G^{(s)}$.

The initial data for (17), denoted $\mathring{\widetilde{U}}$, consists of the original initial data $\mathring{U}$ for (9), and the values of $\partial_0^k U\big|_{\{x^0=0\}}$ determined by the above procedure for $k = 2, \dots, 29$.

Remark 2. The above procedure determines all derivatives of U, evaluated at $x^0 = 0$, in terms of the initial conditions $\mathring{U}$. It follows that if the initial data $\mathring{U}$ is analytic, a well-known argument using power series can be employed to construct an analytic solution to (9) in a neighborhood of $\{x^0 = 0\}$. These techniques for construction of analytic solutions, however, say nothing about causality.

Having supplied (17) with appropriate initial conditions, we can now invoke Theorem A.10 to conclude the following. There exist a $\widetilde{\mathcal{T}} > 0$, a vector field

$u : [0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3 \to \mathbb{R}^4$, a function $\epsilon : [0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3 \to (0, \infty)$, and a Lorentzian metric g defined on $[0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3$, such that $U = (u^\beta, \epsilon, g_{\alpha\beta})$ satisfies (17) in $[0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3$ and takes the initial data $\mathring{\widetilde{U}}$ on $\{x^0 = 0\}$. Moreover, $(u, \epsilon, g) \in G^{2,(s)}([0, \widetilde{\mathcal{T}}) \times \mathbb{R}^3)$ and this solution is unique in this class.

(We note that in invoking Theorem A.10, we are using that the intersections of the cones determined by the polynomials $p_i(\xi)$ have non-empty interiors (recall definition A.4). This follows from the above expressions, but it can also be verified from the explicit computations in section 3.5.)

The conclusions that $\epsilon > 0$ and g is a Lorentzian metric follow by continuity in the x^0 variable, since these conditions are true at $x^0 = 0$.

Now we move to obtain a solution to (9) in $\mathbb{R}^4$. The argument is similar to the one in [30], thus we shall go over it briefly.

Let $\{\mathring{U}_k\}_{k=1}^\infty$ be a sequence of analytic initial conditions for the system (9) converging in $G^{(s)}(\{x^0 = 0\})$ to $\mathring{U}$. For each k, let (u_k, ϵ_k, g_k) be the analytic solution to (9), defined in a neighborhood of $\{x^0 = 0\}$, and taking on the initial data $\mathring{U}_k$ (see Remark 2). Let $\mathring{\widetilde{U}}_k$ be the initial data for (17) obtained from $\mathring{U}_k$ and compatible with (9), i.e., the one derived by the inductive procedure previously described. Then, $\mathring{\widetilde{U}}_k \to \mathring{\widetilde{U}}$ in $G^{(s)}(\{x^0 = 0\})$. In light of the compatibility of $\mathring{\widetilde{U}}_k$, and because (17) was derived from (9) via diagonalization, the solutions (u_k, ϵ_k, g_k) also satisfy (17). Furthermore, this solution to (17) also agrees with the one given by Theorem A.10 (since this theorem also applies for analytic data, i.e., $s = 1$). The energy-type of estimates proved by Leray and Ohya [28] guarantee then that $(u_k, \epsilon_k, g_k) \to (u, \epsilon, g)$ in $G^{(s)}$ and that (u, ϵ, g) satisfy the original system (9). By construction, (u, ϵ, g) take on the initial data $\mathring{U}$. □

Remark 3. The initial conditions for the VECF system have to satisfy the Einstein constraint equations (recall Definition 2.1). The initial conditions $\mathring{U}$ satisfy the constraints in the region $\mathcal{W}$ in light of the way that $\mathring{U}$ was constructed out of $\mathcal{I}|_{\mathcal{U}}$. This is, naturally, necessary for the eventual construction of a full solution to the VECF system. However, purely from the point of view of (9) in $\mathbb{R}^4$, initial condition can be prescribed freely, i.e., they do not have to satisfy any constraints. Therefore, the existence of the analytic initial data $\mathring{U}_k$ follows simply by the density of analytic functions in $G^{(s)}$. Also by density, we can guarantee that the components $(\mathring{\epsilon}_0)_k$ and $(\mathring{g}_{\alpha\beta})_k$ in $\mathring{U}_k$ satisfy $(\mathring{\epsilon}_0)_k > 0$ and that $(\mathring{g}_{\alpha\beta})_k$ is a Lorentzian metric.

Remark 4. The above calculations involving $(a_2^2 - 2a_2 - 8) \geq 0$ show why we have the technical assumption $a_2 \geq 4$. As our calculations were presented already with $a_1 = 4$ in place, they do not reveal the reason for this assumption, which as follows. Computing the characteristic determinant with general a_1 produces a very complicated expression with some terms proportional to $a_1 - 4$. These terms vanish when $a_1 = 4$, and the corresponding expression simplifies to (11). This can be seen explicitly in Appendix C.

3.5. **Causality.** Having obtained solutions, we now investigate the causality of equations (9). As in section 3.4, we use results and terminology recalled in the appendix.

Lemma 3.2. *The solution $U = (u, \epsilon, g)$ to (9) given in Proposition 1 is causal, in the following sense. For any $x \in [0, \mathcal{T}) \times \mathbb{R}^3$, $(u(x), \epsilon(x), g(x))$ depends only on $\mathring{U}|_{\{x^0=0\} \cap J^-(x)}$, where $J^-(x)$ is the causal past of x (with respect to the metric g).*

Proof. Fix $x \in [0, \mathcal{T}) \times \mathbb{R}^3$. The characteristic determinant of (9) at x is given by (11), with the obvious replacement of $\widehat{U}$ by U and evaluated at x; the polynomials $p_i(U(x), \xi) \equiv p_i(x, \xi)$, $i = 1, \dots, 4$, are given by expressions (12) to (15), again with the obvious replacement by $U(x)$. By the same argument used in section 3.4 to prove that the $p_i(\xi)$'s are hyperbolic polynomials on $\{x^0 = 0\}$, namely, that $g_{\alpha\beta}$ is near the Minkowski metric, we know that the polynomials $p_i(x, \xi)$ are hyperbolic (perhaps after shrinking $\mathcal{T}$ if necessary).

Denote by $V_i(x)$ the characteristic cone $\{p_i(x, \xi) = 0\}$, and by $\Gamma_i^{*,\pm}(x)$ the corresponding (forward and backward) convex cones (on the cotangent space). Let $K^{*,\pm}(x)$ be the (forward and backward) time-like interiors of the light-cone $\{g^{\mu\nu}(x)\xi_\mu\xi_\nu = 0\}$. We need to show that $K^{*,\pm}(x) \subseteq \Gamma_i^{*,\pm}(x)$ (see Remark 12). This is straightforward for $i = 1$ and $i = 4$.

Assume for a moment that g is the Minkowski metric at x and that $u^\lambda u_\lambda = -1$ (note that we have not proved yet that u remains normalized for $x^0 > 0$). The roots of $\{p_2(x, \xi) = 0\}$ are given by (16), changing $\widehat{u}$ by u, which we can write as

$$\xi_{0,\pm} = s_\pm(u, \theta)\sqrt{\underline{\xi}^2}, \tag{18}$$

where

$$s_\pm(u, \theta) = -\frac{1}{1 + (a_2 - 1)(1 + \underline{u}^2)} \Big((a_2 - 1)\sqrt{\underline{u}^2}\cos\theta\sqrt{1 + \underline{u}^2} \pm\sqrt{a_2 + (a_2 - 1)\underline{u}^2 - (a_2 - 1)\underline{u}^2\cos^2\theta}\Big),$$

θ is the angle between $\underline{u}$ and $\underline{\xi}$ in $\mathbb{R}^3$, we used $\underline{u} \cdot \underline{\xi} = \sqrt{\underline{u}^2}\sqrt{\underline{\xi}^2}\cos\theta$, and we omitted the dependence of u and θ on x for simplicity.

Equation (18) determines the two halves of the characteristic cone $V_2(x)$ in the cotangent space at x. We will have that $K^{*,\pm}(x) \subseteq \Gamma_2^{*,\pm}(x)$ if the slopes $s_\pm$ satisfy $-1 < s_\pm(u, \theta) < 1$ for each u and θ. To see that this is the case, compute

$$s_\pm(u, 0) = s_\pm(u, 2\pi) = -\frac{\pm\sqrt{a_2} + (a_2 - 1)\sqrt{\underline{u}^2(1 + \underline{u}^2)}}{1 + (a_2 - 1)(1 + \underline{u}^2)},$$

and observe that this expression is always between -1 and 1 for $a_2 \geq 4$. We seek the maxima and minima of $s_\pm(u, \theta)$ for $0 < \theta < 2\pi$. Computing the derivative with respect to θ and solving for $\sin\theta$, we find $\sin\theta = 0$, i.e., $\theta = \pi$. We readily verify that $-1 < s_\pm(u, \pi) < 1$, thus $-1 < s_\pm(u, \theta) < 1$. Since this last condition is open, the result remains true when g is sufficiently close to the Minkowski metric and u sufficiently close to unitary, which is the case if $\mathcal{T}$ is taken sufficiently small. The same argument shows that $K^{*,\pm}(x) \subseteq \Gamma_3^{*,\pm}(x)$, where again one uses the condition $a_2 \geq 4$.

We conclude that for any $x \in [0, \mathcal{T}) \times \mathbb{R}^3$, we have $K^{*,\pm}(x) \subseteq \bigcap_{i=1}^4 \Gamma_i^{*,\pm}(x)$, and the result now follows from Theorem A.11 and Remark 12. □

Remark 5. The characteristics associated with $p_1(\xi)$ and $p_4(\xi)$ are of course those of the flow lines and gravitational waves. The characteristics associated with $p_3(\xi)$ and $p_2(\xi)$ are interpreted, respectively, as sound waves and shear waves. The latter is sometimes called a second sound wave and is present also in the Müller-Israel-Stewart theory [22]. It is useful to compare these characteristics to those of the ideal fluid. In the latter case we have the flow lines and the sound cone (i.e., the characteristics of the sound waves; see [17] for a detailed discussion of the role of the sound cone in the relativistic Euler equations). Here it is as if the sound cone had "split" into two sound-type characteristics. This resembles what happens in magnetohydrodynamics: there two different characteristics are present for the magnetoacoustic waves, namely, the so-called fast and slow magnetoacoustic waves (see [1] for details).

3.6. **Existence and causality for the system in $\mathbb{R} \times \Sigma$.** Here we show how the solution found in section 3.4 can be used to construct a causal solution in a region of $\mathbb{R} \times \Sigma$, thus effectively proving Theorem 2.2. Recall that we embedded Σ into $\mathbb{R} \times \Sigma$.

Remark 6. Consider the solution $U = (u, \epsilon, g)$ to (9) obtained in Proposition 1. Let p be a point on $\{x^0 = 0\} \times \Sigma$ and $\mathcal{W}$ be as in section 3.3. Let $D_g^+(\mathcal{W}) \subseteq [0, \mathcal{T}) \times \mathbb{R}^3$ be the future domain of dependence of $\mathcal{W}$ in the metric g, where replacing $\mathcal{W}$ with a smaller set if necessary, we can assume that $x^0 < \mathcal{T}$ for every $(x^0, x^1, x^2, x^3) \in D_g^+(\mathcal{W})$. In the coordinates on $D_g^+(\mathcal{W})$ induced from the coordinates on $[0, \mathcal{T}) \times \mathcal{W}$, the solution U is in $G^{(2,s)}$ The solution will remain in $G^{(2,s)}$ upon coordinate changes that are Gevrey regular [32]. Note that there are plenty of such coordinate changes in that a smooth manifold always admits a maximal compatible analytic atlas.

Lemma 3.3. *It holds that $u^\lambda u_\lambda = -1$ in $D_g^+(\mathcal{W})$.*

Proof. The vector field u satisfies (8), whose explicit form is

$$u_\lambda u^\alpha u^\mu \nabla_\mu \nabla_\mu u^\lambda + u^\alpha \nabla_\alpha u_\lambda u^\mu \nabla_\mu u^\lambda = 0.$$

This can be written as

$$\frac{1}{2} u^\alpha u^\mu \nabla_\alpha \nabla_\mu (u_\lambda u^\lambda) = 0.$$

This is an equation for the scalar $u_\lambda u^\lambda$. The operator $u^\alpha u^\mu \nabla_\alpha \nabla_\mu$ satisfies the assumptions of Theorem A.10. Therefore, $u_\alpha u^\alpha = -1$ in $D_g^+(\mathcal{W})$ if this condition is satisfied initially, which is the case by construction. □

Lemma 3.4. *For every $q \in \Sigma$ there exists a neighborhood $Z_q \subseteq \Sigma$ of q in Σ and a globally hyperbolic development M_q of $\mathcal{I}|_{Z_q}$, where $M_q \subseteq [0, \mathcal{T}_q) \times \Sigma$ for some $\mathcal{T}_q > 0$.*

Proof. Let p be a point on $\{x^0 = 0\} \times \Sigma$ and $\mathcal{W}$ be as in section 3.3. Since the initial conditions $\mathring{U}$ (where $\mathring{U}$ is as in section 3.3) agree on $\mathcal{W}$ with those from the initial data $\mathcal{I}$, in view of Lemma 3.2, we conclude that U is a solution to the reduced Einstein equations within $D_g^+(\mathcal{W})$. It is well-known that a solution to the reduced equations within $D_g^+(\mathcal{W})$ is also a solution to the full Einstein's equations if and only if the constraints are satisfied, which is the case by the definition of $\mathcal{I}$. Because p was an arbitrary point, the result is proven. □

We now glue the different M_q's in order to obtain a global (in space) solution.

Proposition 2. *Let $q, r \in \Sigma$, Z_q and Z_r be neighborhoods of q and r as in lemma 3.4, with globally hyperbolic developments M_q and M_r of $\mathcal{I}|_{Z_q}$ and $\mathcal{I}|_{Z_r}$, respectively, and corresponding solutions $U_q = (u_q, \epsilon_q, g_q)$ and $U_r = (u_r, \epsilon_r, g_r)$ of the VECF equations. Assume that $Z_q \cap Z_r \neq \varnothing$. Then, for any $w \in Z_q \cap Z_r$, there exist neighborhoods $\mathcal{U}_q$ and $\mathcal{U}_r$ of w in M_q and M_r, respectively, and a diffeomorphism $\psi : \mathcal{U}_q \to \mathcal{U}_r$ such that $U_q = \psi^*(U_r)$.*

Proof. We shall construct harmonic coordinates for g_q in a neighborhood of w in M_q as follows. Identifying (a portion of) Σ with its embedding in M_q, take normal coordinates $(V, \{y^i\})$ for g_0 on Σ centered at w, where g_0 comes from the initial data $\mathcal{I}$. Note that the initial data is Gevrey regular in the $\{y^i\}$ coordinates (see the argument in section 3.2). We can thus assume that U_q is in $G^{(2,s)}$ (see Remark 6)

On $[0, \mathcal{T}_q) \times V$, where $\mathcal{T}_q > 0$ is some small number such that U_q is defined on $[0, \mathcal{T}_q) \times V$, we introduce coordinates $\{y^\alpha\}$, $y^0 \in [0, \infty)$. Consider family of initial-value problems parametrized by α:

$$\begin{aligned}\nabla^\mu \nabla_\mu f^{(i)} &= 0,\\ f^{(i)}(0, y^1, y^2, y^3) &= y^i,\\ \partial_0 f^{(i)}(0, y^1, y^2, y^3) &= 0,\end{aligned}$$

and

$$\begin{aligned}\nabla^\mu \nabla_\mu f^{(0)} &= 0,\\ f^{(0)}(0, y^1, y^2, y^3) &= 0,\\ \partial_0 f^{(0)}(0, y^1, y^2, y^3) &= 1,\end{aligned}$$

where ∇ is the covariant derivative in the metric g_q. This problem has a Gevrey regular solution in a neighborhood of w in $[0, \mathcal{T}_q) \times V$, and a standard implicit function type of argument shows that the functions $x^\alpha \equiv f^{(\alpha)}$ define (harmonic) coordinates near w. We now consider the change of coordinates $x = x(y) : [0, \mathcal{T}_q') \times V' \to W \subseteq [0, \infty) \times \mathbb{R}^3$, $x = (x^0, x^1, x^2, x^3)$, where V' is a neighborhood of w in V, $\mathcal{T}' > 0$ is determined by the foregoing conditions guaranteeing the existence of the coordinates $\{x^\alpha\}$, and W is an open set containing the origin. Pulling U_q back to W via x^{-1}, it follows from these constructions that $(x^{-1})^*(U_q)$ satisfies the reduced Einstein equations in W. Since U_q originally satisfied (2) and (4) as well, we conclude that it is a solution to (9) in W.

We can repeat the above argument to obtain wave coordinates $\{z^\alpha\}$ for g_r. Because $(V, \{y^i\})$ is intrinsically determined by g_0, and M_q and M_r induce on $Z_q \cap Z_r$ the same initial data, the map z agrees with x on $\{0\} \times V'$ (in the region where both are defined). From these facts, we conclude that $(x^{-1})^*(U_q)$ and $(z^{-1})^*(U_r)$ (i) are solutions to (9) in some domain $[0, t) \times Y \subseteq [0, \infty) \times \mathbb{R}^3$ containing the origin, and (ii) take the same initial data on $\{0\} \times Y$.

We have shown that (9) enjoys uniqueness and causality. Thus, considering possibly a smaller region that is globally hyperbolic for both $(x^{-1})^*(g_q)$ and $(z^{-1})^*(g_r)$, we conclude that $(x^{-1})^*(U_q) = (z^{-1})^*(U_r)$, so that $U_q = (z^{-1} \circ x)^*(U_r)$, as desired. □

Using Proposition 2, we can now identify overlapping globally hyperbolic developments, thus obtaining a globally hyperbolic development of $\mathcal{I}$ as stated in Theorem 2.2. Causality follows essentially from Lemma 3.2: by the foregoing, we can assume that M is diffeomorphic to $[0, \mathcal{T}) \times \Sigma$ for some $\mathcal{T} > 0$. Shrinking $\mathcal{T}$ if necessary, we reduce the problem to local coordinates, in which case we can employ wave coordinates. Causality, as stated in Theorem 2.2, is preserved by diffeomorphisms, thus the result follows from the causality of the reduced system guaranteed by Lemma 3.2. This finishes the proof of Theorem 2.2.

4. **Proof of Theorem 2.3.** The proof of Theorem 2.3 is essentially contained in the above. In the case of a Minkowski background, the system reduces to

$$m(U, \partial)U = \mathfrak{q}(U),$$

where m is as in (10), $U = (u^\beta, \epsilon)$ and $\mathfrak{q}(U)$ is as in (9) with the appropriate changes for this 5×5 system. The system can be analyzed as in section 3.4. We can do this directly in $\mathbb{R}^4$, without the complications of constructing the initial data $\mathring{U}$. The characteristic determinant is given by $p_1(\xi)p_2(\xi)p_3(\xi)$, where these polynomials are as before, with the simplification that now we need not carry out any near-Minkowski arguments. Without the matrix $g^{\mu\nu}\partial^2_{\mu\nu}$ coming from Einstein's equations, the Gevrey index of the system is $\frac{7}{6}$, and analogues of Proposition 1 and Lemma 3.2 establish the result.

Appendix A. **Tools of weakly hyperbolic systems.** For the reader's convenience, we state in this appendix the results about Leray-Ohya systems (sometimes called weakly hyperbolic systems) that are used in the proof of Theorem 2.2. These results have been established by Leray and Ohya in [27, 28] for the case of systems with diagonal principal part, and extended by Choquet-Bruhat in [6] to more general systems. These works build upon the classical work of Leray on hyperbolic differential equations [26]. The reader can consult these references for the proofs of the results stated below. Further discussion can be found (without proofs) in [7, 10, 12]. Related results can also be found in [34].

We start by recalling some standard notions and fixing the notation that will be used throughout. Given $T > 0$, let $X = [0, T] \times \mathbb{R}^n$. By ∂^k we shall denote any k^{th} order derivative. We shall denote coordinates on X by $\{x^\alpha\}_{\alpha=0}^n$, thinking of $x^0 \equiv t$ as the time-variable. We use the multi-index notation to write

$$\partial^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \equiv \partial^{\alpha_0}_{x^0} \partial^{\alpha_1}_{x^1} \partial^{\alpha_2}_{x^2} \cdots \partial^{\alpha_n}_{x^n},$$

where $|\alpha| = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

A.1. **Gevrey spaces.** In this section we review the definition of Gevrey spaces. Roughly speaking, a function is of Gevrey class if it obeys inequalities similar, albeit weaker, than those satisfied by analytic functions. One of the crucial properties of Gevrey spaces for their use in general relativity is that they admit compactly supported functions.

Definition A.1. Let $s \geq 1$. We say that $f : \mathbb{R}^n \to \mathbb{C}$ belongs to the Gevrey space $G^{(s)}(\mathbb{R}^n)$ if

$$\sup_\alpha \frac{1}{(1 + |\alpha|)^s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}^{\frac{1}{1+|\alpha|}} < \infty.$$

Let $K \subset \mathbb{R}^n$ be the cube of unit side. We say that f belongs to the local Gevrey space $G^{(s)}_{loc}(\mathbb{R}^n)$ if

$$\sup_\alpha \frac{1}{(1 + |\alpha|)^s} \left(\sup_K \|\partial^\alpha f\|_{L^2(K)} \right)^{\frac{1}{1+|\alpha|}} < \infty,$$

where $\sup_K$ is taken over all side one cubes K in $\mathbb{R}^n$.

We note that the case $s = 1$, i.e., $G^{(1)}(\mathbb{R}^n)$, corresponds to the space of analytic functions.

We next introduce the space of maps defined on X whose derivatives up to order m belong to $G^{(s)}(\{x^0 = t\})$, $0 \leq t \leq T$.

Definition A.2. On X, denote $S_t = \{x^0 = t\}$. Let $s \geq 1$, and let $m \geq 0$ be an integer. We denote by $\overline{\alpha}$ a multi-index $\alpha = (\alpha_0, \dots, \alpha_n)$ for which $\alpha_0 = 0$. We define $G^{m,(s)}(X)$ as the set of maps $f : X \to \mathbb{C}$ such that

$$\sup_{\overline{\alpha},\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1+|\overline{\alpha}|)^s} \left\| \partial^{\beta+\overline{\alpha}} f \right\|_{L^2(S_t)}^{\frac{1}{1+|\overline{\alpha}|}} < \infty.$$

Let Y be an open set of $\mathbb{R}^d$. We define $G^{m,(s)}(X \times Y)$ as the set of maps $f : X \times Y \to \mathbb{C}$ such that

$$\sup_{\overline{\alpha},\, \gamma,\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1+|\overline{\alpha}|+|\gamma|)^s} \left\| \sup_{y \in Y} \left| \partial_x^{\beta+\overline{\alpha}} \partial_y^\gamma f \right| \right\|_{L^2(S_t)}^{\frac{1}{1+|\overline{\alpha}|+|\gamma|}} < \infty.$$

Let $K_t \subset S_t$ be the cube whose sides have unit length. The spaces $G^{m,(s)}_{loc}(X)$ and $G^{m,(s)}_{loc}(X \times Y)$ are defined as the set of maps $f : X \to \mathbb{C}$ and $f : X \times Y \to \mathbb{C}$, respectively, such that

$$\sup_{\overline{\alpha},\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1+|\overline{\alpha}|)^s} \left(\sup_{K_t} \left\| \partial^{\beta+\overline{\alpha}} f \right\|_{L^2(K_t)} \right)^{\frac{1}{1+|\overline{\alpha}|}} < \infty,$$

and

$$\sup_{\overline{\alpha},\, \gamma,\, |\beta| \leq m,\, 0 \leq t \leq T} \frac{1}{(1+|\overline{\alpha}|+|\gamma|)^s} \left(\sup_{K_t} \left\| \sup_{y \in Y} \left| \partial_x^{\beta+\overline{\alpha}} \partial_y^\gamma f \right| \right\|_{L^2(K_t)} \right)^{\frac{1}{1+|\overline{\alpha}|+|\gamma|}} < \infty,$$

where $\sup_{K_t}$ is taken over all cubes of side one within S_t.

Remark 7. Definitions A.1 and A.2 are easily generalized to vector and tensor fields in $\mathbb{R}^n$ and X, and to open subsets of $\mathbb{R}^n$ and X. In particular, replacing $\mathbb{R}^n$ by an open set Ω and X by $[0,T] \times \Omega$ in the above definitions we obtain the corresponding spaces for Ω. This allows one to define Gevrey spaces on manifolds. If M is a differentiable manifold, we say that $f : M \to \mathbb{C}$ belongs to $G^{(s)}(M)$ if for every $p \in M$ there exists a coordinate chart (x, U) about p such that $f \circ x^{-1} \in G^{(s)}(\Omega)$, where $\Omega = x(U)$. This definition generalizes for vector and tensor fields.

Remark 8. The reason to treat X and Y differently in definitions of $G^{(s)}(X \times Y)$ and $G^{m,(s)}(X \times Y)$ is that, in the theorems of section A.2, we need to distinguish between the regularity with respect to the space-time X and the regularity with respect to the parametrization of the initial data.

Remark 9. We could similarly define for manifolds the analog of the other Gevrey spaces introduce above. However, this can be somewhat cumbersome and not always natural. In particular, the spaces $G^{m,(s)}$ require a distinguished coordinate that plays the role of time. This can always be done locally, and it can be done for globally hyperbolic manifolds if we fix a particular foliation in terms of space-like slices (as done, e.g., in [10, 12]), although it is debatable how canonical this is. Here we prefer to avoid extra complications, i.e., we in fact only need the definition of $G^{(s)}(\Sigma)$, which is used for the construction of appropriate local coordinates and the construction of the initial data for the system in $\mathbb{R}^4$ (sections 3.2 and 3.3) and in the results of section 3.6. The bulk of the proofs are carried out for the system in $\mathbb{R}^4$, where all the different Gevrey spaces play a role. It follows that the solution in $\mathbb{R}^4$ is in particular smooth, giving rise to a smooth globally hyperbolic development. Note that for the conclusion of Theorem 2.2 it is not needed to assert that the full solution enjoys certain Gevrey regularity.

For more about Gevrey spaces, see, e.g., [28, 39]. We remark that the terminology "local" and the notation G_{loc} are not standard.

A.2. **The Cauchy problem.** Let $a = a(x, \partial^k)$, $x \in X$, be a linear differential operator of order k. We can write

$$a(x, \partial^k) = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha,$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index. Let $p(x, \partial^k)$ be the principal part of $a(x, \partial^k)$, i.e.,

$$p(x, \partial^k) = \sum_{|\alpha| = k} a_\alpha(x) \partial^\alpha.$$

At each point $x \in X$ and for each co-vector $\xi \in T^*_x X$, where T^*X is the cotangent bundle of X, we can associate a polynomial of order k in the cotangent space $T^*_x X$ obtained by replacing the derivatives by $\xi \in T^*_x X$. More precisely, for each k^{th} order derivative in $a(x, \partial^k)$, i.e.,

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

$|\alpha| = k$, we associate the polynomial

$$\xi^\alpha \equiv \xi_0^{\alpha_0} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n},$$

where $\xi = (\xi_0, \xi_1, \xi_2, \ldots, \xi_n) \in T^*_x X$, forming in this way the polynomial

$$p(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha.$$

Clearly, $p(x, \xi)$ is a homogeneous polynomial of degree k. It is called the characteristic polynomial (at x) of the operator a.

The cone $V_x(p)$ of p in $T^*_x X$ is defined by the equation

$$p(x, \xi) = 0.$$

Definition A.3. With the above notation, $p(x, \xi)$ is called a hyperbolic polynomial (at x) if there exists $\zeta \in T^*_x X$ such that every straight line through ζ that does not contain the origin intersects the cone $V_x(p)$ at k real distinct points. The differential operator $a(x, \partial^k)$ is called a hyperbolic operator (at x) if $p(x, \xi)$ is hyperbolic.

Leray proved in [26] that (if X is at least three-dimensional) if $p(x, \xi)$ is hyperbolic at x, then the set of points ζ satisfying the condition of Definition A.3 forms the interior of two opposite half-cones $\Gamma^{*,+}_x(a)$, $\Gamma^{*,-}_x(a)$, with $\Gamma^{*,\pm}_x(a)$ non-empty, with boundaries that belong to $V_x(p)$.

Remark 10. Another way of stating Definition A.3 is as follows. Given $\zeta \in T_x X$, consider a non-zero vector θ that is not parallel to ζ and form the line $\lambda\zeta + \theta$, where $\lambda \in \mathbb{R}$ is a parameter. We then require this line to intersect the cone $V_x(p)$ at k distinct real points. An equivalent definition of hyperbolic polynomials is as follows [9]: $p(x,\xi)$ is hyperbolic at x if for each non-zero $\xi = (\xi_0, \dots, \xi_n) \in T^*_x X$, the equation $p(x,\xi) = 0$ has k distinct real roots $\xi_0 = \xi_0(\xi_1, \dots, \xi_n)$.

With applications to systems in mind, we next consider the $N \times N$ diagonal linear differential operator matrix

$$A(x,\partial) = \begin{pmatrix} a^1(x,\partial^{k_1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a^N(x,\partial^{k_N}) \end{pmatrix}.$$

Each $a^J(x,\partial^{k_J})$, $J = 1, \dots, N$ is a linear differential operator of order k_J.

Definition A.4. The operator $A(x,\partial)$ is called Leray-Ohya hyperbolic (at x) if:

(i) The characteristic polynomial $p^J(x,\xi)$ of each $a^J(x,\partial^{k_J})$ is a product of hyperbolic polynomials, i.e.

$$p^J(x,\xi) = p^{J,1}(x,\xi) \cdots p^{J,r_J}(x,\xi),\ J = 1, \dots, N,$$

where each $p^{J,q}(x,\xi)$, $q = 1, \dots, r_J$, $J = 1, \dots, N$, is a hyperbolic polynomial.

(ii) The two opposite convex half-cones,

$$\Gamma^{*,+}_x(A) = \bigcap_{J=1}^{N} \bigcap_{q=1}^{r_J} \Gamma^{*,+}_x(a^{J,q}), \text{ and } \Gamma^{*,-}_x(A) = \bigcap_{J=1}^{N} \bigcap_{q=1}^{r_J} \Gamma^{*,-}_x(a^{J,q}),$$

have a non-empty interior. Here, $\Gamma^{*,\pm}_x(a^{J,q})$ are the half-cones associated with the hyperbolic polynomials $p^{J,q}(x,\xi)$, $q = 1, \dots, r_J$, $J = 1, \dots, N$.

Remark 11. When the above hyperbolicity properties hold for every x, we call the corresponding operators hyperbolic (we can also talk about hyperbolicity in an open set, a certain region, etc.). When we say that an operator is Leray-Ohya hyperbolic on the whole space (or in an open set, etc.), this means not only that Definition A.4 applies for every x, but also that the numbers r_J and the degree of the polynomials $p^{J,q}(x,\xi)$, $q = 1, \dots, r_J$, $J = 1, \dots, N$, do not change with x.

Definition A.5. We define the dual convex half-cone $C^+_x(A)$ at $T_x X$ as the set of $v \in T_x X$ such that $\xi(v) \geq 0$ for every $\xi \in \Gamma^{*,+}_x(A)$; $C^-_x(A)$ is analogously defined, and we set $C_x(A) = C^+_x(A) \cup C^-_x(A)$. If the convex cones $C^+_x(A)$ and $C^-_x(A)$ can be continuously distinguished with respect to $x \in X$, then X is called time-oriented (with respect to the hyperbolic form provided by the operator A). A path in X is called future (past) time-like with respect to A if its tangent at each point $x \in X$ belongs to $C^+_x(A)$ ($C^-_x(A)$), and future (past) causal if its tangent at each point $x \in X$ belongs or is tangent to $C^+_x(A)$ ($C^-_x(A)$). A regular surface Σ is called space-like with respect to A if $T_x\Sigma$ ($\subset T_x X$) is exterior to $C_x(A)$ for each $x \in \Sigma$. It follows that for a time-oriented X, the concepts of causal past, future, domains of dependence and influence of a set can be defined in the same way one does when the manifold is endowed with a Lorentzian metric. We refer the reader to [26] for details. Here we need only the following: the causal past $J^-(x)$ of a point $x \in X$ is the set of points that can be joined to x by a past causal curve.

Remark 12. The definitions in Definition A.5 endow X with a causal structure provided by the operator A. Despite the similar terminology, however, it should be noticed that all of the above definitions depend only on the structure of the operator A, and do not require an a priori Lorentzian metric on X. The case of interest in general relativity, however, is when the causal structure of the spacetime is connected with that of A. In this regard, the following observation is useful. Suppose that X has a Lorentzian metric g. For causal solutions of the systems of equations here described (see Theorem A.11 below) to be causal in the sense of general relativity, one needs that, for all $x \in X$, $C_x^{\pm}(A) \subseteq K_x^{\pm}$, where $K_x^{\pm}$ are the two halves of the light-cone $\{g_{\mu\nu}\xi^\mu\xi^\nu \leq 0\}$. By duality, this is equivalent to saying that in the cotangent spaces we have $K_x^{*,\pm} \subseteq \Gamma_x^{*,+}(A)$, where $K_x^{*,\pm}$ are the two halves of the dual light-cone $\{g^{\mu\nu}\xi_\mu\xi_\nu \leq 0\}$.

Next, we consider the following quasi-linear system of differential equations

$$A(x, U, \partial)U = B(x, U), \tag{19}$$

where $A(x, U, \partial)$ is the $N \times N$ diagonal matrix

$$A(x, U, \partial) = \begin{pmatrix} a^1(x, U, \partial^{k_1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a^N(x, U, \partial^{k_N}) \end{pmatrix},$$

with $a^J(x, U, \partial^{k_J})$, $J = 1, \ldots, N$ differential operators of order k_J. $B(x, U)$ is the vector

$$B(x, U) = (b^J(x, U)),\ J = 1, \ldots, N,$$

and the vector

$$U(x) = (U^I(x)),\ I = 1, \ldots, N$$

is the unknown. Notice that because a_J is allowed to depend on U, the above system is in general non-linear.

Definition A.6. The system $A(x, U, \partial)U = B(x, U)$ is called a Leray system if it is possible to attach to each unknown u^I an integer $m_I \geq 0$, and to each equation J of the system an integer $n_J \geq 0$, such that:

(i) $k_J = m_J - n_J$, $J = 1, \ldots, N$;

(ii) the functions b^J and the coefficients of the differential operators a^J are[9] functions of x, of u^I, and of the derivatives of u^I of order at most $m_I - n_J - 1$, $I, J = 1 \ldots, N$. If for some I and some J, $m_I - n_J < 0$, then the corresponding a^J and b^J do not depend on u^I.

Remark 13. The indices m_I and n_J in Definition A.6 are defined up to an additive integer.

Definition A.7. A Leray-Ohya system (with diagonal principal part) is a Leray system where the matrix A is Leray-Ohya hyperbolic. In the quasi-linear case, since the operators a depend on U, we need to specify a function U that is plugged into $A(x, U, \partial)$ in order to compute the characteristic polynomials. In this case we talk about a Leray-Ohya system for the function U. The primary case of interest is when U assumes the values of the given Cauchy data.

When considering a quasi-linear system, we write $p(x, U, \xi)$ and similar expressions to indicate the dependence on U.

We now formulate the Cauchy problem for Leray systems.

[9]The regularity required for the coefficients a^J and b^J depends on particular applications and context. For instance, for Theorem A.10 the required regularity is specified. Similarly, in Definition A.8, one needs to take derivatives of these quantities up to order n_J, thus they need to be at least as many times differentiable.

Definition A.8. Let Σ be a regular hypersurface in X, which we assume for simplicity to be given by $\{x^0 = 0\}$. The Cauchy data on Σ for a Leray system in X consists of the values of $U = (u^I)$ and their derivatives up to order $m_I - 1$ on Σ, i.e., $\partial^\alpha u^I\big|_\Sigma$, $|\alpha| \leq m_I - 1$, $I = 1, \dots, N$. The Cauchy data is required to satisfy the following compatibility conditions. If $V = (v^I)$ is an extension of the Cauchy data defined in a neighborhood of Σ, i.e. $\partial^\alpha v^I\big|_\Sigma = \partial^\alpha u^I\big|_\Sigma$, $|\alpha| \leq m_I - 1$, $I = 1, \dots, N$, then the difference $a^J(x, V, \partial)U - b^J(x, V)$ and its derivatives of order less than n_J vanish on Σ, for $J = 1, \dots, N$. When to a Leray system $A(x, U, \partial)U = B(x, U)$ we prescribe initial data satisfying these conditions, we say that we have a Cauchy problem for $A(x, U, \partial)U = B(x, U)$.

Notice that by definition, the Cauchy data for a Leray system satisfies the aforementioned compatibility conditions. We also introduce the following notions related to the Cauchy problem for a Leray system.

Assumption 2. Consider the Cauchy problem for a Leray system $A(x, U, \partial)U = B(x, U)$. Let Y be an open set of $\mathbb{R}^L$, where L equals the number of derivatives of u^J of order less or equal to $\max_I m_I - n_J$, $J = 1, \dots, N$, and such that Y contains the closure of the values taken by the Cauchy data on Σ. It is convenient to consider $A(x, U, \partial)$ as a differential operator defined over $X \times Y$, as follows. We shall assume that there exists a differential operator $\widetilde{A}(x, y, \partial)$ defined over $X \times Y$ with the following property. If $(x, y) \in X \times Y$ and $V = (v^J)$ is a sufficiently regular function on Σ such that $y = (\partial^{\max_I m_I - n_J} v^J(x))_{J=1,\dots,N}$, then $A(x, V(x), \partial) = \widetilde{A}(x, y, \partial)$. We shall write $A(x, y, \partial)$ for $\widetilde{A}(x, y, \partial)$.

Definition A.9. Consider the Cauchy problem for a Leray system $A(x, U, \partial)U = B(x, U)$. Let Σ and Y be as in Definition A.8 and Assumption 2, respectively. Denote by $\mathcal{A}^s(\Sigma, I)$ the set of $V = (v^J) \in G^{(s)}(\Sigma)$, $J = 1, \dots, N$, such that $(\partial^{\max_I m_I - n_J} v^J(x))_{J=1,\dots,N} \in Y$ for all $x \in \Sigma$.

We are now ready to state the results of this appendix. We use the above notation and definitions in the statement of the theorems below.

Theorem A.10 (Existence and uniqueness)**.** *Consider the Cauchy problem for (19). Suppose that the Cauchy data is in $G^{(s)}(\Sigma)$, and that*

$$a^J(\cdot, \cdot, \partial^{k_J}) \in G^{n_J,(s)}_{loc}(X \times Y), \text{ and } b^J(\cdot, \cdot) \in G^{n_J,(s)}(X \times Y).$$

Suppose that for any $V \in \mathcal{A}^s(\Sigma, Y)$ the system is Leray-Ohya hyperbolic with indices m_I and n_J; thus for all $x \in \Sigma$, each $p^J(x, V, \xi)$ is the product of r_J hyperbolic polynomials,

$$p^J(x, V, \xi) = p^{J,1}(x, V, \xi) \cdots p^{J,r_J}(x, V, \xi),\ J = 1, \dots, N.$$

Suppose that each $p^{J,q+1}(x, V, \xi)$, $q = 0, \dots, r_J - 1$, depends on at most $m_I - m_{J,q} - r_I + q$ derivatives of v^I, $I = 1, \dots, N$, where

$$m_{J,q} = n_J + \deg(p^{J,1}) + \cdots + \deg(p^{J,q}),\ m_{J,r_J} = m_J,\ m_{J,0} = n_J.$$

Above, $\deg(p^{J,q})$ is the degree, in ξ, of the polynomial $p^{J,q}(x, V, \xi)$.

Denote by $a^J_{q+1}(x, y, \partial)$ the differential operator associated with $p^{J,q+1}$. Assume that

$$a^J_{q+1}(\cdot, \cdot, \partial) \in G^{m_{J,q} - q,(s)}_{loc}(X \times Y).$$

Let $0 \leq g_I \leq r_I$ be the smallest integers such that $a^J(x, V, \partial^{m_J - n_J})$ and $b^J(x, V)$ depend on at most $m_I - n_J - r_I + g_I$ derivatives of v^I, $I = 1, \dots, N$, $J = 1, \dots, N$. Finally, assume that

$$1 \leq s \leq \frac{r_J}{g_J} \quad \text{and} \quad \frac{n}{2} + r^J < n_J,\ J = 1, \dots, N.$$

Then, there exists a $T' > 0$ and a solution $U = (u^I)$ to the Cauchy problem for (19) and defined on $[0, T') \times \mathbb{R}^n \subseteq X$. The solution satisfies

$$u^I \in G^{m_I,(s)}([0, T') \times \mathbb{R}^n),\ I = 1, \dots, N.$$

Furthermore, the solution is unique in this regularity class.

Theorem A.11 (Causality)**.** *Assume the same hypotheses of Theorem A.10, and suppose further that*

$$1 \leq s < \frac{r_J}{g_J},\ J = 1, \dots, N.$$

Let T' and U be as in the conclusion of Theorem A.10. Then, if T' is sufficiently small, the operator $A(x, U, \partial)$ is Leray-Ohya hyperbolic (thus the causal past of a point is well-defined), and for each $x \in [0, T') \times \mathbb{R}^n$, $U(x)$ depends only on $U_0|_{J^-(x) \cap \Sigma}$, where U_0 is the Cauchy data.

Remark 14. Theorem A.10 assumes that the system is Leray-Ohya hyperbolic for $V \in \mathcal{A}(\Sigma, Y)$, which is essentially the space of values near the initial data. (Naturally, it would not make sense to require the system to be Leray-Ohya hyperbolic for the yet to be proven to exist solution U.) Once U is constructed, one can then ask whether the system is Leray-Ohya hyperbolic for U. This will be the case if T' is small, since in this case the values of U will be close to those of the initial data by continuity, guaranteeing that $U(x) \in \mathcal{A}(\Sigma, Y)$.

Theorems A.10 and A.11 are proven in [28] (reprinted in [29]).

We now consider a system whose principal part is not necessarily diagonal. The definition of a Leray system depends only on the existence of the indices m_I and n_J with the stated properties, and thus can be extended to non-diagonal systems.

Definition A.12. Consider a system of N partial differential equations and N unknowns in X, and denote the unknown as $U = (u^I)$, $I = 1, \dots, N$. The system is a (not necessarily diagonal in the principal part) Leray system if it is possible to attach to each unknown u^I a non-negative integer m_I and to each equation a non-negative integer n_J, such that the system reads

$$h^J_I(x, \partial^{m_K - n_J - 1} u^K, \partial^{m_I - n_J}) u^I + b^J(x, \partial^{m_K - n_J - 1} u^K) = 0,\ J = 1, \dots, N. \quad (20)$$

Here, $h^J_I(x, \partial^{m_K - n_J - 1} u^K, \partial^{m_I - n_J})$ is a homogeneous differential operator of order $m_I - n_J$ (which can be zero), whose coefficients depend on at most $m_K - n_J - 1$ derivatives of u^K, $K = 1, \dots N$, and there is a sum over I in $h^J_I(\cdot) u^I$. The remaining terms, $b^J(x, \partial^{m_K - n_J - 1} u^K)$, also depend on at most $m_K - n_J - 1$ derivatives of u^K, $K = 1, \dots N$. As before, these indices are defined only up to an overall additive integer.

As done above, for a given sufficiently regular U, $h^J_I(x, \partial^{m_K - n_J - 1} U^K, \partial^{m_I - n_J})$ are well-defined linear operators, and we can ask about their hyperbolicity properties. The case of interest will be, again, when we evaluate these operators at some given Cauchy data.

Write (20) in matrix form as

$$H(x,U,\partial)U = B(x,U). \tag{21}$$

Definition A.13. The characteristic determinant of (21) at $x \in X$ and for a given U is the polynomial $p(x,\xi)$ in the co-tangent space T_x^*X, $\xi \in T_x^*X$, given by

$$p(x,U,\xi) = \det(H(x,U,\xi)). \tag{22}$$

Note that p is a homogeneous polynomial of degree

$$\ell \equiv \sum_{I=1}^N m_I - \sum_{J=1}^N n_J.$$

Under appropriate conditions, (21) can be transformed into a Leray-Ohya system of the form (19), i.e., with diagonal principal part. More precisely, we have the following.

Theorem A.14 (Diagonalization)**.** *Consider (21). Suppose that the characteristic determinant (22) at a given U is not identically zero, and it is the product of Q hyperbolic polynomials, i.e.,*

$$p(x,U,\xi) = p_1(x,U,\xi)\cdots p_Q(x,U,\xi).$$

Let d_q be the degree of $p_q(x,U,\xi)$, $q = 1,\dots,Q$, and suppose that

$$\max_q d_q \geq \max_I m_I - \min_J n_J.$$

Finally, assume that

$$\ell \geq \max_I m_I - \min_J n_J.$$

Then, there exists a $N\times N$ matrix $C(x,U,\partial)$ of differential operators whose coefficients depend on U, such that

$$C(x,U,\partial)H(x,U,\partial)U = \mathbb{I}\, p(x,U,\partial)U + \widetilde{B}_1(x,U),$$

and

$$C(x,U,\partial)B(x,U) = \widetilde{B}_2(x,U),$$

where $\mathbb{I}$ is the $N\times N$ identity matrix, $p(x,U,\partial)$ is the differential operator associated with $p(x,U,\xi)$, and $\widetilde{B}_1(x,U)$ and $\widetilde{B}_2(x,U)$ depend on at most $\ell-1$ derivatives of U, as do the coefficients of the operator $p(x,U,\xi)$. Furthermore, there is a choice of indices that makes the system

$$\mathbb{I}\, p(x,U,\partial)U = \widetilde{B}_2(x,U) - \widetilde{B}_1(x,U) \tag{23}$$

into a Leray system. In particular, if the intersections $\cap_q \Gamma_x^{*,+}(a^q)$ and $\cap_q \Gamma_x^{*,-}(a^q)$, where $\Gamma_x^{*,\pm}(a^q)$ are the half-cones associated with the hyperbolic polynomials $p_q(x,U,\xi)$, have non-empty interiors, then (23) is a Leray-Ohya system with diagonal principal part in the sense of definition A.7.

Theorem A.14 is proven in [6].

Definition A.15. Under the hypotheses of Theorem A.14, the number $\frac{Q}{Q-1}$ is called the Gevrey index of the system.

Remark 15. Suppose that (23) forms a Leray-Ohya system in the sense of definition A.7, i.e., the half-cones have non-empty interiors as stated in Theorem A.14. It can then be shown [6] that a value of s sufficient to apply Theorems A.10 and A.11 is $1 \leq s < \frac{Q}{Q-1}$.

Let us make a brief comment about the proofs of the above results. Theorem A.10 is proven as follows. First, one solves the associated linear problem. This is done by a method of majorants reminiscent of the Cauchy-Kowalevskaya theorem. One uses the fact that Gevrey functions admit a formal series expansion that provides a consistent way of constructing successive approximating solutions to the problem. The non-linear problem is then treated via a fixed point argument, upon solving successive linear problems. Theorem A.11 is obtained by a Holmgren type of argument. We remark that the assumption that $p^{J,q+1}(x,V,\xi)$, $q=0,\ldots,r_J-1$, depends on at most $m_I - m_{J,q} - r_I + q$ derivatives of v^I, $I=1,\ldots,N$, ensures that the coefficients of the associated differential operators $a^{J,q+1}(x,U,\partial)$ do not depend on too many derivatives of U, as it should be in the treatment of quasi-linear equations.

Theorem A.14 is based on the following identity:

$$c^T a = \det(a), \tag{24}$$

where a is an $N \times N$ invertible matrix and c^T the transpose of the co-factor matrix. At the level of differential operators, this identity produces the lower order terms $\widetilde{B}_1$. One then needs to match the order of the resulting differential operators and lower order terms with appropriate indices satisfying the definition of a Leray system. This is possible under the conditions on d_q and ℓ stated in the theorem.

Appendix B. **Derivation of the equations of motion.** In this section we give the derivation of (6) and (7). The derivation of (6) is standard and we include it here for the reader's convenience, thus let us start with (6). Let

$${}^{(0)}t_{\alpha\beta} = \frac{4}{3}u_\alpha u_\beta \epsilon + \frac{1}{3}g_{\alpha\beta}\epsilon, \tag{25}$$

and denote the third to ninth terms in (1) by ${}^{(1)}t_{\alpha\beta}$ to ${}^{(7)}t_{\alpha\beta}$, respectively. Explicitly,

$$\begin{aligned}
{}^{(1)}t_{\alpha\beta} &= -\eta\pi_\alpha^\mu\pi_\beta^\nu(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda),\\
{}^{(2)}t_{\alpha\beta} &= \lambda(u_\alpha u^\mu\nabla_\mu u_\beta + u_\beta u^\mu \nabla_\mu u_\alpha),\\
{}^{(3)}t_{\alpha\beta} &= \frac{1}{3}\chi\pi_{\alpha\beta}\nabla_\mu u^\mu,\\
{}^{(4)}t_{\alpha\beta} &= \chi u_\alpha u_\beta \nabla_\mu u^\mu,\\
{}^{(5)}t_{\alpha\beta} &= \frac{\lambda}{4\epsilon}(u_\alpha \pi_\beta^\mu \nabla_\mu\epsilon + u_\beta\pi_\alpha^\mu\nabla_\mu\epsilon),\\
{}^{(6)}t_{\alpha\beta} &= \frac{3\chi}{4\epsilon}u_\alpha u_\beta u^\mu\nabla_\mu\epsilon,\\
{}^{(7)}t_{\alpha\beta} &= \frac{\chi}{4\epsilon}\pi_{\alpha\beta}u^\mu\nabla_\mu\epsilon,
\end{aligned}$$

so that

$$T_{\alpha\beta} = {}^{(0)}t_{\alpha\beta} + {}^{(1)}t_{\alpha\beta} + \cdots {}^{(7)}t_{\alpha\beta}.$$

B.1. **Calculation of $\nabla_\alpha {}^{(1)}t^\alpha_\beta$.** We have

$$\begin{aligned}\nabla_\alpha {}^{(1)}t^\alpha_\beta &= -\eta\pi^{\alpha\mu}\pi^\nu_\beta(\nabla_\alpha\nabla_\mu u_\nu + \nabla_\alpha\nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\alpha\nabla_\lambda u^\lambda) \\ &\quad + \nabla_\alpha(\eta\pi^{\alpha\mu}\pi^\nu_\beta)(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda).\end{aligned} \tag{26}$$

Compute

$$\begin{aligned}\pi^\nu_\beta\nabla_\alpha\nabla_\mu u_\nu &= (g^\nu_\beta + u_\beta u^\nu)\nabla_\alpha\nabla_\mu u_\nu = \nabla_\alpha\nabla_\mu u_\beta + u_\beta u^\nu\nabla_\alpha\nabla_\mu u_\nu \\ &= \nabla_\alpha\nabla_\mu u_\beta + u_\beta\nabla_\alpha(u^\nu\nabla_\mu u_\nu) - u_\beta\nabla_\alpha u^\nu\nabla_\mu u_\nu \\ &= \nabla_\alpha\nabla_\mu u_\beta - u_\beta\nabla_\alpha u^\nu\nabla_\mu u_\nu,\end{aligned}$$

so that

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta\nabla_\alpha\nabla_\mu u_\nu &= -\eta\pi^{\alpha\mu}(\nabla_\alpha\nabla_\mu u_\beta - \nabla_\alpha u^\nu\nabla_\mu u_\nu) \\ &= -\eta(g^{\alpha\mu} + u^\alpha u^\mu)\nabla_\alpha\nabla_\mu u_\beta + \eta\pi^{\alpha\mu}\nabla_\alpha u^\nu\nabla_\mu u_\nu \\ &= -\eta g^{\alpha\mu}\nabla_\alpha\nabla_\mu u_\beta + u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + \eta\pi^{\alpha\mu}\nabla_\alpha u^\nu\nabla_\mu u_\nu.\end{aligned} \tag{27}$$

Similarly, we find

$$\begin{aligned}\pi^{\alpha\mu}\nabla_\alpha\nabla_\nu u_\mu &= (g^{\alpha\mu} + u^\alpha u^\mu)\nabla_\alpha\nabla_\nu u_\mu = g^{\alpha\mu}\nabla_\alpha\nabla_\nu u_\mu + u^\alpha u^\mu\nabla_\alpha\nabla_\nu u_\mu \\ &= \nabla_\alpha\nabla_\nu u^\alpha - u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu,\end{aligned}$$

so that

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta\nabla_\alpha\nabla_\nu u_\mu &= -\eta\pi^\nu_\beta(\nabla_\alpha\nabla_\nu u^\alpha - u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu) \\ &= -\eta g^\nu_\beta\nabla_\alpha\nabla_\nu u^\alpha - \eta u_\beta u^\nu\nabla_\alpha\nabla_\nu u^\alpha + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu.\end{aligned} \tag{28}$$

But

$$\nabla_\alpha\nabla_\nu u^\alpha = \nabla_\nu\nabla_\alpha u^\alpha + R_{\nu\alpha}u^\alpha,$$

so that (28) becomes

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta\nabla_\alpha\nabla_\nu u_\mu &= -\eta g^\nu_\beta(\nabla_\nu\nabla_\alpha u^\alpha + R_{\nu\alpha}u^\alpha) - \eta u_\beta u^\nu(\nabla_\nu\nabla_\alpha u^\alpha + R_{\nu\alpha}u^\alpha) \\ &\quad + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu \\ &= -\eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha - \eta g^\nu_\beta R_{\nu\alpha}u^\alpha - \eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha \\ &\quad - \eta u_\beta u^\nu R_{\nu\alpha}u^\alpha + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu.\end{aligned} \tag{29}$$

Next compute

$$\begin{aligned}-\eta\pi^{\alpha\mu}\pi^\nu_\beta(-\frac{2}{3}g_{\mu\nu}\nabla_\alpha\nabla_\lambda u^\lambda) &= \frac{2}{3}\eta\pi^{\alpha\mu}\pi_{\beta\mu}\nabla_\alpha\nabla_\lambda u^\lambda \\ &= \frac{2}{3}\eta\pi^\alpha_\beta\nabla_\alpha\nabla_\lambda u^\lambda = \frac{2}{3}\eta(g^\alpha_\beta + u^\alpha u_\beta)\nabla_\alpha\nabla_\lambda u^\lambda \\ &= \frac{2}{3}\eta g^\alpha_\beta\nabla_\alpha\nabla_\lambda u^\lambda + \frac{2}{3}\eta u_\beta u^\alpha\nabla_\alpha\nabla_\lambda u^\lambda.\end{aligned} \tag{30}$$

Plugging (27), (29), and (30) into (26) we find

$$\begin{aligned}\nabla_\alpha {}^{(1)}t^\alpha_\beta &= -\eta g^{\alpha\mu}\nabla_\alpha\nabla_\mu u_\beta - \eta u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + \eta u_\beta\pi^{\alpha\mu}\nabla_\alpha u_\nu\nabla_\mu u^\nu - \eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha \\ &\quad - \eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha - \eta R_{\beta\alpha}u^\alpha - \eta u_\beta R_{\nu\alpha}u^\nu u^\alpha + \eta\pi^\nu_\beta u^\alpha\nabla_\alpha u^\mu\nabla_\nu u_\mu \\ &\quad + \frac{2}{3}\eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha + \frac{2}{3}\eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha \\ &\quad + \nabla_\alpha(\eta\pi^{\alpha\mu}\pi^\nu_\beta)(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda).\end{aligned}$$

We now group the first two terms, the fourth term with the ninth term, and the fifth term with the tenth term, to find

$$\begin{aligned}\nabla_\alpha {}^{(1)}t^\alpha_\beta = -\eta(g^{\alpha\mu} + u^\alpha u^\beta)\nabla_\alpha\nabla_\mu u_\beta - \frac{1}{3}\eta g^\nu_\beta \nabla_\nu\nabla_\alpha u^\alpha \\ - \frac{1}{3}\eta u_\beta u^\nu \nabla_\nu \nabla_\alpha u^\alpha + {}^{(1)}B_\beta,\end{aligned} \tag{31}$$

where

$$\begin{aligned}{}^{(1)}B_\beta = \eta u_\beta \pi^{\alpha\mu}\nabla_\alpha u_\nu \nabla_\mu u^\nu - \eta R_{\beta\alpha}u^\alpha - \eta u_\beta R_{\nu\alpha}u^\nu u^\alpha + \eta\pi^\nu_\beta u^\alpha \nabla_\alpha u^\mu \nabla_\nu u_\mu \\ + \nabla_\alpha(\eta\pi^{\alpha\mu}\pi^\nu_\beta)(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}g_{\mu\nu}\nabla_\lambda u^\lambda).\end{aligned} \tag{32}$$

B.2. **Calculation of $\nabla_\alpha {}^{(2)}t^\alpha_\beta$.** Compute

$$\begin{aligned}\nabla_\alpha {}^{(2)}t^\alpha_\beta &= \nabla_\alpha\big[\lambda(u^\alpha u^\mu \nabla_\mu u_\beta + u_\beta u^\mu \nabla_\mu u^\alpha)\big] \\ &= \lambda(u^\alpha u^\mu \nabla_\alpha\nabla_\mu u_\beta + u_\beta u^\mu \nabla_\alpha \nabla_\mu u^\alpha) + \nabla_\alpha(\lambda u^\alpha u^\mu)\nabla_\mu u_\beta \\ &+ \nabla_\alpha(\lambda u_\beta u^\mu)\nabla_\mu u^\alpha.\end{aligned}$$

Using $\nabla_\alpha\nabla_\mu u^\alpha = \nabla_\mu\nabla_\alpha u^\alpha + R_{\mu\alpha}u^\alpha$ we find

$$\nabla_\alpha {}^{(2)}t^\alpha_\beta = \lambda u^\alpha u^\mu \nabla_\alpha\nabla_\mu u_\beta + \lambda u_\beta u^\mu \nabla_\mu\nabla_\alpha u^\alpha + {}^{(2)}B_\beta, \tag{33}$$

where

$${}^{(2)}B_\beta = \lambda u_\beta R_{\mu\alpha}u^\mu u^\alpha + \nabla_\alpha(\lambda u^\alpha u^\mu)\nabla_\mu u_\beta + \nabla_\alpha(\lambda u_\beta u^\mu)\nabla_\mu u^\alpha. \tag{34}$$

B.3. **Calculation of $\nabla_\alpha {}^{(3)}t^\alpha_\beta$.** Compute

$$\nabla_\alpha {}^{(3)}t^\alpha_\beta = \nabla_\alpha\big(\frac{1}{3}\pi^\alpha_\beta \nabla_\mu u^\mu\big) = \frac{1}{3}\chi\pi^\alpha_\beta\nabla_\alpha\nabla_\mu u^\mu + \frac{1}{3}\nabla_\alpha(\chi\pi^\alpha_\beta)\nabla_\mu u^\mu,$$

so that

$$\nabla_\alpha {}^{(3)}t^\alpha_\beta = \chi\frac{1}{3}g^\mu_\beta\nabla_\mu\nabla_\alpha u^\alpha + \frac{1}{3}\chi u_\beta u^\mu \nabla_\mu\nabla_\alpha u^\alpha + {}^{(3)}B_\beta, \tag{35}$$

where

$${}^{(3)}B_\beta = \frac{1}{3}\nabla_\alpha(\chi\pi^\alpha_\beta)\nabla_\mu u^\mu. \tag{36}$$

B.4. **Calculation of $\nabla_\alpha {}^{(4)}t^\alpha_\beta$.** Compute

$$\begin{aligned}\nabla_\alpha {}^{(4)}t^\alpha_\beta &= \nabla_\alpha\big(\chi u^\alpha u_\beta \nabla_\mu u^\mu\big) \\ &= \chi u_\beta u^\mu \nabla_\mu\nabla_\alpha u^\alpha + {}^{(4)}B_\beta,\end{aligned} \tag{37}$$

where

$${}^{(4)}B_\beta = \nabla_\alpha(\chi u^\alpha u_\beta)\nabla_\mu u^\mu. \tag{38}$$

B.5. **Calculation of $\nabla_\alpha {}^{(5)}t^\alpha_\beta$.** Compute

$$\begin{aligned}\nabla_\alpha {}^{(5)}t^\alpha_\beta &= \nabla_\alpha\Big[\frac{\lambda}{4\epsilon}(u^\alpha \pi^\mu_\beta \nabla_\mu \epsilon + u_\beta \pi^{\alpha\mu}\nabla_\mu\epsilon\Big] \\ &= \frac{\lambda}{4\epsilon} u^\alpha \pi^\mu_\beta \nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta\pi^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\big[\frac{\lambda}{4\epsilon}(u^\alpha\pi^\mu_\beta + u_\beta\pi^{\alpha\mu})\big]\nabla_\mu\epsilon \\ &= \frac{\lambda}{4\epsilon}u^\alpha g^\mu_\beta \nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu \nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta g^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon \\ &+ \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\big[\frac{\lambda}{4\epsilon}(u^\alpha\pi^\mu_\beta + u_\beta\pi^{\alpha\mu})\big]\nabla_\mu\epsilon.\end{aligned}$$

We rearrange the terms, swapping the first and third terms, so that

$$\begin{aligned}\nabla_\alpha {}^{(5)}t^\alpha_\beta = \frac{\lambda}{4\epsilon}u_\beta g^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon \\ + \frac{\lambda}{4\epsilon}u^\alpha g^\mu_\beta\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + {}^{(5)}B_\beta,\end{aligned} \tag{39}$$

where

$$ {}^{(5)}B_\beta = \nabla_\alpha\big[\frac{\lambda}{4\epsilon}(u^\alpha\pi^\mu_\beta + u_\beta\pi^{\alpha\mu})\big]\nabla_\mu\epsilon. \tag{40}$$

B.6. **Calculation of $\nabla_\alpha {}^{(6)}t^\alpha_\beta$.** Compute

$$\begin{aligned}\nabla_\alpha {}^{(6)}t^\alpha_\beta &= \nabla_\alpha\Big[\frac{3\chi}{4\epsilon}u^\alpha u_\beta u^\mu\nabla_\mu\epsilon\Big] = \frac{3\chi}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\Big[\frac{3\chi}{4\epsilon}u^\alpha u_\beta u^\mu\Big]\nabla_\mu\epsilon \\ &= \frac{3\chi}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + {}^{(6)}B_\beta,\end{aligned} \tag{41}$$

where

$$ {}^{(6)}B_\beta = \nabla_\alpha\Big[\frac{3\chi}{4\epsilon}u^\alpha u_\beta u^\mu\Big]\nabla_\mu\epsilon. \tag{42}$$

B.7. **Calculation of $\nabla_\alpha {}^{(7)}t^\alpha_\beta$.** Compute

$$\begin{aligned}\nabla_\alpha {}^{(7)}t^\alpha_\beta &= \nabla_\alpha\Big[\frac{\chi}{4\epsilon}\pi^\alpha_\beta u^\mu\nabla_\mu\epsilon\Big] = \frac{\chi}{4\epsilon}(g^\alpha_\beta + u^\alpha u_\beta)u^\mu\nabla_\alpha\nabla_\mu\epsilon + \nabla_\alpha\Big[\frac{\chi}{4\epsilon}\pi^\alpha_\beta u^\mu\Big]\nabla_\mu\epsilon \\ &= \frac{\chi}{4\epsilon}g^\alpha_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + \frac{\chi}{4\epsilon}u^\alpha u_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + {}^{(7)}B_\beta,\end{aligned} \tag{43}$$

where

$$ {}^{(7)}B_\beta = \nabla_\alpha\Big[\frac{\chi}{4\epsilon}\pi^\alpha_\beta u^\mu\Big]\nabla_\mu\epsilon. \tag{44}$$

B.8. **Calculation of $\nabla_\alpha T^\alpha_\beta$.** Using (1), (25), (31), (33), (35), (37), (39), (41), and (43), we find

$$\begin{aligned}\nabla_\alpha T^\alpha_\beta &= -\eta(g^{\alpha\mu} + u^\alpha u^\beta)\nabla_\alpha\nabla_\mu u_\beta - \frac{1}{3}\eta g^\nu_\beta\nabla_\nu\nabla_\alpha u^\alpha - \frac{1}{3}\eta u_\beta u^\nu\nabla_\nu\nabla_\alpha u^\alpha \\ &+ \lambda u^\alpha u^\mu\nabla_\alpha\nabla_\mu u_\beta + \lambda u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha \\ &+ \chi\frac{1}{3}g^\mu_\beta\nabla_\mu\nabla_\alpha u^\alpha + \frac{1}{3}\chi u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha \\ &+ \chi u_\beta u^\mu\nabla_\mu\nabla_\alpha u^\alpha + \frac{\lambda}{4\epsilon}u_\beta g^{\alpha\mu}\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon \\ &+ \frac{\lambda}{4\epsilon}u^\alpha g^\mu_\beta\nabla_\alpha\nabla_\mu\epsilon + \frac{\lambda}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon + \frac{3\chi}{4\epsilon}u_\beta u^\alpha u^\mu\nabla_\alpha\nabla_\mu\epsilon \\ &+ \frac{\chi}{4\epsilon}g^\alpha_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + \frac{\chi}{4\epsilon}u^\alpha u_\beta u^\mu\nabla_\alpha\nabla_\mu\epsilon + B_\beta,\end{aligned} \tag{45}$$

where the first three terms on the RHS of (45) come from (31), the fourth and fifth from (33), the sixth and seventh from (35), the eighth from (37), the ninth to twelfth from (39), the thirteenth from (41), the fourteenth and fifteenth from (43), and B_β is given by

$$B_\beta = {}^{(1)}B_\beta + {}^{(2)}B_\beta + {}^{(3)}B_\beta + {}^{(4)}B_\beta + {}^{(5)}B_\beta + {}^{(6)}B_\beta + {}^{(7)}B_\beta + \nabla_\alpha {}^{(0)}t^\alpha_\beta, \tag{46}$$

with ${}^{(1)}B_\beta, \dots, {}^{(7)}B_\beta$ given by (32), (34), (36), (38), (40), (42), and (44), respectively, and ${}^{(0)}t^\alpha_\beta$ is given by (25). We now group the terms on the RHS of (45) as follows: the first and the fourth terms, the fifth and the eighth terms, the second and the sixth terms, the third and the seventh terms, the ninth, tenth, and thirteenth terms, the eleventh and fourteenth terms, and the twelfth and fifteenth terms. We obtain:

$$\begin{aligned}\nabla_\alpha T^\alpha_\beta &= (-\eta g^{\alpha\mu} + (\lambda-\eta)u^\alpha u^\mu)\nabla_\alpha\nabla_\mu u_\beta + (\lambda+\chi)u_\beta u^\mu \nabla_\mu \nabla_\alpha u^\alpha \\ &\quad + \frac{1}{3}(-\eta+\chi)g^\mu_\beta \nabla_\mu\nabla_\alpha u^\alpha + \frac{1}{3}(-\eta+\chi)u_\beta u^\mu \nabla_\mu\nabla_\alpha u^\alpha \\ &\quad + \frac{1}{4\epsilon}u_\beta(\lambda g^{\alpha\mu} + (\lambda+3\chi)u^\alpha u^\mu)\nabla_\alpha\nabla_\mu\epsilon + \frac{1}{4\epsilon}(\lambda+\chi)g^\mu_\beta u^\alpha \nabla_\alpha\nabla_\mu\epsilon \\ &\quad + \frac{1}{4\epsilon}(\lambda+\chi)u_\beta u^\alpha u^\mu \nabla_\alpha\nabla_\mu \epsilon + B_\beta,\end{aligned} \tag{47}$$

where the first term on the RHS of (47) comes from the first and the fourth terms on the RHS of (45), the second term on the RHS of (47) comes from the fifth and the eighth terms on the RHS of (45), the third term on the RHS of (47) comes from second and the sixth terms on the RHS of (45), the fourth term on the RHS of (47) comes from the third and the seventh terms on the RHS of (45), the fifth term on the RHS of (47) comes from the ninth, tenth, and thirteenth terms on the RHS of (45), the sixth term on the RHS of (47) comes from the eleventh and fourteenth terms on the RHS of (45), the seventh term on the RHS of (47) comes from and the twelfth and fifteenth terms on the RHS of (45), and we used that $\nabla_\alpha\nabla_\mu\epsilon = \nabla_\mu\nabla_\alpha\epsilon$.

Expanding the covariant derivatives and using Notation 1 gives (7).

B.9. **Derivation of (6).** Let us first write (3) in trace reversed form. Tracing (3) gives

$$R = 4\Lambda - T,$$

where $T = g^{\alpha\beta}T_{\alpha\beta}$. (For (1) we in fact have $T = 0$, as it must be for a conformal tensor. But at this point we are writing Einstein's equations for a general tensor.) Plugging this for R in (3) gives

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta} + \Lambda g_{\alpha\beta}.$$

We now proceed to compute $R_{\alpha\beta}$ in local coordinates. In coordinates, we have

$$R_{\alpha\beta} = \partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} + \Gamma^\lambda_{\alpha\beta}\Gamma^\mu_{\lambda\mu} - \Gamma^\lambda_{\alpha\mu}\Gamma^\mu_{\beta\lambda}.$$

Using the definition of the Christoffel symbols $\Gamma^\lambda_{\alpha\beta}$ gives

$$\begin{aligned}R_{\alpha\beta} &= -\frac{1}{2}g^{\mu\nu}\partial^2_{\mu\nu}g_{\alpha\beta} + \frac{1}{2}(g_{\alpha\lambda}\partial_\beta\Gamma^\lambda + g_{\beta\lambda}\partial_\alpha\Gamma^\lambda) \\ &\quad - \frac{1}{2}(\partial_\beta g^{\lambda\mu}\partial_\lambda g_{\alpha\mu} + \partial_\alpha g^{\lambda\mu}\partial_\lambda g_{\beta\mu}) - \Gamma^\mu_{\alpha\lambda}\Gamma^\lambda_{\beta\mu},\end{aligned}$$

where Γ^λ is given by

$$\Gamma^\lambda = g^{\mu\nu}\Gamma^\lambda_{\mu\nu}.$$

Using that in wave coordinates $\Gamma^\lambda = 0$ and recalling (1), the above gives (6).

Appendix C. **The characteristic determinant.** In this section we derive (11). Because of the structure of the system in (9) it suffices to compute the characteristic determinant of $m(U,\partial)$ in (10). Using Mathematica and (5) we find (we are not assuming $a_1 = 4$ at this point)

$$\det m(\widehat{U},\xi) = \widetilde{p}_1(\widehat{U},\xi)\widetilde{p}_2(\widehat{U},\xi)\widetilde{p}_3(\widehat{U},\xi),$$

where

$$\widetilde{p}_1(\widehat{U},\xi) = \frac{1}{12\widehat{\epsilon}}\eta^4(\widehat{u}^\mu\xi_\mu)^2,$$

$$\begin{aligned}\widetilde{p}_2(\widehat{U},\xi) = \big[&(a_2-1)(\widehat{u}^0)^2\xi_0^2 + (a_2-1)(\widehat{u}^1)^2\xi_1^2 - (\widehat{u}^2)^2\xi_2^2 + a_2(\widehat{u}^2)^2\xi_2^2 - 2\widehat{u}^2\widehat{u}^3\xi_2\xi_3 \\ &+ 2a_2\widehat{u}^2\widehat{u}^3\xi_2\xi_3 - (\widehat{u}^3)^2\xi_3^2 + a_2(\widehat{u}^3)^2\xi_3^2 + \xi_0(2(-1+a_2)\xi_1\widehat{u}^0\widehat{u}^1 \\ &+ 2(a_2-1)\xi_2\widehat{u}^0\widehat{u}^2 - 2\xi_3\widehat{u}^0\widehat{u}^3 + 2a_2\widehat{u}^0\widehat{u}^3\xi_3 - \xi^0) \\ &+\xi_1(2(-1+a_2)\widehat{u}^1\widehat{u}^2\xi_2 + 2(a_2-1)\widehat{u}^1\widehat{u}^3\xi_3 - \xi^1) - \xi_2\xi^2 - \xi_3\xi^3\big]^2\end{aligned}$$

and

$$\begin{aligned}\widetilde{p}_3(\widehat{U},\xi) = &-6(-2a_1\widehat{u}_0\widehat{u}^0 - a_2\widehat{u}_0\widehat{u}^0 + 2a_1a_2\widehat{u}_0\widehat{u}^0 + a_2^2\widehat{u}_0\widehat{u}^0 - 2a_1\widehat{u}_1\widehat{u}^1 - a_2\widehat{u}_1\widehat{u}^1 \\ &+ 2a_1a_2\widehat{u}_1\widehat{u}^1 + a_2^2\widehat{u}_1\widehat{u}^1 - 2a_1\widehat{u}_2\widehat{u}^2 - a_2\widehat{u}_2\widehat{u}^2 + 2a_1a_2\widehat{u}_2\widehat{u}^2 + a_2^2\widehat{u}_2\widehat{u}^2 \\ &- 2a_1\widehat{u}_3\widehat{u}^3 - a_2\widehat{u}_3\widehat{u}^3 + 2a_1a_2\widehat{u}_3\widehat{u}^3 + a_2^2\widehat{u}_3\widehat{u}^3)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 \\ &+ \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^4 \\ &- 2(-a_2\widehat{u}_0 + 4a_1a_2\widehat{u}_0 + 3a_2^2\widehat{u}_0)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^0 \\ &- 2(-a_2\widehat{u}_1 + 4a_1a_2\widehat{u}_1 + 3a_2^2\widehat{u}_1)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^1 \\ &- 2(-a_2\widehat{u}_2 + 4a_1a_2\widehat{u}_2 + 3a_2^2\widehat{u}_2)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^2 \\ &- 2(-a_2\widehat{u}_3 + 4a_1a_2\widehat{u}_3 + 3a_2^2\widehat{u}_3)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^3 \\ &+ 5(3a_1\widehat{u}_0\widehat{u}^0 + 2a_2\widehat{u}_0\widehat{u}^0 + a_1a_2\widehat{u}_0\widehat{u}^0 + 3a_1\widehat{u}_1\widehat{u}^1 + 2a_2\widehat{u}_1\widehat{u}^1 \\ &+ a_1a_2\widehat{u}_1\widehat{u}^1 + 3a_1\widehat{u}_2\widehat{u}^2 + 2a_2\widehat{u}_2\widehat{u}^2 + a_1a_2\widehat{u}_2\widehat{u}^2 \\ &+ 3a_1\widehat{u}_3\widehat{u}^3 + 2a_2\widehat{u}_3\widehat{u}^3 + a_1a_2\widehat{u}_3\widehat{u}^3)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 \\ &+ \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^2(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3) \\ &+ (3a_1\widehat{u}_0 + 2a_2\widehat{u}_0 + a_1a_2\widehat{u}_0)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^0(\xi_0\xi^0 \\ &+ \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3) + (3a_1\widehat{u}_1 + 2a_2\widehat{u}_1 + a_1a_2\widehat{u}_1)(\xi_0\widehat{u}^0 \\ &+ \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^1(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3) \\ &+ (3a_1\widehat{u}_2 + 2a_2\widehat{u}_2 + a_1a_2\widehat{u}_2)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 \\ &+ \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^2(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3) \\ &+ (3a_1\widehat{u}_3 + 2a_2\widehat{u}_3 + a_1a_2\widehat{u}_3)(\xi_0\widehat{u}^0 \\ &+ \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^3(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3) \\ &+ (4a_2\widehat{u}_0\widehat{u}^0 - a_1a_2\widehat{u}_0\widehat{u}^0 + 4a_2\widehat{u}_1\widehat{u}^1 - a_1a_2\widehat{u}_1\widehat{u}^1 + 4a_2\widehat{u}_2\widehat{u}^2\end{aligned} \tag{48}$$

$$- a_1 a_2 \widehat{u}_2 \widehat{u}^2 + 4a_2 \widehat{u}_3 \widehat{u}^3 - a_1 a_2 \widehat{u}_3 \widehat{u}^3)(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3)^2.$$

It is not difficult to see, after some manipulations, that $\widetilde{p}_2(\widehat{U},\xi)$ is precisely $p_2(\widehat{U},\xi)$, i.e., (13). Let us now analyze $\widetilde{p}_3(\widehat{U},\xi)$. The first term in $\widetilde{p}_3(\widehat{U},\xi)$, that spans lines 2 to 5 in (C), is proportional to $(\widehat{u}^\mu\xi_\mu)^4$. The terms from lines 6 to 9 combined are also proportional to $(\widehat{u}^\mu\xi_\mu)^4$. Indeed, the term on the sixth line can be written as

$$\begin{aligned} &-2(-a_2\widehat{u}_0 + 4a_1a_2\widehat{u}_0 + 3a_2^2\widehat{u}_0)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\xi^0 \\ &\quad = -2(-a_2 + 4a_1a_2 + 3a_2^2)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)^3\widehat{u}_0\xi^0, \end{aligned}$$

and similarly we can group $\widehat{u}_i$ with ξ^i in the terms on the seventh to ninth line. Factoring then the common factor in lines 6 to 9 gives a term cubic in $\widehat{u}^\mu\xi_\mu$ times the term

$$\widehat{u}_0\xi^0 + \widehat{u}_1\xi^1 + \widehat{u}_2\xi^2 + \widehat{u}_3\xi^3.$$

But this last term equals $\widehat{u}^\mu\xi_\mu$, which can then be grouped with the cubic term in $\widehat{u}^\mu\xi_\mu$ producing a term proportional to $(\widehat{u}^\mu\xi_\mu)^4$, as claimed.

The next term in $\widetilde{p}_3(\widehat{U},\xi)$, spanning lines 10 to 13 in (C) is proportional to $(\widehat{u}^\mu\xi_\mu)^2$.

We claim that the terms spanning lines 14 to 20, when combined, produce a term proportional to $(\widehat{u}^\mu\xi_\mu)^2$. To see this, note that as written the terms in lines 14 to 20 all have a factor $\widehat{u}_0\xi^0 + \widehat{u}_1\xi^1 + \widehat{u}_2\xi^2 + \widehat{u}_3\xi^3$, which equals $\widehat{u}^\mu\xi_\mu$. The term that begins on line 14 of (C) can be written as

$$\begin{aligned} &(3a_1\widehat{u}_0 + 2a_2\widehat{u}_0 + a_1a_2\widehat{u}_0)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)\xi^0(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3) \\ &\quad = (3a_1 + 2a_2 + a_1a_2)(\xi_0\widehat{u}^0 + \xi_1\widehat{u}^1 + \xi_2\widehat{u}^2 + \xi_3\widehat{u}^3)(\xi_0\xi^0 + \xi_1\xi^1 + \xi_2\xi^2 + \xi_3\xi^3)\widehat{u}_0\xi^0, \end{aligned}$$

and similarly we can combine $\widehat{u}_i$ with ξ^i in the other terms in lines 15 to 20. Factoring then the common factor to all terms in lines 14 to 20 produces a term linear in $\widehat{u}^\mu\xi_\mu$ times $\widehat{u}_0\xi^0 + \widehat{u}_1\xi^1 + \widehat{u}_2\xi^2 + \widehat{u}_3\xi^3 \equiv u^\mu\xi_\mu$, hence a term quadratic in $\widehat{u}^\mu\xi_\mu$, as claimed.

Therefore, we see that all terms in $\widetilde{p}_3(\widehat{U},\xi)$ contain a factor of $(\widehat{u}^\mu\xi_\mu)^2$, except for the last term which spans lines 21 and 22. This last term, however, vanishes identically if $a_1 = 4$. In this case we can factor $(\widehat{u}^\mu\xi_\mu)^2$ from $\widetilde{p}_3(\widehat{U},\xi)$. We combine the factored $(\widehat{u}^\mu\xi_\mu)^2$ with $\widetilde{p}_1(\widehat{U},\xi)$, producing $p_1(\widehat{U},\xi)$, i.e., (12), and the remainder from $\widetilde{p}_3(\widehat{U},\xi)$ produces $p_3(\widehat{U},\xi)$, i.e., (14).

Remark 16. Without setting $a_1 = 4$, the above factorization procedure can be used to show that $\widetilde{p}_3(\widehat{U},\xi)$ factors as

$$A(\widehat{u}^\mu\xi_\mu)^4 + B(\widehat{u}^\mu\xi_\mu)^2\xi^\lambda\xi_\lambda + C(\xi^\lambda\xi_\lambda)^2,$$

where A, B, and C depend on a_1 and a_2. We would like to factor this quartic polynomial as a product of (real) degree two polynomials, since then we can analyze its roots explicitly. The above choice $a_1 = 4$ does exactly this. But other choices of a_1 and a_2 also lead to the desired factorization, as showed in [3].

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Nonlinear Constraints on Relativistic Fluids Far from Equilibrium

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New constraints are found that must necessarily hold for Israel-Stewart-like theories of fluid dynamics to be causal far away from equilibrium. Conditions that are sufficient to ensure causality, local existence, and uniqueness of solutions in these theories are also presented. Our results hold in the full nonlinear regime, taking into account bulk and shear viscosities (at zero chemical potential), without any simplifying symmetry or near-equilibrium assumptions. Our findings provide fundamental constraints on the magnitude of viscous corrections in fluid dynamics far from equilibrium.

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Introduction.—Relativistic fluid dynamics is essential to the state-of-the-art modeling of the quark-gluon plasma (QGP) formed in ultrarelativistic heavy-ion collisions (see [1–3]). However, despite its wide use and significant success, it remains unclear why such a fluid dynamical description is applicable given that local deviations from equilibrium in nucleus-nucleus collisions can be very large, especially at early times [4–6]. In fact, typical fluidlike signatures involving anisotropic flow [7] persist even in small systems formed in proton-nucleus and proton-proton collisions at sufficiently high multiplicity [8–14]. Such findings have motivated a series of new investigations on the foundations of relativistic viscous fluid dynamics [15–18] and their subsequent extension toward the far-from-equilibrium regime relevant for heavy-ion collisions [19–45].

The viscous fluid description of the QGP is currently based on ideas from Israel and Stewart (IS) [46,47] (see, also, Mueller [48]), who proposed a way to fix the long-standing acausality [49] and instability [50] problems of the relativistic generalization of Navier-Stokes (NS) equations derived by Eckart [51] and Landau and Lifshitz [52]. The general mechanism introduced by IS to try to avoid such issues assumes that dissipative currents such as the shear stress tensor, $\pi_{\mu\nu}$, and the bulk scalar, Π, are new degrees of freedom [53,54] which obey nonlinear relaxation equations describing how such quantities relax to their relativistic NS limits within relaxation time scales τ_π and τ_Π. The same principle is also at play in modern formulations of fluid dynamics put forward by Ref. [55] and Ref. [56], which are currently employed in numerical simulations (see, for instance, [57]).

It is well known that the IS-like theories are linearly stable around equilibrium [58–61]. But physically sensible relativistic theories of fluid dynamics must also be causal, i.e., the equations of motion must be hyperbolic, and the propagation of information must be, at most, the speed of light [62]. Also, the Cauchy problem must be locally well posed [63], i.e., given initial conditions, one must show that the equations admit a unique solution. A common misconception in the field is that IS-like theories have already been proven to be causal a long time ago in Refs. [46,58,59,64–66]. This is not the case. Those early works only considered linearized disturbances around equilibrium, where the background fields $\pi_{\mu\nu}$ and Π vanish and the corresponding linear disturbances are small. Such a linearized analysis says nothing about the nonlinear regime, even for small $\pi_{\mu\nu}$ and Π. The far-from-equilibrium regime, in particular, is necessarily nonlinear as $\pi_{\mu\nu}$ and Π can be as large as the local equilibrium pressure P.

Hence, it is not known if IS theories are, indeed, sensible in the regime probed by high energy hadronic collisions. Understanding the far-from-equilibrium properties of such theories is also crucial to reliably assess the role of viscous effects in early universe cosmology [67]. Here, we make essential steps toward solving this critical problem by finding conditions (in the form of simple algebraic inequalities that can be checked at every step of the evolution) that must necessarily hold for IS-like theories to be causal in the nonlinear regime. We also present conditions that are sufficient to ensure causality, local existence, and uniqueness of solutions of IS-like theories. Our results are the first in the literature that hold in the full nonlinear regime, with bulk and shear viscosities (at zero chemical potential), in three spatial dimensions, without recurring to any symmetry or near-equilibrium assumptions.

The equations of motion.—Using the Landau frame definition of the hydrodynamic variables [52], the energy-momentum tensor of the fluid can be written as [We use units $c=\hbar=k_B=1$. The spacetime metric signature is $(-+++)$. Greek indices run from 0 to 3, Latin indices from 1 to 3.] $T^{\mu\nu}=\varepsilon u^\mu u^\nu + (P+\Pi)\Delta^{\mu\nu}+\pi^{\mu\nu}$, where u^μ is the fluid's four-velocity (with $u_\mu u^\mu=-1$), ε is the energy density, $P=P(\varepsilon)$ is the equilibrium pressure defined by an equation of state, $\Delta_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu$ is the projector orthogonal to the flow, $g_{\mu\nu}$ is the spacetime metric, $\pi_{\mu\nu}=\pi_{\nu\mu}$, $\pi^{\mu\nu}u_\mu=0$, and $\Delta_{\mu\nu}\pi^{\mu\nu}=0$. We focus on high energy collisions, and thus, we only investigate, here, the case of zero chemical potentials. Conservation of energy and momentum implies that $\nabla_\mu T^{\mu\nu}=0$, which can be written as ($c_s^2=dP/d\varepsilon$ is the equilibrium speed of sound squared)

$$u^\alpha\nabla_\alpha\varepsilon+(\varepsilon+P+\Pi)\nabla_\alpha u^\alpha+\pi^\alpha_\mu\nabla_\alpha u^\mu=0,$$
$$(\varepsilon+P+\Pi)u^\beta\nabla_\beta u_\alpha+c_s^2\Delta^\beta_\alpha\nabla_\beta\varepsilon+\Delta^\beta_\alpha\nabla_\beta\Pi+\Delta^\beta_\alpha\nabla_\mu\pi^\mu_\beta=0. \tag{1}$$

Here, we consider the case where the dissipative currents $\{\pi^{\mu\nu},\Pi\}$ satisfy the following equations (Note that our metric signature is different than in [56].), derived using the Denicol-Niemi-Molnar-Rischke formalism [56], and commonly used in heavy-ion collision applications

$$\tau_\Pi u^\mu\nabla_\mu\Pi+\Pi=-\zeta\nabla_\mu u^\mu-\delta_{\Pi\Pi}\Pi\nabla_\mu u^\mu-\lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \tag{2a}$$

$$\tau_\pi\Delta^{\mu\nu}_{\alpha\beta}u^\lambda\nabla_\lambda\pi^{\alpha\beta}+\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}-\delta_{\pi\pi}\pi^{\mu\nu}\nabla_\alpha u^\alpha-\tau_{\pi\pi}\pi^{\langle\mu}_\alpha\sigma^{\nu\rangle\alpha}-\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}, \tag{2b}$$

where $\sigma^{\mu\nu}=\Delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha u^\beta$ is the shear tensor, $\Delta^{\mu\nu}_{\alpha\beta}=(\Delta^\mu_\alpha\Delta^\nu_\beta+\Delta^\mu_\beta\Delta^\nu_\alpha)/2-\frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$, $A^{\langle\mu}_\lambda B^{\nu\rangle\lambda}=\Delta^{\mu\nu}_{\alpha\beta}A^{\alpha\lambda}B^\beta_\lambda$, and η, ζ are the shear and bulk viscosities, respectively. All the transport coefficients, $\{\eta,\zeta,\tau_\Pi,\tau_\pi,\delta_{\Pi\Pi},\lambda_{\Pi\pi},\delta_{\pi\pi},\tau_{\pi\pi},\lambda_{\pi\Pi}\}$, can depend on the ten dynamical variables $\{\varepsilon,u_\mu,\pi_{\mu\nu},\Pi\}$ (so, in principle, they may even depend on the dissipative tensors) but not on their derivatives. Explicit expressions for transport coefficients in models can be found, for instance, in [56,68,69].

We note that $\{\eta,\zeta,\tau_\pi,\tau_\Pi\}$ are the only coefficients that remain after linearization around equilibrium where $\pi^{\mu\nu}=0$ and $\Pi=0$. This shows why linearized analyses [58,59] necessarily miss the effects from the other coefficients, $\{\delta_{\Pi\Pi},\lambda_{\Pi\pi},\delta_{\pi\pi},\tau_{\pi\pi},\lambda_{\pi\Pi}\}$, which contribute to the nonlinear evolution. However, other nonlinear terms such as $\pi_{\mu\nu}\pi^{\mu\nu}$, Π^2, $\pi^{\mu\nu}\Pi$, $\pi^{\langle\mu}_\alpha\pi^{\nu\rangle\alpha}$, which appear in [56], could have been trivially added to the equations as they do not contribute to a causality analysis since they do not involve derivatives of the fields. Nevertheless, there are still some other nonlinear terms that can be considered such as $\pi^{\langle\mu}_\alpha\Omega^{\nu\rangle\alpha}$, where $\Omega_{\mu\nu}=(\Delta^\alpha_\mu\nabla_\alpha u_\nu-\Delta^\alpha_\nu\nabla_\alpha u_\mu)/2$ is the vorticity and, also, $\Omega^{\langle\mu}_\alpha\Omega^{\nu\rangle\alpha}$ [3]. The former will be investigated in a separate publication. The latter contributes with derivatives of the fields to the principal part of the system of equations and, thus, a different analysis than presented here would be required.

Causality.—Causality is the concept in relativity theory asserting that no information propagates faster than the speed of light and no closed timelike curves exist (so the future cannot influence the past). See Refs. [70–74] for a mathematically precise definition of causality. Causality can be investigated by determining the characteristic manifolds associated with a system of partial differential equations [75,76]. Let us write equations (1)–(2) as $A^\alpha\nabla_\alpha\Psi=F(\Psi)$, where we defined the vector $\Psi=(\varepsilon,u^\nu,\Pi,\pi^{0\nu},\pi^{1\nu},\pi^{2\nu},\pi^{3\nu})$, the 22×22 matrix

$$A^\alpha=\begin{bmatrix} u^\alpha & \rho\delta^\alpha_\nu+\pi^\alpha_\nu & 0_{1\times1} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4}\\ c_s^2\Delta^{\mu\alpha} & \rho u^\alpha\delta^\mu_\nu-\pi^\alpha_\nu u^\mu & \Delta^{\mu\alpha} & \delta^\alpha_0 I_4 & \delta^\alpha_1 I_4 & \delta^\alpha_2 I_4 & \delta^\alpha_3 I_4\\ 0_{4\times1} & E^\alpha_\nu & \tau_\Pi u^\alpha & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4}\\ 0_{4\times1} & C^{0\delta\alpha}_\nu & 0_{4\times1} & \tau_\pi u^\alpha I_4 & 0_{4\times4} & 0_{4\times4} & 0_{4\times4}\\ 0_{4\times1} & C^{1\delta\alpha}_\nu & 0_{4\times1} & 0_{4\times4} & \tau_\pi u^\alpha I_4 & 0_{4\times4} & 0_{4\times4}\\ 0_{4\times1} & C^{2\delta\alpha}_\nu & 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & \tau_\pi u^\alpha I_4 & 0_{4\times4}\\ 0_{4\times1} & C^{3\delta\alpha}_\nu & 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & \tau_\pi u^\alpha I_4 \end{bmatrix}, \tag{3}$$

and $F(\Psi)$ is a vector that does not contain derivatives of the variables. Above, we also defined $\rho = \varepsilon + P + \Pi$, $E^{\alpha}_{\nu} = (\zeta + \delta_{\Pi\Pi}\Pi)\delta^{\alpha}_{\nu} + \lambda_{\Pi\pi}\pi^{\alpha}_{\nu}$, $B^{\mu\lambda\alpha}_{\nu} = \frac{1}{2}(\Delta^{\mu\alpha}\delta^{\lambda}_{\nu} + \Delta^{\lambda\alpha}\delta^{\mu}_{\nu} - \frac{2}{3}\Delta^{\mu\lambda}\delta^{\alpha}_{\nu})$, and

$$C^{\sigma\delta\alpha}_{\nu} = \left[(2\eta + \lambda_{\pi\Pi}\Pi)\delta^{\sigma}_{\mu}\delta^{\delta}_{\lambda} + \frac{\tau_{\pi\pi}}{2}\pi^{\sigma}_{\lambda}\delta^{\delta}_{\mu} + \frac{\tau_{\pi\pi}}{2}\pi^{\delta}_{\lambda}\delta^{\sigma}_{\mu}\right]B^{\mu\lambda\alpha}_{\nu} - \frac{\tau_{\pi\pi}}{3}\Delta^{\sigma\delta}\pi^{\alpha}_{\nu} + \delta_{\pi\pi}\pi^{\sigma\delta}\delta^{\alpha}_{\nu} - \tau_{\pi}(\pi^{\sigma}_{\nu}u^{\delta} + \pi^{\delta}_{\nu}u^{\sigma})u^{\alpha}.$$

The characteristic surfaces $\{\Phi(x) = 0\}$ are determined by the principal part of the equations by solving the characteristic equation $\det(A^{\alpha}\xi_{\alpha}) = 0$, with $\xi_{\alpha} = \nabla_{\alpha}\Phi$ [77]. The system is causal if, for any ξ_i, it holds that (C1) the roots $\xi_0 = \xi_0(\xi_i)$ of the characteristic equation are real and (C2) $\xi_{\alpha} = (\xi_0, \xi_i)$ is spacelike or lightlike. Condition (C2) implies that the characteristic surfaces $\{\Phi(x) = 0\}$ are timelike or lightlike, indicating that no information is superluminal. For instance, for an ideal fluid (where $\Pi = 0$ and $\pi_{\mu\nu} = 0$), the characteristic velocities are determined by the speed of sound and causality implies that $c_s^2 \leq 1$ [63].

From (3), it is clear that the characteristics associated with the evolution depend on the dissipative tensors $\{\pi^{\mu\nu}, \Pi\}$. Therefore, the true causal behavior of IS theories is necessarily a far-from-equilibrium property of the fluid, and linear analyses around equilibrium cannot be used to establish causality and well posedness in IS theories. The computation of the characteristics defined by (3), which is needed for a causality analysis, is extremely involved and is presented in the Supplemental Material [78]. Below, we present the main consequences of such calculations.

Let Λ_{α}, $\alpha = 0, 1, 2, 3$, be the eigenvalues of the π^{μ}_{ν}. The eigenvalues are such that $\Lambda_0 = 0$, since u_{μ} is in the kernel of π^{μ}_{ν} $(u_{\mu}\pi^{\mu}_{\nu} = 0)$, and $\Lambda_1 + \Lambda_2 + \Lambda_3 = 0$, so that the trace is kept zero. Without loss of generality, let us take $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3$ with $\Lambda_1 \leq 0 \leq \Lambda_3$. Now, we state our assumptions, which are the following: (A1) for the transport coefficients and relaxation times, suppose that $\tau_{\Pi}, \tau_{\pi} > 0$ and $\eta, \zeta, \tau_{\pi\pi}, \delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \delta_{\pi\pi}, \lambda_{\pi\Pi}, c_s^2 \geq 0$; (A2) for the fluid variables, suppose that $\varepsilon > 0$, $P \geq 0$, and $\varepsilon + P + \Pi > 0$; finally, we also assume that (A3) $\varepsilon + P + \Pi + \Lambda_a > 0$, $a = 1, 2, 3$. Then, the following conditions are necessary for causality, i.e., if any of the inequalities below is not satisfied, then the system is not causal:

$$(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{1}{2}\tau_{\pi\pi}|\Lambda_1| \geq 0, \tag{4a}$$

$$\varepsilon + P + \Pi - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_{\pi}}\Lambda_3 \geq 0, \tag{4b}$$

$$\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_{\pi}}(\Lambda_a + \Lambda_d) \geq 0, \quad a \neq d, \tag{4c}$$

$$\varepsilon + P + \Pi + \Lambda_a - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_{\pi}}(\Lambda_d + \Lambda_a) \geq 0, \quad a \neq d, \tag{4d}$$

$$\frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_d + \frac{1}{6\tau_{\pi}}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_{\Pi}} + (\varepsilon + P + \Pi + \Lambda_d)c_s^2 \geq 0, \tag{4e}$$

$$\varepsilon + P + \Pi + \Lambda_d - \frac{1}{2\tau_{\pi}}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_{\pi}}\Lambda_d - \frac{1}{6\tau_{\pi}}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] - \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_{\Pi}} - (\varepsilon + P + \Pi + \Lambda_d)c_s^2 \geq 0, \tag{4f}$$

where (4c)–(4f) must hold for $a, d = 1, 2, 3$. The proof that (4) are necessary conditions for causality under assumptions (A1)–(A3) is given in the Supplemental Material [78]. Here, we discuss the significance of this result.

We stress that assumptions (A1) and (A2) are standard in heavy-ion collision applications [57], and (A3) is a very natural assumption since $P + \Pi + \Lambda_a$ for $a = 1, 2, 3$ may be interpreted as the pressure in each spatial axis in the local rest frame. Thus, if (A3) is violated, the theory would have a pathology in the sense that fluid elements would have negative inertia, i.e., the acceleration is opposite to the force given by the negative of the gradient of pressure. Furthermore, it is natural to make assumptions that hold close to equilibrium, and since (A2) guarantees $\varepsilon + P + \Pi > 0$, for small deviations from equilibrium, Λ_a will be small, giving $\varepsilon + P + \Pi + \Lambda_a > 0$. That said, we stress that, although (A3) is expected to hold near equilibrium, it is, itself, not a near-equilibrium assumption.

Conditions (4) could never have been found using a linearized analysis, as they depend on Π and Λ_a, both of which vanish in equilibrium. Consequently, if, in any fluid dynamic simulation in heavy-ion collisions that employs (1)–(2), the necessary conditions above are not fulfilled, causality is necessarily violated. It is important to point out that this causality violation has nothing to do with the ability of numerical schemes to produce a solution, a point we shall return to at the end of the Letter.

While the above conditions must hold for the system to be causal, they are not sufficient conditions, i.e., by themselves, conditions (A1)–(A3) and (4) do not assure the system to be causal (see the Supplemental Material

[78]). Therefore, it is important to have conditions that are sufficient for causality. In this regard, assume, again, that (A1)–(A3) hold. Then, the following conditions are sufficient to ensure that causality holds, i.e., if they are satisfied, then the system is causal:

$$(\varepsilon+P+\Pi-|\Lambda_1|)-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\geq 0, \quad (5a)$$

$$(2\eta+\lambda_{\pi\Pi}\Pi)-\tau_{\pi\pi}|\Lambda_1|>0, \quad (5b)$$

$$\tau_{\pi\pi}\leq 6\delta_{\pi\pi}, \quad (5c)$$

$$\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\geq 0, \quad (5d)$$

$$\frac{1}{3\tau_\pi}[4\eta+2\lambda_{\pi\Pi}\Pi+(3\delta_{\pi\pi}+\tau_{\pi\pi})\Lambda_3] + \frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi}+|\Lambda_1|+\Lambda_3 c_s^2 + \frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3} \leq(\varepsilon+P+\Pi)(1-c_s^2), \quad (5e)$$

$$\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi+(\tau_{\pi\pi}-6\delta_{\pi\pi})|\Lambda_1|] + \frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi}+(\varepsilon+P+\Pi-|\Lambda_1|)c_s^2\geq 0, \quad (5f)$$

$$1\geq\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\left[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right]^2}, \quad (5g)$$

$$\frac{1}{3\tau_\pi}[4\eta+2\lambda_{\pi\Pi}\Pi-(3\delta_{\pi\pi}+\tau_{\pi\pi})|\Lambda_1|] + \frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi}+(\varepsilon+P+\Pi-|\Lambda_1|)c_s^2 \geq\frac{(\varepsilon+P+\Pi+\Lambda_2)(\varepsilon+P+\Pi+\Lambda_3)}{3(\varepsilon+P+\Pi-|\Lambda_1|)} \times\left\{1+\frac{2\left[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)+\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]}{\varepsilon+P+\Pi-|\Lambda_1|}\right\}, \quad (5h)$$

where condition (5h) can be dropped if $\delta_{\pi\pi}=\tau_{\pi\pi}=0$. The detailed proof can be found in the Supplemental Material [78]. Since (4) must hold for causality, they must be satisfied for any set of conditions that imply causality, and it is possible to verify that (5) imply (4) under assumptions (A1)–(A3). When shear viscous effects are neglected, (5) reduces to the conditions for the bulk viscosity case found in [79].

Conditions (A1)–(A3) and (5) also ensure the unique local solvability of the initial-value problem in the class of quasianalytic functions: given initial data of sufficient regularity satisfying (5), there exists a unique solution to the nonlinear equations taking the given initial data, defined for a certain time interval. Thus, if (A1)–(A3) and (5) hold, the evolution of the viscous fluid is guaranteed to be well defined and causal even far from equilibrium where the gradients (and, hence, $\pi^{\mu\nu}$ and Π) are large. This is relevant for determining the properties of hydrodynamic attractors [20] under general flow conditions [27,80] and, also, for an overall validation of fluid dynamics descriptions of extreme systems, such as proton-proton collisions. Moreover, while, here, we focus on applications to heavy-ion collisions, so $g_{\mu\nu}$ is the Minkowski metric, the methods of [79] can be adapted to show that our conclusions hold when (1)–(2) are coupled to Einstein's equations. Hence, our results are also crucial for determining the far-from-equilibrium behavior of viscous fluids in general relativity, which may be relevant to neutron star mergers [81]. The technical details of these statements are provided in the Supplemental Material [78].

When we linearize the equations around the equilibrium, terms involving $\tau_{\pi\pi}, \delta_{\Pi\Pi}, \lambda_{\Pi\pi}, \delta_{\pi\pi}, \lambda_{\pi\Pi}$ drop out and, thus, (A1) can be replaced by $\tau_\pi, \tau_\Pi>0$, $\eta, \zeta, c_s^2\geq 0$, and (A2) and (A3) can be replaced by $\varepsilon+P>0$ and $P\geq 0$. Then, conditions (5) become necessary and reduce to $\varepsilon+P>0$, $\varepsilon+P-(\eta/\tau_\pi)\geq 0$, and $1/(\varepsilon+P)(4\eta/3\tau_\pi+\zeta/\tau_\Pi)\leq 1-c_s^2$. These conditions coincide with the corresponding well-known results previously found in [58,59] that ensure causality and stability in the linearized regime around equilibrium.

It is instructive to compare the causal propagation modes of the full nonlinear theory determined here with that of the dynamics linearized about equilibrium. Linearizing Eqs. (1)–(2) around equilibrium, we find four distinct modes of propagation which correspond to the flow lines, the sound waves, and shear waves at two distinct speeds. These are the same ones found in the previous works [46,58,59,64–66], where, there, the authors find, in addition, a second longitudinal mode (second sound) due to the fact that their equations also include a conserved current. In the nonlinear case, we found six distinct propagation modes corresponding to the flow lines, the sound waves, and shear waves at four distinct speeds. This, again, highlights how one misses an important part of the dynamics by looking only at linearizations around equilibrium: there are two additional speeds allowed for the shear waves that collapse onto the remaining two upon linearization, so that these additional velocities are not visible in the linearized analysis.

We presented two sets of conditions for causality, namely, conditions that are necessary and conditions that are sufficient. Further studies must be done to discover conditions that are necessary and sufficient, i.e., conditions that ensure the system to be causal if and only if they hold. This is an extremely challenging task given the complexity of the characteristic equation in the nonlinear problem.

Conformal limit.—To obtain some physical understanding about our nonlinear constraints, consider a conformal fluid [55], i.e., $\Pi = 0$, $P = \varepsilon/3$, $\delta_{\pi\pi} = 4\tau_\pi/3$, with η/s and $\tau_\pi T$ being constants (here, $T \sim \varepsilon^{1/4}$ is the temperature and $s \sim T^3$ is the equilibrium entropy density). Assume, for simplicity, that all the other transport coefficients vanish (as in [82]). The necessary conditions in (4) then impose that $\Lambda_a/(\varepsilon + P) \geq -1 + (\eta/s)(1/\tau_\pi T)$, so none of the eigenvalues of π^μ_ν can be too negative. Also, when $\Lambda_a/(\varepsilon + P) > -1 + (\eta/s)(1/\tau_\pi T)$, the eigenvalues are also limited from above since (4e) gives $\Lambda_a/(\varepsilon + P) \leq 1 - (2/\tau_\pi T)(\eta/s)$. Using typical values motivated by heavy-ion collision applications, $\eta/s = 1/(4\pi)$ [83] and $\tau_\pi T = 5\eta/s$ [84], one then finds $-4/5 < \Lambda_a/(\varepsilon + P) \leq 3/5$. This implies that the relative magnitude of the shear stress tensor, $\sqrt{\pi_{\mu\nu}\pi^{\mu\nu}/(\varepsilon + P)^2}$, cannot be arbitrarily large. Using a NS initial condition where $\pi_{\mu\nu} \sim -2\eta\sigma_{\mu\nu}$ at the initial time τ_0, the corresponding normalized eigenvalues would be parametrically given by $\Lambda_a/(\varepsilon + P) \sim (\eta/s)/[\tau_0 T(\tau_0, \vec{x})]$ (assuming $\sigma_{\mu\nu} \sim 1/\tau_0$ in the initial state). Given that our conditions imply that, roughly, $|\Lambda_a/(\varepsilon + P)| \lesssim 1$, for $\tau_0 = 0.6$ fm and $\eta/s = 1/(4\pi)$, causality issues will be found where $T(\tau_0, \vec{x}) \lesssim 30$ MeV, which is below the typical values for the freeze-out temperature. However, in initial state models where the initialized $\pi_{\mu\nu}/(\varepsilon + P)$ is large and strongly deviates from NS, the estimate above does not apply, and causality violations may appear in hot regions of the plasma as well. A detailed numerical study is needed to assess the importance of our results to current simulations of heavy-ion collisions.

Conclusions.—In this Letter, we established, for the first time, that causality, in fact, holds for the full set of nonlinear equations in IS-like theories without the need for symmetry assumptions and in the presence of both shear and bulk viscosity. All our conditions are simple algebraic inequalities among the dynamical variables that can be easily checked in a given system or simulation. Previous attempts to go beyond the linear regime were restricted to $1 + 1$ dimensions [60] or assumed strong symmetry conditions [61,85], which, in practice, also corresponds to partial differential equations with only one spatial variable. Without such restrictions, the only other work where nonlinear causality has been shown for IS-like systems is [79], which only included bulk viscous effects. We have also studied the Cauchy problem for (1)–(2), establishing that it is well defined, so that it is meaningful to talk about solutions.

Prior to our work, unless a numerical code was specifically tailored to detect causality violations of the underlying equations, which typically is not a feature present in standard codes, one could only identify whether a numerical simulation of (1)–(2) violated causality if this caused (a) a breakdown of the simulation, (b) a manifestly spurious solution, or (c) clear nonphysical behavior. These constraints are all too weak, as we now explain. For illustration, consider $-\partial_t^2\psi + (1 + \psi)\Delta\psi = 0$, where Δ is the Laplacian. This is a nonlinear wave equation with (nonlinear) speed given by $\sqrt{1 + \psi}$ for (For $\psi < -1$, the equation is no longer a wave equation, becoming elliptic, and it is a degenerate wave equation when $\psi = -1$.) $\psi > -1$. Indeed, the characteristics are given by $\xi_0 = \pm\sqrt{1 + \psi}|\vec{\xi}|$. Therefore, the solutions are not causal when $\psi > 0$, but are causal for $-1 < \psi \leq 0$. Nevertheless, the equation remains hyperbolic as long as $\psi > -1$. Standard hyperbolic theory (see, e.g., [86]) ensures that, given smooth initial data $\psi|_{t=0}$ and $\partial_t\psi|_{t=0}$, there exists a unique smooth solution defined for some time. So any numerical scheme that is able to track the unique solution will produce results in both the acausal and causal cases $\psi > 0$ and $-1 < \psi \leq 0$, respectively. This makes it extremely difficult to infer violations of causality using (a) or (b) as criteria. Exactly the same situation can happen in simulations of (1)–(2). We also note that linearizing the equation about the "equilibrium" $\psi = 0$ gives $-\delta\psi_{tt} + \Delta\delta\psi = 0$, which is always causal, reinforcing, again, the idea that causality cannot always be obtained from linearizations.

Criteria (c) has also limited applicability. First, there are different mechanisms that can produce nonphysical solutions. Thus, it is still important to understand whether unphysical behavior is being caused by causality violation or some other mechanism, such as running beyond the limit where the effective description is valid. Second, relativistic fluids in the far-from-equilibrium regime, such as the QGP, may exhibit unexpected behavior, so one needs to be careful to differentiate genuine exotic features from those that are consequences of running a simulation in a superluminal regime. This may be particularly relevant to heavy-ion simulations where the values of the fields drop extremely rapidly at the edges of the QGP at early times and in the cold or dilute regions of plasma where a rescaling of dissipative tensors has been employed [87–90]. Third, numerical simulations of relativistic fluids must be based on equations of motion that respect causality, a fundamental physical principle in relativity.

The results we presented here are an important step in addressing all these difficulties, as one can check if (A1)–(A3), (4), or (5) hold at any moment in numerical simulations [Comparing with the example of the equation for ψ above, this would be similar to monitoring the value of $\sqrt{1 + \psi}$: if $\psi > 1$, then the system is not causal, which is the analog of (5), whereas causality is guaranteed if $-1 < \psi \leq 0$, which is the analog of (5).] since all the quantities involved in our inequalities can be readily extracted in numerical simulations [3]. We also note that our results apply, in particular, to the initial conditions, so (4) and (5) can be used to rule out initial conditions that violate causality or to select initial conditions for which causality holds. This can be particularly relevant to further constrain the physical assumptions behind the modeling of initial conditions in QGP simulations. There are many subtleties involved in numerically solving the IS equations, including possible violations of causality caused entirely as an artifact of the numerical simulation. Thus, it is important to distinguish between such numerically caused unphysical phenomena from true violation of causality of the underlying equations. Our new causality criteria can be instrumental in such analyses.

In sum, in this Letter, we established, for the first time in the literature, conditions to settle the longstanding questions concerning causality in Israel-Stewart theories in the nonlinear, far-from-equilibrium regime. As such, our general results provide the most stringent tests to date for determining the validity of relativistic fluid dynamic approaches in heavy-ion collisions, astrophysics, and cosmology.

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Nonlinear Constraints on Relativistic Fluids Far From Equilibrium SUPPLEMENTAL MATERIAL

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In this Supplemental Material, in Section II we provide the proof that conditions (4) are necessary for causality, in Section III we provide the proof that conditions (5) are sufficient for causality, and in Section IV we establish local existence and uniqueness of solutions to the initial-value problem for equations (1)-(2). All these results depend on a careful analysis of the roots of the characteristic equation $\det(A^\alpha\xi_\alpha)=0$. Thus, we first present in Section I a suitable factorization of $\det(A^\alpha\xi_\alpha)$. In Section V we show that conditions (4), albeit necessary, are not sufficient for causality. In Section VI we provide the formal definition of causality and comment on why, in our case, it can be reduced to conditions (C1) and (C2). Since causality is intrinsically tied to concepts of relativity theory, we refer to the standard literature (e.g., [1]) for further background. Throughout this Supplemental Material, we continue to use the notation and definitions of the paper.

I. THE CHARACTERISTIC EQUATION

Letting $b=u^\alpha\xi_\alpha$, $v^\mu=\Delta^{\mu\nu}\xi_\nu$, and $w^\mu=\pi^{\mu\nu}\xi_\nu$, the characteristic determinant can be written as

$$\det(A^\alpha\xi_\alpha)=b^{13}\tau_\pi^{16}\tau_\Pi\det\begin{bmatrix} b & \rho\xi_\nu+w_\nu \\ bc_s^2v^\mu & \rho b^2\delta^\mu_\nu-bw_\nu u^\mu-\frac{\bar{C}^\mu_\nu}{\tau_\pi}-\frac{v^\mu\tilde{E}_\nu}{\tau_\Pi}\end{bmatrix}=b^{14}\tau_\pi^{16}\tau_\Pi\det[M], \tag{S1}$$

where $M=[M^\mu_\nu]_{4\times4}$ with $M^\mu_\nu=\rho b^2\delta^\mu_\nu-bw_\nu u^\mu-\frac{\bar{C}^\mu_\nu}{\tau_\pi}-\frac{v^\mu\tilde{E}_\nu}{\tau_\Pi}-c_s^2v^\mu(\rho\xi_\nu+w_\nu)$, $\tilde{E}_\nu=E^\alpha_\nu\xi_\alpha=(\zeta+\delta_{\Pi\Pi}\Pi)\,\xi_\nu+\lambda_{\Pi\pi}w_\nu$, and

$$\begin{aligned}\bar{C}^\delta_\nu \;&=\; C^{\sigma\delta\alpha}_\nu\xi_\alpha\xi_\sigma=\frac{1}{2}\left[(2\eta+\lambda_{\pi\Pi}\Pi)\xi_\mu\delta^\delta_\lambda+\frac{\tau_{\pi\pi}}{2}w_\lambda\delta^\delta_\mu+\frac{\tau_{\pi\pi}}{2}\pi^\delta_\lambda\xi_\mu\right]\left(v^\mu\delta^\lambda_\nu+v^\lambda\delta^\mu_\nu-\frac{2}{3}\Delta^{\mu\lambda}\xi_\nu\right)\\ &\quad-\frac{\tau_{\pi\pi}}{3}v^\delta w_\nu+\delta_{\pi\pi}w^\delta\xi_\nu-b\tau_\pi(w_\nu u^\delta+b\pi^\delta_\nu).\end{aligned} \tag{S2}$$

Since $\pi^{\mu\nu}$ is symmetric and traceless, it can be diagonalized, so the eigenvalue problem $\pi^\mu_\nu e^\nu_A=\Lambda_A e^\mu_A$, with $A=0,1,2,3$, defines an orthonormal set of eigenvectors $e^\mu_{A=0}=u^\mu$, $e^\mu_{A=a}=e^\mu_a$ with real eigenvalues Λ_a for $a=1,2,3$ in the sense that $g_{\mu\nu}e^\mu_Ae^\nu_B=\eta_{AB}$ where $\eta_{AB}=diag(-1,1,1,1)$. The eigenvalues are such that $\Lambda_0=0$ and $\Lambda_1+\Lambda_2+\Lambda_3=0$. Without any loss of generality, let us take $\Lambda_1\leq\Lambda_2\leq\Lambda_3$ with $\Lambda_1\leq0\leq\Lambda_3$ so that the trace is kept zero (note that if $\pi^\mu_\nu\neq0$, this allows degeneracies to occur with multiplicity up to two). Since $\{e^\mu_A\}$ is a complete set in $\mathbb{R}^4$, we may define a tetrad of dual vectors $\{e^A_\nu\}$ by setting $e^A_\nu\equiv\eta^{AB}(e_B)_\nu$ so that[1] $\delta^B_A=e^\nu_Ae^B_\nu$. Also, the following completeness relation holds: $\delta^\mu_\nu=\sum_A e^\mu_Ae^A_\nu=-u^\mu u_\nu+\sum_a e^\mu_a(e_a)_\nu$. Therefore, the components of any four-vector z^μ relative to the tetrad $\{e^\mu_A\}$ are defined by $z^A\equiv z^\nu e^A_\nu$. We can then use this to define $v_A\equiv e^\mu_Av_\mu$ and $\xi_A\equiv e^\mu_A\xi_\mu$. Given that $\xi^\mu=-bu^\mu+\sum_a v^ae^\mu_a$ $(a=1,2,3)$ one finds that $\xi_{A=0}=-\xi^{A=0}=b$ while $\xi_a=v_a$. Furthermore, $w_A\equiv e^\mu_Aw_\mu=e^\mu_A\pi_{\mu\nu}\xi^\nu=\Lambda_A\xi_A=\Lambda_Av_A$, where

[1] From now on, repeated Latin indexes are not summed unless explicitly stated.

we used that $\Lambda_0 = 0$ and again $\xi_a = v_a$ (note also that $v^a = v_a$ since $\eta_{ab} = \delta_{ab}$). Using these observations, we can show that the determinant $\det(M)$ needed for the characteristics in (S1) is given by

$$\det(M) = \det(E^{-1}ME) = m_0 m_1 m_2 m_3$$
$$\times \left[1 - \sum_a \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho+\Lambda_a)c_s^2\right\}\hat{v}_a^2}{\bar{m}_a}\right.$$
$$\left. - \frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{\substack{a,b\\a<b}} \frac{(\Lambda_a - \Lambda_b)^2 \hat{v}_a^2 \hat{v}_b^2}{\bar{m}_a \bar{m}_b}\right], \tag{S3}$$

where $E = [e^\mu_A]_{4\times4}$, $E^{-1} = [e^B_\nu]_{4\times4}$, and $E^{-1}ME = [e^A_\mu M^\mu_\nu e^\nu_B]_{4\times4}$. Also, we defined above $m_0 = \rho\left(b^2 - \sum_a \mathfrak{g}_a v_a^2\right)$, $\mathfrak{g}_a = \frac{2(2\eta+\lambda_{\pi\Pi}\Pi)+\tau_{\pi\pi}\Lambda_a}{4\rho\tau_\pi}$, $m_a = (\rho+\Lambda_a)b^2 - \frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)(v\cdot v) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_a v\cdot v + \sum_c \Lambda_c v_c^2\right)$, $\hat{v}_a = v_a/\sqrt{v\cdot v}$ (assuming $v \neq 0$), and $\bar{m}_0 = m_0/(v\cdot v), \bar{m}_a = m_a/(v\cdot v)$. Note that $\sum_a \hat{v}_a^2 = \sum_a v_a^2/(v\cdot v) = 1$ since $v\cdot v = v^\mu v_\mu = \sum_a v_a^2$. Assuming $v \neq 0$ is allowed because $v = 0$ does not lead to nontrivial roots $b \neq 0$ of the characteristic equation if assumptions (A1)–(A3) hold.

The roots ξ of $\det(A^\alpha \xi_\alpha) = 0$ in Eq. (S1) are the 14 roots from $b = u^\alpha \xi_\alpha = 0$ and the 8 roots from $\det(M) = 0$ in Eq. (S3) which consist of the 2 roots from $m_0 = 0$ and the 6 roots coming from the zeros of

$$f(k) = \bar{m}_1 \bar{m}_2 \bar{m}_3 G(k), \tag{S4}$$

where we defined $k \equiv b^2/v\cdot v$ and

$$G(k) = 1 - \sum_a \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho+\Lambda_a)c_s^2\right\}\hat{v}_a^2}{\bar{m}_a}$$
$$- \frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{\substack{a,b\\a<b}} \frac{(\Lambda_a - \Lambda_b)^2 \hat{v}_a^2 \hat{v}_b^2}{\bar{m}_a \bar{m}_b}. \tag{S5}$$

In this notation $\det(M) = m_0(v\cdot v)^3 f(k)$ because we used the definition $\bar{m}_a = m_a/v\cdot v$. Note that although $G(k)$ has $\bar{m}_a$ appearing in denominators, these are canceled by the multiplication of $G(k)$ by $\bar{m}_1\bar{m}_2\bar{m}_3$ in the definition of $f(k)$. Thus, $f(k)$ is a polynomial of degree 3 in k (of degree 6 in b) and is defined for all values of $k \in \mathbb{R}$. Then, it is possible to factorize $f(k)$ as

$$f(k) = \left[\prod_{a=1}^{3}(\varepsilon + P + \Pi + \Lambda_a)\right](k - k_1)(k - k_2)(k - k_3), \tag{S6}$$

where k_1, k_2, k_3 as the three roots of $f(k)$. Note that for the sake of brevity, we have suppressed the dependence on $\hat{v}$ in writing $G(k)$ and $f(k)$ (to be more precise, these should have been written as $G(k,\hat{v}), f(k,\hat{v})$).

Conditions (C1) and (C2) for causality demand that all the 22 roots $\xi_0 = \xi_0(\xi_i)$ of $\det(A^\alpha\xi_\alpha) = 0$ are real and satisfy $\xi_\alpha\xi^\alpha = -b^2 + v\cdot v \geq 0$, i.e., $0 \leq k \leq 1$. The 14 roots $b = 0$ are causal. Thus, the rest the analysis of necessary conditions in Section II will focus on the remaining roots defined by $f(k) = 0$. We summarize this in the following important statement:

> The system is causal if and only if for all for all $\hat{v}$ on the unit sphere, the roots of $\bar{m}_0(k,\hat{v}) = 0$ and $f(k,\hat{v}) = 0$ are real and $0 \leq k \leq 1$. (C3)

II. DERIVATION OF NECESSARY CONDITIONS FOR CAUSALITY

Here we establish that conditions (4) are necessary (but not sufficient, see Section V) for causality. More precisely, we establish the following Theorem.

Theorem 1. *Let* $\Psi = (\varepsilon, u^\nu, \Pi, \pi^{0\nu}, \pi^{1\nu}, \pi^{2\nu}, \pi^{3\nu})_{\nu=0,\ldots,3}$ *be a smooth solution to equations (1)-(2) in Minkowski space, with* $u_\mu u^\mu = -1$ *and* $\pi_{\mu\nu}$ *satisfying* $\pi^\mu_\mu = 0$ *and* $u^\mu \pi_{\mu\nu} = 0$. *Suppose that (A1)-(A3) hold. If any of conditions (4) is not satisfied, then* Ψ *is not causal in the sense of Definition 4 (see Section VI).*

Proof of Theorem 1: Our derivation of necessary conditions for causality is via the following reasoning. Causality requires that conditions (C1) and (C2) hold for all ξ_i. Thus, in order to violate causality, it suffices to show that for some ξ_i, (C1) or (C2) fails. Suppose now that we find a condition, say Z, for which we can exhibit one ξ_i such that (C1) or (C2) fail, i.e., we obtain the statement "Z implies non-causality." This statement is logically equivalent to "Causality implies non-Z." This means that non-Z is a necessary condition for causality: if it is violated, the system is not causal. In our case, conditions like Z will be inequalities among the scalars of the problem (e.g., the relaxation times, eigenvalues Λ_a, etc.) of the form $A > B$, whose negation is then $A \leq B$. The latter is then the necessary condition we are looking for: if $A \leq B$ does not hold, the system is not causal.

Recall that (C1) and (C2) is equivalent to (C3), so in view of the foregoing discussion, we aim to violate (C3). With the choice $\hat{v}_a = \delta_{ad}$, one can write $m_0 = \rho(v \cdot v)(k - \mathfrak{g}_d) = 0$. Under our assumptions, the only root is $k = \mathfrak{g}_d$. Since we need $0 \leq k \leq 1$, as discussed, and since $\mathfrak{g}_1 \leq \mathfrak{g}_2 \leq \mathfrak{g}_3$, causality if violated if $\mathfrak{g}_1 < 0$, leading to condition (4a), or if $\mathfrak{g}_3 > 1$, leading to condition (4b).

As for the roots of $f(k)$, we may note that now in $f(k) = \bar{m}_1 \bar{m}_2 \bar{m}_3 G(k)$ we have

$$\bar{m}_a = (\varepsilon + P + \Pi + \Lambda_a)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_a + \Lambda_d) \tag{S7}$$

and

$$G(k) = 1 - \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} + (\rho + \Lambda_d)c_s^2\right\}}{\bar{m}_d} \tag{S8}$$

because we have set $\hat{v}_a = \delta_{ad}$. We may therefore rewrite

$$f(k) = \bar{m}_a \bar{m}_b \left[\bar{m}_d - \left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} + (\rho + \Lambda_d)c_s^2\right\}\right], \tag{S9}$$

where $a \neq b$ and $a, b \neq d$. Setting each of the factors m_a, m_b equal to zero, we obtain the roots

$$k = \frac{\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_a + \Lambda_d)}{\varepsilon + P + \Pi + \Lambda_a}, \quad a \neq d. \tag{S10}$$

Causality is violated if $k < 0$, leading to condition (4c), of if $k > 1$, leading to condition (4d). The remaining root in (S9) is obtained when the term in brackets vanishes, giving

$$\begin{aligned} k = {} & \frac{\frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d}{\varepsilon + P + \Pi + \Lambda_d} \\ & + \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_d] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi} + (\rho + \Lambda_d)c_s^2\right\}}{\varepsilon + P + \Pi + \Lambda_d}. \end{aligned} \tag{S11}$$

Causality is violated if $k < 0$, leading to (4e), or if $k > 1$, leading to (4f). This finishes the proof. □

III. DERIVATION OF SUFFICIENT CONDITIONS FOR CAUSALITY

Here we establish that conditions (5) are sufficient for causality. More precisely, we have:

Theorem 2. *Let $\Psi = (\varepsilon, u^\nu, \Pi, \pi^{0\nu}, \pi^{1\nu}, \pi^{2\nu}, \pi^{3\nu})_{\nu=0,\ldots,3}$ be a smooth solution to equations (1)-(2) in Minkowski space, with $u_\mu u^\mu = -1$ and $\pi_{\mu\nu}$ satisfying $\pi^\mu_\mu = 0$ and $u^\mu \pi_{\mu\nu} = 0$. Suppose that (A1)-(A3) and (5) hold. Then Ψ is causal in the sense of Definition 4 (see Section VI).*

Proof of Theorem 2: As discussed in Section I, the 14 roots $b = 0$ are causal and do not need any further treatment. The remaining 8 roots that come from $\det(M) = 0$ are, again, the two roots of m_0 and the six roots of $f(k)$ defined in (S4). We begin by analyzing the two roots of m_0. Recalling that $v = 0$ does not lead a nontrivial root of $\det(A^\alpha \xi_\alpha) = 0$, we see that the roots of m_0 are given by $b^2 = k = \sum_a \mathfrak{g}_a \hat{v}_a^2$. For these roots we need to check (according to (C3)) that

$$0 \le \sum_a \mathfrak{g}_a \hat{v}_a^2 \le 1. \tag{S12}$$

(A3) together with conditions (5a) and (5b) give $0 \le \mathfrak{g}_1 \le \mathfrak{g}_2 \le \mathfrak{g}_3 \le 1$. From $\mathfrak{g}_1 \le \sum_a \mathfrak{g}_a \hat{v}_a^2 \le \mathfrak{g}_3$, we see that (S12) is satisfied.

Now we analyze the remaining 6 roots of $\det(M) = 0$ coming from $f(k)$ defined in Eq. (S4) and written explicitly as a polynomial in (S6). We will show further below that the three roots k_i in (S6) are real. But let us first show that any real root of f must lie within $[0, 1]$. Since f is a cubic polynomial, it either has only one real root, say s_1, or three real roots, in which case we can order them as $k_1 \le k_2 \le k_3$ in (S6). Invoking (5a), we see that in the first case f is negative to the left of s_1 and positive to its right, and in the second case that f is a growing cubic polynomial except in the interval between the roots k_1 and k_3. In either situation, any real root will be between 0 and 1 if

$$f(k < 0) < 0, \tag{S13}$$

and

$$f(k > 1) > 0. \tag{S14}$$

Let us first verify the inequality (S14). For $k > 1$

$$\bar{m}_a(k > 1) \ge k(\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3, \tag{S15}$$

where we have used $-2|\Lambda_1| \le \Lambda_a + \sum_c \Lambda_c \hat{v}_c^2 \le 2\Lambda_3$. Now, observe that

$$k(\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3 > (\varepsilon + P + \Pi - |\Lambda_1|) - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3$$

for $k > 1$, hence the condition (5a) lead us to $\bar{m}_a(k \ge 1) > 0$. This guarantees that

$$\bar{m}_1(k > 1)\bar{m}_2(k > 1)\bar{m}_3(k > 1) > 0.$$

To obtain $f(k > 1) > 0$ in (S14), we therefore need $G(k > 1) > 0$. By means of (5c) and (5d),

$$\begin{aligned} &-\sum_a \frac{\left\{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho + \Lambda_a)c_s^2\right\}\hat{v}_a^2}{\bar{m}_a(k > 1)} \\ &> -\frac{\frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_3] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi} + (\varepsilon + P + \Pi + \Lambda_3)c_s^2}{\varepsilon + P + \Pi - |\Lambda_1| - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3} \end{aligned} \tag{S16}$$

as well as

$$-\frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{a<b}\frac{(\Lambda_a - \Lambda_b)^2\hat{v}_a^2\hat{v}_b^2}{\bar{m}_a(k > 1)\bar{m}_b(k > 1)}$$

$$> -\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3-\Lambda_1)^2}{\left[\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]^2}, \tag{S17}$$

and thus,

$$G(k>1) > 1-\frac{\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi+(6\delta_{\pi\pi}-\tau_{\pi\pi})\Lambda_3]+\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi}+(\varepsilon+P+\Pi+\Lambda_3)c_s^2}{\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3}$$
$$-\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\left[\varepsilon+P+\Pi-|\Lambda_1|-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]^2}. \tag{S18}$$

Note that we have used $\max_{a,b}(\Lambda_a-\Lambda_b)^2=(\Lambda_3-\Lambda_1)^2=(\Lambda_3+|\Lambda_1|)^2$, which follows from the ordering of the eigenvalues Λ_a. Hence (5e) implies $G(k)>0$ for $k>1$.

It now remains to verify the inequality (S13). In this case, when $k<0$

$$\bar{m}_a(k<0) = -|k|(\varepsilon+P+\Pi+\Lambda_a)-\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_a+\sum_c\Lambda_c\hat{v}_c^2\right)$$
$$< -\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)+\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|. \tag{S19}$$

From condition (5b), one has that $\bar{m}_a(k\leq 0)<0$. Then,

$$f(k<0)=\bar{m}_1(k<0)\bar{m}_2(k<0)\bar{m}_3(k<0)G(k<0)<0$$

if, and only if, $G(k<0)>0$. Due to $\bar{m}_a(k\leq 0)<0$ together with (5c) and (5d), we obtain that

$$\sum_a\frac{\left\{\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi+(6\delta_{\pi\pi}-\tau_{\pi\pi})\Lambda_a]+\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi}+(\rho+\Lambda_a)c_s^2\right\}\hat{v}_a^2}{-\bar{m}_a(k<0)}$$
$$> \frac{\frac{1}{6\tau_\pi}[2\eta+\lambda_{\pi\Pi}\Pi-(6\delta_{\pi\pi}-\tau_{\pi\pi})|\Lambda_1|]+\frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}|\Lambda_1|}{\tau_\Pi}+(\varepsilon+P+\Pi-|\Lambda_1|)c_s^2}{-m_a(k<0)}. \tag{S20}$$

Condition (5f) guarantees that $\sum_a\ldots>0$ in the above inequality. Moreover,

$$-\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\sum_{a<b}\frac{(\Lambda_a-\Lambda_b)^2\hat{v}_a^2\hat{v}_b^2}{\bar{m}_a(k<0)\bar{m}_b(k<0)}$$
$$> -\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\left[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right]^2}. \tag{S21}$$

where we used (S19) and $(\Lambda_3+|\Lambda_1|)^2=\max_{a,b}(\Lambda_a-\Lambda_b)^2$ again. Now, since

$$G(k<0) > 1-\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_3+|\Lambda_1|)^2}{\left[\frac{1}{2\tau_\pi}(2\eta+\lambda_{\pi\Pi}\Pi)-\frac{\tau_{\pi\pi}}{2\tau_\pi}|\Lambda_1|\right]^2}, \tag{S22}$$

we have $G(k<0)>0$ from condition (5g), finally implying $f(k<0)<0$.

It remains to establish the reality of the roots k_i in (S6). To do that, let us write $G(k)$ as

$$G(k) \;=\; 1-\sum_a\frac{R_a\hat{v}_a^2}{\bar{m}_a}-\sum_{\substack{a,b\\a<b}}\frac{S_{ab}\hat{v}_a^2\hat{v}_b^2}{\bar{m}_a\bar{m}_b} \tag{S23}$$

and

$$\bar{m}_a = \rho_a k - r_a, \tag{S24}$$

where

$$\begin{aligned}
R_a &= \frac{1}{6\tau_\pi}[2\eta + \lambda_{\pi\Pi}\Pi + (6\delta_{\pi\pi} - \tau_{\pi\pi})\Lambda_a] + \frac{\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\rho + \Lambda_a)c_s^2 && \text{(S25)}\\
S_{ab} &= \frac{12\delta_{\pi\pi} - \tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_a - \Lambda_b)^2, && \text{(S26)}\\
\rho_a &= \rho + \Lambda_a = \varepsilon + P + \Pi + \Lambda_a, && \text{(S27)}\\
r_a &= \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_a + \sum_c \Lambda_c \hat{v}_c^2\right). && \text{(S28)}
\end{aligned}$$

Note, in particular, that $\bar{r}_1 \leq r_a \leq \bar{r}_3$, where $\bar{r}_{1,3} \equiv \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) + \frac{\tau_{\pi\pi}\Lambda_{1,3}}{2\tau_\pi} > 0$ from (5b). By applying conditions (5) one has that $R_a, S_{ab}, \rho_a, r_a \geq 0$. Then, $f(k)$ can be written as

$$\begin{aligned}
f(k) &= \bar{m}_1\bar{m}_2\bar{m}_3 - \bar{m}_1\bar{m}_2 R_3\hat{v}_3^2 - \bar{m}_2\bar{m}_3 R_1\hat{v}_1^2 - \bar{m}_3\bar{m}_1 R_2\hat{v}_2^2 - \bar{m}_1 S_{23}\hat{v}_2^2\hat{v}_3^2 - \bar{m}_2 S_{13}\hat{v}_1^2\hat{v}_3^2\\
&\quad - \bar{m}_3 S_{12}\hat{v}_1^2\hat{v}_2^2\\
&= a_3 k^3 + a_2 k^2 + a_1 k + a_0, && \text{(S29)}
\end{aligned}$$

where

$$\begin{aligned}
a_0 &= -\big(r_1r_2r_3 + r_1r_2R_3\hat{v}_3^2 + r_2r_3R_1\hat{v}_1^2 + r_1r_3R_2\hat{v}_2^2 - r_1S_{23}\hat{v}_2^2\hat{v}_3^2 - r_2S_{13}\hat{v}_1^2\hat{v}_3^2\\
&\quad - r_3S_{12}\hat{v}_1^2\hat{v}_2^2\big), && \text{(S30)}\\
a_1 &= \rho_1r_2r_3 + \rho_2r_1r_3 + \rho_3r_1r_2 + (\rho_1r_2 + \rho_2r_1)R_3\hat{v}_3^2 + (\rho_2r_3 + \rho_3r_2)R_1\hat{v}_1^2\\
&\quad + (\rho_3r_1 + \rho_1r_3)R_2\hat{v}_2^2 - \rho_1S_{23}\hat{v}_2^2\hat{v}_3^2 - \rho_2S_{13}\hat{v}_1^2\hat{v}_3^2 - \rho_3S_{12}\hat{v}_1^2\hat{v}_2^2, && \text{(S31)}\\
a_2 &= -(\rho_1\rho_2r_3 + \rho_1\rho_3r_2 + \rho_2\rho_3r_1 + \rho_1\rho_2R_3\hat{v}_3^2 + \rho_2\rho_3R_1\hat{v}_1^2 + \rho_1\rho_3R_2\hat{v}_2^2), && \text{(S32)}\\
a_3 &= \rho_1\rho_2\rho_3. && \text{(S33)}
\end{aligned}$$

In view of (5), we have $a_3 > 0$ and $a_2 < 0$. Since all coefficients of $f(k)$ are real, then at least one of the roots must be real, say $k = s_1 \in \mathbb{R}$ is the real root. Then, we know that the other two roots s_2 and s_3 are real or complex conjugate, i.e., $s_3^* = s_2$. Let us assume that s_2 and s_3 can be imaginary and set $s_{2,3} = k_R \pm ik_I$, $k_I \neq 0$. By using Vieta's formula $s_1 + s_2 + s_3 = -\frac{a_2}{a_3} = \frac{|a_2|}{a_3} > 0$ we obtain that

$$\frac{|a_2|}{a_3} - 1 \leq 2k_R = \frac{|a_2|}{a_3} - s_1 \leq \frac{|a_2|}{a_3}. \tag{S34}$$

Thus, the following condition holds,

$$\frac{3\rho_1(\bar{r}_1 + R_1)}{\rho_2\rho_3} - 1 < 2k_R < \frac{3\rho_3(\bar{r}_3 + R_3)}{\rho_1\rho_2} \tag{S35}$$

because the real root $s_1 \in [0,1]$ when (5a)–(5g) apply, as we have already showed. Since we are assuming $s_{2,3} = k_R \pm ik_I$, where $k_I \neq 0$, we have that $\bar{m}_a(s_{2,3}) = \rho_a k_R - r_a \pm ik_I$ cannot be zero (unless $k_I = 0$ and the roots are real). Consequently, from (S23) we obtain that $f(s_{2,3}) = 0$ lead us to $G(s_{2,3}) = 0$, where $s_{2,3}$ must obey the above conditions implied by f being a cubic polynomial, in particular the condition on k_R in (S35). Thus, let us split $G(s_{2,3})$ in (S19) into $G_R(s_{2,3}) + iG_I(s_{2,3})$, where $G_R(s_{2,3}) = \Re[G(s_{2,3})]$ and $G_I(s_{2,3}) = \Im[G(s_{2,3})]$. In particular,

$$G_I(s_{2,3}) = \pm k_I \sum_a \frac{\hat{v}_a^2}{|\bar{m}_a|^2}\left[\rho_a R_a + \sum_{\substack{b\\ b>a}} \frac{[\rho_a(\rho_b k_R - \bar{r}_b) + \rho_b(\rho_a k_R - \bar{r}_a)]S_{ab}\hat{v}_b^2}{|\bar{m}_b|^2}\right]. \tag{S36}$$

To show that the roots are real, if suffices to have $G_I(s_{2,3}) \neq 0$. We distinguish two cases. If $S_{ab} = 0$ then $G_I(s_{2,3}) \neq 0$ because we assumed $k_I \neq 0$. This means that in this case the roots must all be real. On the other hand, if $S_{ab} \neq 0$ and $\rho_1 R_1 - \bar{r}_3 > 0$, then Eq. (S36) also gives $G_I(s_{2,3}) \neq 0$, because then the sum over b in (S36) is > 0. To check that $\rho_1 R_1 - \bar{r}_3 > 0$, note first that (5a) guarantees that $\rho_a > r_a$. Then, by means of (S35), we obtain that

$$\rho_1 k_R - \bar{r}_3 \;>\; \frac{\rho_1}{2}\left(\frac{3\rho_1(R_1+\bar{r}_1)}{\rho_2\rho_3} - 1 - \frac{2\bar{r}_3}{\rho_1}\right) \geq 0 \tag{S37}$$

because of condition (5h), and this implies $\rho_1 k_R - \bar{r}_3 > 0$. Since we have already showed that any real root of $f(k)$ must lie within $[0,1]$, this finishes our proof. □

IV. LOCAL EXISTENCE AND UNIQUENESS

In this Section, we establish the local existence and uniqueness of solutions to the Cauchy problem. Below, $\mathcal{G}$ is the space of Gevrey functions or quasi-analytic functions.

Theorem 3. *Consider the Cauchy problem for equations (1)-(2) in Minkowski space, with initial data* $\mathring{\Psi} = (\mathring{\varepsilon}, \mathring{u}^\nu, \mathring{\Pi}, \mathring{\pi}^{0\nu}, \mathring{\pi}^{1\nu}, \mathring{\pi}^{2\nu}, \mathring{\pi}^{3\nu})_{\nu=0,\dots,3}$ *given on* $\{t=0\}$*. Assume that the data satisfies the constraints*[2] $\mathring{u}^\nu \mathring{u}_\nu = -1$*,* $\mathring{u}^\nu$ *is future-pointing,* $\mathring{\pi}^\nu_\nu = 0$*, and* $\mathring{\pi}^\nu_\mu \mathring{u}^\mu = 0$*. Suppose that (A1)-(A3) and (5) hold for* $\mathring{\Psi}$ *in a strict form (i.e.* $<$ *instead of* $\leq$*,* $>$ *instead of* $\geq$*). Finally, assume that* $\mathring{\Psi} \in \mathcal{G}^\delta(\{t=0\})$*, where* $1 \leq \delta < 20/19$*. Then, there exist a* $T > 0$ *and a unique* $\Psi = (\varepsilon, u^\nu, \Pi, \pi^{0\nu}, \pi^{1\nu}, \pi^{2\nu}, \pi^{3\nu})_{\nu=0,\dots,3}$ *defined on* $[0,T) \times \mathbb{R}^3$ *such that* Ψ *is a solution to (1)-(2) in* $[0,T)\times\mathbb{R}^3$ *and* $\Psi = \mathring{\Psi}$ *on* $\{t=0\}$*. Moreover, the solution* Ψ *is causal in the sense of Definition 4* *(see Section VI)*.

Proof of Theorem 3: The calculations provided in Section I and in the proof of Theorem 2 imply that, under the assumptions, the characteristic polynomial of the system evaluated at the initial data is a product of strictly hyperbolic polynomials. One also sees that intersection of the interior of the characteristic cones defined by these strictly hyperbolic polynomials has non-empty interior and lies outside the light-cone defined by the metric. Under these circumstances we can apply theorems A.18, A.19, and A.23 of [2] to conclude the result (the remaining assumptions of these theorems are easily verified in our case). □

For the sake of brevity, we refer readers to [3] for a definition of $\mathcal{G}^\delta$, making only the following remarks. The case of $\delta = 1$ corresponds to the space of analytic functions, of which $\mathcal{G}^\delta$ with $\delta > 1$ is a generalization. This is why $\mathcal{G}$ is sometimes referred to as the space of quasi-analytic functions. The usefulness of Gevrey functions to the study of hyperbolic problems is at least two-fold. On the one hand, one can prove very general existence and uniqueness theorems for Gevrey data given on a non-characteristic surface that are akin to the Cauchy-Kovalewskaya theorem for analytic data. On the other hand, an advantage of Gevrey maps over analytic ones is that one can construct Gevrey functions that are compactly supported; hence one can appeal to the type of localization arguments that are so useful in the study of hyperbolic equations. This is particularly important when one is considering coupling to Einstein's equations.

While typical evolution problems consider solutions in more general function spaces than $\mathcal{G}^\delta$, we stress that ours is the very first existence and uniqueness result for equations (1)-(2). In other words, while it is desirable to extend our result to more general function spaces, Theorem 3 is important because it shows, for the very first time in the literature, that the initial value problem for equation (1)-(2) is well-defined, so that it is meaningful to talk about solutions.

We remark that the diagonalization of $\pi_{\mu\nu}$ was carried out in terms of orthonormal frames which can be defined for any Lorentzian metric. Also, our computations are manifestly covariant. Thus, the results of

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[2] Alternatively, we could have only unconstrained data be prescribed and obtain the full set of data from the stated constraints. For example, we could have $\mathring{u}^i$ prescribed and define u^0 so that $\mathring{u}^\nu$ is unit time-like and future pointing.

Theorems 1, 2, and 3, remain true in a general globally hyperbolic space-time, as mentioned in the main text. Moreover, as also mentioned in the main text, the result extends to the case when (1)-(2) are coupled to Einstein's equations. This follows by computing the characteristic determinant of the coupled system and observing that it factors into the product of the characteristic determinant of (1)-(2), which we analyzed here, and the characteristic determinant of Einstein's equations. The argument is the same as given in [4].

V. INSUFFICIENCY OF CONDITIONS FOR CAUSALITY

In this Section, we show that conditions (4), albeit necessary, are not sufficient for causality. We do this by showing that causality can be violated if we only assume (A1)-(A3) and (4).

Thus, suppose that (A1)-(A3) and (4) hold. Consider the case where (Ins1) $\delta_{\pi\pi} = \tau_{\pi\pi}/4$, $\delta_{\Pi\Pi} = 0$, $\zeta + \lambda_{\Pi\pi}\Lambda_a \geq 0$, $\frac{\lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi} > 0$, and $1 - c_s^2 - \frac{\tau_{\pi\pi}}{3\tau_\pi} - \frac{\lambda_{\Pi\pi}}{\tau_\Pi} < 0$. Also, the parameters as well as c_s^2 obey the necessary conditions (4). Assume also that (Ins2) $\Lambda_3 = \Lambda_2 > 0$, i.e., Λ_3 is a degenerated eigenvalue. Then, we may write

$$G(k) = 1 - \sum_a \frac{R_a \hat{v}_a^2}{\bar{m}_a} - \sum_{\substack{a,b \\ a<b}} \frac{S_{ab}\hat{v}_a^2\hat{v}_b^2}{\bar{m}_a\bar{m}_b}, \tag{S38}$$

where

$$R_a = \frac{1}{6\tau_\pi}\left[2\eta + \lambda_{\pi\Pi}\Pi + \frac{\tau_{\pi\pi}}{2}\Lambda_a\right] + \frac{\zeta + \lambda_{\Pi\pi}\Lambda_a}{\tau_\Pi} + (\varepsilon + P + \Pi + \Lambda_a)c_s^2 \tag{S39}$$

and

$$S_{ab} = \frac{\tau_{\pi\pi}}{6\tau_\pi}\left(\frac{\Lambda_{\Pi\pi}}{\tau_\Pi} + c_s^2 - \frac{\tau_{\pi\pi}}{12\tau_\pi}\right)(\Lambda_a - \Lambda_b)^2. \tag{S40}$$

From (4a) together with the above choices we have that $R_a, S_{ab} > 0$. Now, let us define

$$\tilde{m}_a \equiv \varepsilon + P + \Pi + \Lambda_a - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_a. \tag{S41}$$

Then, (4f) can be written as

$$\tilde{m}_d - R_d \geq 0, \tag{S42}$$

culminating into $\tilde{m}_d > 0$. Note that this must hold for any $d = 1, 2, 3$. Let us consider the case where a_1 is such that $\tilde{m}_{a_1} - R_{a_1} = \min_d(\tilde{m}_d - R_d)$. Thus, if (S42) is verified for $d = a_1$, it is automatically true for for all $d = 1, 2, 3$. Now, we may choose the constraint in the parameters such that (Ins3) $\tilde{m}_{a_1} - R_{a_1} = 0$, what is in accord with (S42). The remainder of this proof relies on the choice $\hat{v}_{a_1} = \sqrt{1-\epsilon^2}$, $\hat{v}_{a_2} = \epsilon$, and $\hat{v}_{a_3} = 0$ for $\epsilon \in (0, 1)$. Moreover, we make the assumption (Ins4) that if $a_1 = 3, 2$, then $a_2 = 2, 3$ while if $a_1 = 1$, then a_2 can be either 2 or 3. Thus, one can clearly see that

$$\begin{aligned} f(k) &= \bar{m}_{a_3}\left(\bar{m}_{a_1}\bar{m}_{a_2} - \bar{m}_{a_1}R_{a_2}\epsilon^2 - \bar{m}_{a_2}R_{a_1}(1-\epsilon^2) - S_{a_1a_2}\epsilon^2(1-\epsilon^2)\right), \qquad \text{(S43)} \\ \bar{m}_d &= (\varepsilon + P + \Pi + \Lambda_d)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}\left[\Lambda_d + \Lambda_{a_1}(1-\epsilon^2) + \Lambda_{a_2}\epsilon^2\right] \\ &= \bar{m}_d^0 - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_{a_2} - \Lambda_{a_1})\epsilon^2, \qquad \text{(S44)} \end{aligned}$$

where we defined

$$\bar{m}_d^0 \equiv (\varepsilon + P + \Pi + \Lambda_d)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_d + \Lambda_{a_1}).$$

From (4d) one may easily verify that $\bar{m}^0_d(k \geq 1) \geq 0$. In particular,

$$\bar{m}^0_{a_1}(k=1) = \tilde{m}_{a_1} > 0 \tag{S45}$$

from (S42), while $\bar{m}^0_{a_2,a_3}(k=1) > \tilde{m}_{a_2,a_3} > 0$. (Ins2) enables us to write (note that $a_2 \neq 1$ according to (Ins4))

$$\Lambda_{a_1}(1-\epsilon^2) + \Lambda_{a_2}\epsilon^2 \begin{cases} = \Lambda_3 = \Lambda_2, \text{ if } a_1 = 2, a_2 = 3 \text{ or } a_1 = 3, a_2 = 2, \\ < \Lambda_3, \text{ if } a_1 = 1 \; \forall \; \epsilon \in (0,1) \end{cases}, \tag{S46}$$

which results into

$$\bar{m}_d \geq (\varepsilon + P + \Pi + \Lambda_d)k - \frac{1}{2\tau_\pi}(2\eta + \lambda_{\pi\Pi}\Pi) - \frac{\tau_{\pi\pi}}{4\tau_\pi}(\Lambda_d + \Lambda_3), \tag{S47}$$

and gives $\bar{m}_d(k \geq 1) \geq 0$ due to (4d) and $\bar{m}_{2,3}(k \geq 1) > 0$ because $\tilde{m}_d > 0$ from (S42).

The roots of f are the roots of $\bar{m}_{a_3}$ and the roots in the term in brackets in (S43). Let us define it as

$$\begin{aligned} \tilde{f}(k) &\equiv \bar{m}_{a_1}\bar{m}_{a_2} - \bar{m}_{a_1}R_{a_2}\epsilon^2 - \bar{m}_{a_2}R_{a_1}(1-\epsilon^2) - S_{a_1a_2}\epsilon^2(1-\epsilon^2) \\ &= \bar{m}_{a_1}\bar{m}_{a_2}G(k), \end{aligned} \tag{S48}$$

where

$$G(k) = 1 - \frac{R_{a_2}\epsilon^2}{\bar{m}_{a_2}} - \frac{R_{a_1}(1-\epsilon^2)}{\bar{m}_{a_1}} - \frac{S_{a_1a_2}\epsilon^2(1-\epsilon^2)}{\bar{m}_{a_1}\bar{m}_{a_2}}. \tag{S49}$$

Note that since $\epsilon \in (0,1)$, the terms $\bar{m}_{a_1,a_2}(\bar{k})$ cannot be zero if $\bar{k}$ is a root of $\tilde{f}$ due to the term $S_{a_1a_2}$. Also, because $\tilde{f}(k) = (\rho + \Lambda_{a_1})(\rho + \Lambda_{a_2})k^2 + \mathcal{O}(k)$ is a positive function to the right of the larger real root due to (Ins3), then $\tilde{f}(k > 1) > 0$, or equivalently $G(k > 1) > 0$, guarantees that there is no real root for $k > 1$. Because (Ins1) leads to $R_a, S_{ab} > 0$, and since $\bar{m}_a(k > 1) > \bar{m}_a(k = 1)$, then condition $G(k > 1) > 0$ is equivalent to $G(k = 1) \geq 0$. In other words we must have that

$$1 - \frac{R_{a_2}\epsilon^2}{\bar{m}_{a_2}(k=1)} - \frac{R_{a_1}(1-\epsilon^2)}{\bar{m}_{a_1}(k=1)} - \frac{S_{a_1a_2}\epsilon^2(1-\epsilon^2)}{\bar{m}_{a_1}(k=1)\bar{m}_{a_2}(k=1)} \geq 0. \tag{S50}$$

Since $\epsilon < 1$ we can expand (S45) in powers of it and, after using (S45) and (Ins3), obtain the causality condition

$$\left\{1 - \frac{\tau_{\pi\pi}}{4\tau_\pi\tilde{m}_{a_1}}(\Lambda_{a_2} - \Lambda_{a_1}) - \frac{R_{a_2}}{\bar{m}^0_{a_2}(k=1)} - \frac{S_{a_1a_2}}{\tilde{m}_{a_1}\bar{m}^0_{a_2}(k=1)}\right\}\epsilon^2 + \mathcal{O}(\epsilon^4) \geq 0. \tag{S51}$$

Now, by writing

$$\bar{m}^0_{a_2}(k=1) = \tilde{m}_{a_1} + (\Lambda_{a_2} - \Lambda_{a_1})\left(1 - \frac{\tau_{\pi\pi}}{4\tau_\pi}\right)$$

and

$$R_{a_2} = R_{a_1} + (\Lambda_{a_2} - \Lambda_{a_1})\left(c_s^2 + \frac{\tau_{\pi\pi}}{12\tau_\pi} + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\right),$$

and by means of (Ins3) we may rewrite

$$1 - \frac{R_{a_2}}{\bar{m}^0_{a_2}(k=1)} = \frac{\Lambda_{a_2} - \Lambda_{a_1}}{\bar{m}^0_{a_2}(k=1)}\left(1 - c_s^2 - \frac{\tau_{\pi\pi}}{3\tau_\pi} - \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\right) \leq 0. \tag{S52}$$

Note that (S52) is negative or zero because of (Ins2), (Ins3), and (Ins4). From (Ins2) and (Ins4), if $a_1 = 2, 3$, then $a_2 = 3, 2$ and $\Lambda_{a_2} - \Lambda_{a_1} = 0$ while if $a_1 = 1$, then $a_2 = 2, 3$ and $\Lambda_{a_2} - \Lambda_1 > 0$, resulting in $\Lambda_{a_2} - \Lambda_{a_1} \geq 0$, while (Ins1) makes (S52) negative or zero. As a consequence of (S52), the term proportional to ϵ^2 in the LHS of (S51) is negative and, for some small value of $\epsilon \in (0,1)$ it must become the leading term, turning the LHS of (S51) strictly negative. Then, one concludes that the system is not causal and the necessary conditions (4) are not sufficient.

VI. FORMAL DEFINITION OF CAUSALITY AND CONDITIONS (C1) AND (C2)

Here we present the precise mathematical definition of causality and how it relates to conditions (C1) and (C2). Causality can be defined as follows (see [5, page 620] or [6, Theorem 10.1.3] for more details).

Definition 4. Let $(\mathcal{M}, g)$ be the Minkowski space. Consider in $\mathcal{M}$ a system of partial differential equations for an unknown ψ, which we write as $\mathcal{P}\psi = 0$, where $\mathcal{P}$ is a differential operator (which is allowed to depend on ψ)[3]. Let φ be a solution to the system. We say that φ is causal if the following holds true: given a Cauchy surface $\Sigma \subset \mathcal{M}$, for any point x in the future of Σ, $\varphi(x)$ depends only on $\varphi|_{J^-(x)\cap\Sigma}$, where $J^-(x)$ is the causal past of x.

The case of most interest is when the Cauchy surface is the hypersurface $\{t = 0\}$ where initial data is prescribed. We also note that since we are working in Minkowski space, $J^-(x)$ is simply the past light-cone with vertex at x. The situation in Definition 4 is illustrated in Fig. 1. In particular, causality implies that $\varphi(x)$ remains unchanged if the the values of φ along Σ are altered only outside $J^-(x)\cap\Sigma$. Observe that this definition says that $\varphi(x)$ can only be influenced by points in the past of x that are causally connected to x, so no information is allowed to propagate faster than the speed of light.

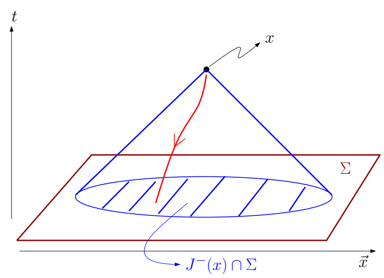

FIG. 1: (color online) Illustration of causality. $J^-(x)$ is the past light-cone with vertex at x. Points inside $J^-(x)$ can be joined to a point x in space-time by a causal past directed curve (e.g. the red line). The value of $\varphi(x)$ depends only on $\varphi|_{J^-(x)\cap\Sigma}$.

The connection between Definition 4 and conditions (C1) and (C2) is via the characteristics of the system $\mathcal{P}\psi = 0$. It is beyond the scope of this Supplemental Material to provide a detailed description of the connections between Definition 4 and the system's characteristics. We refer readers to Appendix A of [2], [7, Chapter VI], and [8]. Here, we restrict ourselves to the following comments. Finite speed of propagation is a property of hyperbolic equations. For such equations, there exist domains of dependence that show precisely how the value of a solution at a point x is determined solely by values within a domain of dependence in the past with "vertex" at x (this is exactly the generalization of the past light-cone). The domain of dependence, in turn, is determined by the system's characteristics. While it is mathematically possible for hyperbolic equations to exhibit domains of dependence where information propagates faster than the speed of light (see, again, our discussion in the Conclusion), for solutions to be causal (i.e., to not have faster-than-light signals), the domains of dependence must always lie inside the light-cones. This is equivalent to the statement (C1) and (C2) that we have used. Definition 4 can be generalized to arbitrary globally hyperbolic spaces, which is needed for the aforementioned generalization of our Theorems to this setting. Again, we refer to Appendix A of [2], [7, Chapter VI], and [8].

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[3] In coordinates, this system of PDEs would be represented by $P^I_K \psi^K = 0$, $I, K = 1, \ldots, N$, where $\{\psi^K\}_{K=1}^N$ are local representations of ψ, and $\mathcal{P}^I_K$ are differential operators (possibly depending on ψ^K).

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First-Order General-Relativistic Viscous Fluid Dynamics

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We present the first generalization of Navier-Stokes theory to relativity that satisfies all of the following properties: (a) the system coupled to Einstein's equations is causal and strongly hyperbolic; (b) equilibrium states are stable; (c) all leading dissipative contributions are present, i.e., shear viscosity, bulk viscosity, and thermal conductivity; (d) nonzero baryon number is included; (e) entropy production is non-negative in the regime of validity of the theory; (f) all of the above hold in the nonlinear regime without any simplifying symmetry assumptions. These properties are accomplished using a generalization of Eckart's theory containing only the hydrodynamic variables, so that no new extended degrees of freedom are needed as in Müller-Israel-Stewart theories. Property (b), in particular, follows from a more general result that we also establish, namely, sufficient conditions that when added to stability in the fluid's rest frame imply stability in any reference frame obtained via a Lorentz transformation All of our results are mathematically rigorously established. The framework presented here provides the starting point for systematic investigations of general-relativistic viscous phenomena in neutron star mergers.

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I. INTRODUCTION

Relativistic fluid dynamics has been successfully used as an effective description of long wavelength, long time phenomena in a multitude of different physical systems, ranging from cosmology [1] to astrophysics [2] and also high-energy nuclear physics [3]. In the latter, relativistic viscous fluid dynamics has played an essential role in the description of the dynamical evolution of the quark-gluon plasma formed in ultrarelativistic heavy-ion collisions [4] and also in the quantitative extraction of its transport properties (see, for instance, Ref. [5]). More recently, with the observation of binary neutron star mergers [6–8], the modeling of the different dynamical stages experienced by the hot and dense matter formed in these collisions requires extending of our current understanding of viscous fluids toward the strong gravity regime where general relativistic effects are important (see, e.g., Refs. [9–14]).

The ubiquitousness of fluid dynamics stems from the existence of general conservation laws (such as energy, momentum, and baryon number) and their consequences to systems where there is a large separation of scales, such that the macroscopic behavior of conserved quantities can be understood without precise knowledge of all the details that govern the system's underlying microscopic properties [15]. Ideal fluid dynamics is the extreme situation where dissipative effects are neglected and the theory's basic properties in this limit are reasonably well understood, both in a fixed background as well as when coupling to Einstein's equations is taken into account [2,16,17]. We remark that because all sources of dissipation relevant for our discussion stem from bulk viscosity, shear viscosity, and heat conduction, and following standard practice in the field [3], we will use the terms viscous fluid and dissipative fluid interchangeably. In particular, other sources of dissipation, such as anomalous dissipation [18,19], will not be discussed.

When dissipative effects are taken into account, the behavior of fluids is far less understood (unless stated otherwise, fluids, hydrodynamics, etc., henceforth mean relativistic fluids, relativistic hydrodynamics, etc.), despite the importance of viscous dissipation in cutting-edge scientific experiments such as in studies of the quark-gluon plasma or their expected relevance for neutron star mergers, as mentioned above. Historically, a stumbling block has been the difficulty of modeling dissipative phenomena while

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preserving *causality*. Causality is a central postulate in special and general relativity, stating that the speed which information can propagate in any system cannot be larger than the speed of light [20]. This implies that a solution to the equations of motion at a given spacetime point x is completely determined by the spacetime region that is in the past of and causally connected to x [20–22]. Of course, this property must hold in relativity regardless of whether dissipation is present or not [20]. While causality is typically not an issue for most matter models under reasonable assumptions [21], including the case of ideal fluids [23], ensuring causality of fluid theories in the presence of dissipation turned out to be a major challenge [2].

The challenges one encounters when modeling fluids with dissipation, however, are not restricted to enforcing causality. Another hallmark property of dissipative fluids is *stability*. By this we mean that perturbations of a system that is in thermodynamic equilibrium should decay in time. This expresses the basic intuition that if dissipation is present, the system will dissipate energy and, consequently, small deviations from equilibrium will be damped, leading the dynamics to return to equilibrium within some characteristic timescale. Naturally, in order to implement this idea in a given formalism one needs to specify what is meant by equilibrium and perturbations. We will consider homogeneous (nonrotating) equilibrium states and our perturbations will be plane-wave solutions to the equations of motion linearized about such homogeneous states. Although this is not the most general definition of equilibrium [24], it captures the most basic intuition about how deviations from equilibrium should behave in a dissipative theory and, consequently, in practice this has been the definition most often used in the literature [25,26]. Like causality, stability is a property that is difficult to incorporate in theories of relativistic fluids with dissipation.

Aside from causality and stability, a third fundamental property required for a theory of relativistic viscous fluids is that the equations of motion be *locally well posed*. This means that given initial conditions, there must exist one and only one solution to the equations of motion taking the prescribed initial conditions [27] and defined for some time [28]. Physically, this means that the system has a well-defined evolution determined by the initial conditions. Like causality, local well posedness is a property required of any field theory [2,20–22], but we emphasize it here since, also like causality, this is a property that is difficult to achieve in theories of fluids with dissipation.

Needless to say, there is little use for a theory of fluids that is causal, stable, and locally well posed if it is not able to make connections with real physical phenomena. Thus, a theory of relativistic viscous fluids must in addition be suitable for empirical studies. This means, at the least, that the theory must agree with well-established physical facts, but also that one needs to be able to extract quantitative predictions from such a theory.

The interplay between theory and experiment is, of course, at the heart of physics. In the context of relativistic fluid dynamics, such interplay has been heavily guided by complex numerical simulations [3]. Moreover, it is clear that simulations will continue to be at the center of developments in the field, particularly when it comes to the investigations of viscous effects in neutron star mergers. In this regard, while there is no one-size-fits-all approach for implementing numerical simulations of general relativistic systems [2,29], in the numerical general relativity community one concept that has been very important for the construction of numerical algorithms is that of *strong hyperbolicity* [30]. This means that the principal part of the equations of motion can be diagonalized; see Sec. V for details. Although a discussion of the role of strong hyperbolicity in general relativistic numerical simulations is beyond the scope of this work (the reader can consult the above references for details), we stress that strong hyperbolicity is a highly desirable feature for numerical studies of general relativistic systems (see also Ref. [31] for more discussion on potential caveats of numerical simulations).

In summary, a physically meaningful theory of relativistic viscous fluids must be (I) causal, (II) stable, and (III) locally well posed. In addition, it is highly desirable to have a theory that is (IV) strong hyperbolic.

While property II is, by definition, concerned with the equations linearized about equilibrium in Minkowski background, we emphasize that whenever referring to causality, local well posedness, and strong hyperbolicity, i.e., properties I, III, and IV, we are always talking about the equations of motion in the *full nonlinear regime*. It is important to stress this point because a substantial body of theoretical work in relativistic viscous fluids is restricted to analyzing the equations linearized about equilibrium and, thus, the corresponding claims about causality, local well posedness, etc., are restricted to this particular, linearized-about-equilibrium case (see Sec. II B). Furthermore, for applications in general relativity (in particular the study of viscous effects in neutron star mergers), one is interested in the case where properties I–IV hold with dynamical coupling to Eintein's equations (again, with exception of property II).

At this point, we should stress that when we say that a theory is causal, stable, etc., we do not mean it unconditionally, but rather under a specific set of assumptions. Obviously, one is interested in cases where the assumptions are physically reasonable, even if they do not cover all cases of physical interest. For simplicity, however, in the remaining of this Introduction and in Sec. II, we avoid discussion of specific hypotheses. Thus, when we say that a certain theory is causal, stable, etc., we mean "causal under a specific set of assumptions," and unless stated otherwise, it will be implicitly understood that the assumptions in question are of physical interest. An exception to this will be made only later in Sec. II B, when we summarize the extent to which different theories of viscous fluids satisfy one or more of the properties I–IV, since in this case mentioning the assumptions under which such theories fulfill some of these requirements will be important for comparison among them and also with our results. Even in this case, however, we will refer to those assumptions only at a high level (e.g., we will say that a certain property holds for nonzero shear viscosity but without specifying the precise range of nonzero values that is in fact required for the result to hold). We believe that this will suffice to give the reader a panoramic view of the state of affairs in the field. All the precise assumptions for the results that will be discussed can be found in the references we provide or, in the case of the results of this paper, in the remainder of the text.

The goal of this work is to provide the first example of a theory of relativistic viscous fluids that simultaneously satisfies all the properties I–IV. All our results are mathematically rigorous, hold with or without dynamical coupling to Einstein's equations, are valid in the full nonlinear regime, and do not make any symmetry or simplifying assumptions. We establish these results without the need for additional (extended) variables (see Sec. II B for details).

Section II provides a more or less self-contained exposition of our results and how they fit within studies of relativistic fluids with viscosity. We hope that such an exposition will be helpful to readers interested in the subject here investigated but who are not necessarily specialists in all the topics covered by our methods. In order to keep our account as simple as possible, we carry out the discussion in Sec. II at a high level, writing few formulas and omitting several details, but we provide full references for interested readers. More precisely, in Sec. II A, we discuss some important concepts underlying the investigation of relativistic viscous fluids. None of the ideas discussed in Sec. II A are new, but they play a key role in our constructions. Therefore, it is convenient to revisit such ideas here. In Sec. II B, we review the state of affairs in the field regarding properties I–IV. This review is not intended to be exhaustive; rather, our goal is to provide enough context for our results. Finally, in Sec. II C, we provide a summary and discussion of our results. Specialists might skip Sec. II without compromising understanding (although some specialists might still be interested in some aspects of the discussion in Sec. II C).

Definitions.—The spacetime metric $g_{\mu\nu}$ has a mostly plus signature $(-+++)$. Greek indices run from 0 to 3, latin indices from 1 to 3. The spacetime covariant derivative is denoted as ∇_μ. We use natural units, $c = \hbar = k_B = 1$.

A. Organization of the paper

This paper is organized as follows. In Sec. II we provide an overview of our results and the context surrounding them. In Sec. III, we formulate a generalization of Navier-Stokes (NS) theory using the Bemfica-Disconzi-Noronha-Kovtun (BDNK) formalism [32–35]. In Sec. IV, we provide necessary and sufficient conditions that must be fulfilled by the parameters of the theory for causality to hold. In Sec. V, we prove that the full nonlinear system of equations in general relativity is strongly hyperbolic, the solutions are unique, and the initial-value problem is well posed in general relativity. A new theorem concerning the linear stability properties of relativistic fluids in flat spacetime is given in Sec. VI. We employ this theorem in Sec. VII to obtain conditions that ensure that the new theory presented here is stable. The rigorous mathematical proofs of Theorem I, Proposition I, Theorem II, and Theorem III are found in the Appendixes A, B, C, and D, respectively. Our conclusions and outlook can be found in Sec. VIII.

II. BACKGROUND AND DISCUSSION

A. Definition of out-of-equilibrium variables: Hydrodynamic frames

In the modern perspective, relativistic fluid dynamics is understood as an effective theory for the evolution of conserved densities, such as the energy-momentum tensor $T^{\mu\nu}$. (We could include, in this introductory part, other conserved quantities such as the baryon current J^μ and those associated with higher moments. In fact, conservation of J^μ will be implicitly understood later in the discussion of Secs. II A and II B and thereafter since we will often refer to the presence of a chemical potential. For simplicity, however, we will often refer only to $T^{\mu\nu}$ in this part, since this will suffice for the aspects we want to highlight.) To say that $T^{\mu\nu}$ is conserved means that

$$\nabla_\mu T^{\mu\nu} = 0,$$

which provides equations of motion governing the dynamics of the fluid.

The energy-momentum tensor $T^{\mu\nu}$ is understood as the expectation value of the microscopic quantum operator $\hat{T}^{\mu\nu}$, which is an observable that can be defined for any nonequilibrium state. In equilibrium, the state of the system can be parametrized by the temperature $T_{\rm eq}$, the flow velocity $u^\mu_{\rm eq}$ (observe that this is the four-velocity of the fluid, although we will often refer to it simply as the velocity; the fluid velocity is always assumed to be normalized; see Sec. II B), and the chemical potential $\mu_{\rm eq}$. One of the assumptions that forms the basis of a fluid dynamics description is that for states not very far from equilibrium, the physical observable $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$ can still be parametrized in terms of a "temperature" T, a "flow velocity" u^μ, and a "chemical potential" μ that reduce to $T_{\rm eq}$, $u^\mu_{\rm eq}$, and $\mu_{\rm eq}$ in equilibrium. We write quotation marks to emphasize the fact that the quantities T, u^μ, and μ have no first-principles microscopic definitions. Therefore, while it is useful to interpret T, u^μ, and μ as out-of-equilibrium macroscopic temperature, velocity, and chemical potential, since they are close to T_{eq}, u^{μ}_{eq}, and μ_{eq} and reduce to the latter in equilibrium, we should ultimately understand T, u^{μ}, and μ as auxiliary variables that are used to parametrize the physical observable $T^{\mu\nu}$—the latter enjoying a first-principles, microscopic definition even when the system is out of equilibrium.

It follows that there exists an ambiguity in the definition of the out-of-equilibrium quantities T, u^{μ}, and μ, since there are different ways of parametrizing $T^{\mu\nu}$ subject to the constraint that one recovers the unambiguous parametrization in terms of T_{eq}, u^{μ}_{eq}, and μ_{eq} in equilibrium. In other words, different *out-of-equilibrium* choices of T, u^{μ}, and μ to parametrize $T^{\mu\nu}$ are allowed as long as they agree in equilibrium. This is sometimes expressed by saying that T, u^{μ}, and μ correspond to a "fictitious" temperature, flow velocity, and chemical potential [2,36].

A particular choice of parametrization of $T^{\mu\nu}$ in terms of T, u^{μ}, and μ has been historically called a choice of a *hydrodynamic frame,* or simply frame. (It is a bit unfortunate the word "frame" has also other meanings in relativity theory, e.g., reference frames related by a Lorentz transformation, frames in a tetrad formalism, null frames, or a local rest frame (LRF), etc. However, all these different meanings can be distinguished from the context.) A choice of frame is, therefore, a definition of what one means by temperature, velocity, and chemical potential out of equilibrium. Consequently, a choice of frame is always involved whenever we describe a fluid out of equilibrium in terms of temperature, velocity, and chemical potential. This is still the case even if further, extended variables are introduced (see Sec. II B for the notion of extended variables). The notion of hydrodynamic frame and how it represents a choice of out-of-equilibrium variables is discussed extensively in the literature. An incomplete list is given by Refs. [34,36–51]. References [34,48], in particular, contain a detailed discussion of the topic.

Observe that once $T^{\mu\nu}$ is cast in terms of T, u^{μ}, and μ, the energy-momentum conservation equations $\nabla_{\mu}T^{\mu\nu} = 0$ can be equivalently written as evolution equations for those quantities. We also remark that one can choose other thermodynamic quantities, e.g., the energy density or the pressure, to parametrize $T^{\mu\nu}$, and we will in fact do so later on in the paper. Of course, not all thermodynamic scalars are independent; they are connected by the first law of thermodynamics and a prescription of an equation of state [2]. Obviously, the nonuniqueness in the definition of the variables used to parametrize $T^{\mu\nu}$ out of equilibrium remains if we choose a parametrization in terms of other thermodynamic variables such as the energy density, pressure, etc.

In order to pass from this qualitative argument about the ambiguity of T, u^{μ}, and μ away from equilibrium to a more precise assessment of such ambiguity, one needs to be more specific about how one formalizes the idea that fluid dynamics arises as a long time, long wavelength limit of an underlying microscopic theory, i.e., as a description of the macroscopic dynamics of the system for small deviations from equilibrium. Such a formalization can be accomplished in the framework of the so-called *gradient expansion,* which was used a century ago by Chapman and Enskog in the derivation of fluid dynamics from the (nonrelativistic) Boltzmann equation and that has since then been adapted to the relativistic setting [38]. We remark that the gradient expansion is not the only way to formalize the idea that fluid dynamics is an effective description that emerges from a more fundamental microscopic behavior; see Sec. II B for a discussion of ideas involving the so-called moment expansion and holographic techniques. Nevertheless, the gradient expansion, while not fundamental, is a very convenient and powerful formalism based on effective field theory ideas that allows one to track how different parametrizations of $T^{\mu\nu}$ lead to different fluid descriptions.

The gradient expansion is based on the idea that one can write

$$T^{\mu\nu} = \mathcal{O}(1) + \mathcal{O}(\partial) + \mathcal{O}(\partial^2) + \cdots,$$

where $\mathcal{O}(\partial^n)$ denotes terms with n derivatives of T, u^{μ}, and μ [so, e.g., $\mathcal{O}(\partial^2)$ involves both terms of the form $\partial^2 T$ and $\partial T \partial\mu$, etc.] and $\mathcal{O}(1)$ corresponds to the terms that reduce to $T^{\mu\nu}_{\mathrm{eq}}$, the energy-momentum tensor parametrized in terms of T_{eq}, u^{μ}_{eq}, and μ_{eq}. Schematically, this is an expansion in powers of the Knudsen number $\mathrm{Kn} \sim \ell_{\mathrm{micro}}\partial$, i.e., the ratio between the relevant microscopic scale ℓ_{micro} and the inverse macroscopic scale L, associated with the derivative of the hydrodynamic fields. In this sense, the gradient expansion corresponds to the well-known Knudsen number expansion used in the description of kinetic systems [38,39]. In particular, since the expansion truncated at $\mathcal{O}(1)$ corresponds to ideal hydrodynamics, viscous contributions require considering at least $\mathcal{O}(\partial)$ terms, which is consistent with the basic intuition that dissipation is a phenomenon associated with deviations from equilibrium.

In order to construct a fluid theory out of the gradient expansion, one truncates it at a certain order. This truncation necessarily defines a scale at which the effective description is supposed to be valid, with higher-order effects encoded by the terms neglected in the expansion which are considered outside the limit of validity of the truncated theory. Aside from the truncation order, one also needs to specify the *constitutive relations,* i.e., the specific form of each term $\mathcal{O}(\partial^n)$ in terms of T, u^{μ}, μ, up to the truncation order (see Secs. II B and III for examples). By specifying the truncation order and the constitutive relations, one is in fact defining what is meant by T, u^{μ}, and μ out of equilibrium; i.e., one is making a choice of hydrodynamic frame.

Different frame choices, therefore, correspond to different effective descriptions of the same truncated theory. At this point, it seems almost unnecessary to talk about frames, and one might be tempted to simply say that one has distinct theories of fluids. The key word here, however, is *effective*. Indeed, when we consider two distinct constitutive relations truncated at a given order,

$$T^{\mu\nu} = T^{\mu\nu}(T, u^\alpha, \mu) \quad \text{and} \quad \tilde{T}^{\mu\nu} = \tilde{T}^{\mu\nu}(\tilde{T}, \tilde{u}^\alpha, \tilde{\mu}),$$

one obviously has different fluid theories: the equations of motion $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu \tilde{T}^{\mu\nu} = 0$ are not the same. Consequently (upon writing these conservation laws in terms of T, u^α, μ and $\tilde{T}, \tilde{u}^\alpha, \tilde{\mu}$, respectively), the quantities T, u^α, μ and $\tilde{T}, \tilde{u}^\alpha, \tilde{\mu}$ satisfy different evolution equations and, thus, cannot represent the same definition of temperature, fluid velocity, and chemical potential. However, one needs to keep in mind that the temperature, flow velocity, and chemical potential are not fundamental quantities, whereas the energy-momentum tensor is (it does have a first-principles definition). Thus, $T^{\mu\nu}(T, u^\alpha, \mu)$ and $\tilde{T}^{\mu\nu}(\tilde{T}, \tilde{u}^\alpha, \tilde{\mu})$ differ because they represent *distinct* coarse-grained or low-energy limits of the actual, microscopically uniquely defined, energy-momentum tensor. Therefore, the language of frames signals the key fact that one is always considering one possible effective description among many.

Summarizing, there exists an intrinsic ambiguity in how one parametrizes $T^{\mu\nu}$ in terms of out-of-equilibrium temperature T, velocity u^μ, and chemical potential μ. Such ambiguity simply expresses the fact that these quantities do not have first-principles microscopic definitions away from equilibrium. What is not ambiguous away from equilibrium is the definition of $T^{\mu\nu}$. One resolves this ambiguity by choosing a definition of T, u^μ, μ. Such a choice is known as a frame choice. Different parametrizations of $T^{\mu\nu}$, therefore, correspond to different frame choices. Not all frame choices, however, are equally useful. In our work, we explore suitable definitions of temperature, flow velocity, and chemical potential to construct effective theories describing fluids that lead to sensible theories in terms of satisfying properties I–IV.

At this point, the attentive reader will probably have noticed that much of the above discussion does not depend on relativistic principles. In other words, the fact that there is no first-principles definition of out-of-equilibrium quantities such as temperature, flow velocity, and chemical potential applies to nonrelativistic theories as well. In the nonrelativistic setting, however, there exists a highly successful theory of dissipative (Newtonian) fluids, namely, the Navier-Stokes-Fourier theory. In light of its success, it is fair to say that for all practical purposes, one can take the definitions of out-of-equilibrium quantities in the Navier-Stokes-Fourier theory as the correct ones in a nonrelativistic context. Had an equivalently successful theory of relativistic viscous fluids been available (where success would in particular incorporate properties I–IV), we could similarly take the definitions of out-of-equilibrium quantities in such a theory as the correct ones for all practical purposes. Nevertheless, as we explain in the next section, there is not, at the moment, a theory of relativistic viscous fluids that can claim such a level of success. Hence, exploring how different frame choices can lead to different fluid descriptions becomes a topic of uttermost interest (see Sec. II C).

B. Brief overview of viscous theories

The first proposal for a relativistic viscous fluid theory was done by Eckart [52] in 1940, with a closely related formulation by Landau and Lifshitz [15] in the 1950s. In these works, the authors postulated a form for the energy-momentum tensor (and also of the baryon current J^μ, but, as in the previous section, here we simplify the discussion by focusing on $T^{\mu\nu}$ only) based on ideas from thermodynamics and following a covariant generalization of the nonrelativistic Navier-Stokes-Fourier theory. For example, in Eckart's theory, one has

$$T^{\mu\nu}_{\text{Eckart}} = \varepsilon u^\mu u^\nu + (P - \zeta \nabla_\lambda u^\lambda)\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu - 2\eta\sigma^{\mu\nu},$$

where ε, T, and u^μ are the (out-of-equilibrium) energy density [53], temperature, and velocity of the fluid, with the latter normalized [54] by $u^\mu u_\mu = -1$, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection onto the space orthogonal to u^μ, P is the equilibrium pressure (see below) given by an equation of state (the choice of which depends on the nature of the fluid; for example, for a conformal fluid one has $P = \frac{1}{3}\varepsilon$), ζ is the coefficient of bulk viscosity, η is the coefficient of shear viscosity, $q_\mu = -\kappa T(\Delta^\nu_\mu \nabla_\nu \ln T + u^\nu \nabla_\nu u_\mu)$ represents energy diffusion, with κ being the coefficient of heat conduction, and $\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta}\nabla_\alpha u_\beta$ is the shear tensor, with $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})$ (so $\Delta^{\mu\nu}_{\alpha\beta}$ projects a two-tensor on the space of two-tensors traceless and orthogonal to u^μ). In the absence of viscous effects, when $\zeta = \eta = \kappa = 0$, one recovers the energy-momentum tensor of an ideal fluid.

According to the standard physical interpretation of the energy-momentum tensor of a fluid, the fluid's total pressure is given by $\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$. It is convenient to write the total pressure as a sum of an "equilibrium" part, which is assumed to be given by an equation of state whose functional form follows that assigned to the fluid in the limit when viscous effects are absent, and a "nonequilibrium" part that contains explicitly the viscous contributions. In the case of $T^{\mu\nu}_{\text{Eckart}}$, the latter is given by $-\zeta\nabla_\mu u^\mu$. This term clearly illustrates the fact that only terms of first order in Knudsen number were kept in this case because ζ/P gives the relevant microscopic length scale associated with particle-number changing processes, while $\nabla_\mu u^\mu$ accounts for the inverse length scale associated with the gradient of the hydrodynamic fields.

As said, Eckart and Landau and Lifshitz were seeking a covariant version of the nonrelativistic Navier-Stokes equation compatible with thermodynamic principles, most notably, the second law of thermodynamics; i.e., their choice of $T^{\mu\nu}$ ensured that entropy production (for a suitable definition of out-of-equilibrium entropy) is non-negative. From a modern perspective, however, these theories are better understood as effective theories that arise from a gradient expansion truncated at first order and with a specific choice of hydrodynamic frame, i.e., a specific choice of constitutive relation that parametrizes the energy-momentum tensor in terms of out-of-equilibrium variables. In fact, it is possible to show that the Eckart and Landau and Lifshitz theories can be obtained from kinetic theory as an expansion in gradients truncated at first order [38]. Constraints on the coefficients that appear in such truncated series are found by imposing the second law of thermodynamics. In accordance with the notion of hydrodynamic frames, the specific choices that lead to the theories of Eckart and of Landau and Lifshitz are known in the literature as the Eckart and Landau and Lifshitz frames [2]. One can immediately see that other frame choices are possible for an energy-momentum tensor truncated at first order upon noticing that $T^{\mu\nu}_{\text{Eckart}}$ does not contain all possible terms that are linear in derivatives of T, u^μ, and μ—terms that are allowed in a truncation at first order. Theories arising from a gradient expansion truncated at first order are known as *first-order theories.* The Eckart and Landau theories are, thus, examples of first-order theories.

The Eckart and Landau and Lifshitz theories are very intuitive and natural at first sight. They correspond to immediate covariant generalizations of the nonrelativistic Navier-Stokes-Fourier theory (in fact, they recover it in the nonrelativistic limit), satisfy the second law of thermodynamics, preserve many features present in the ideal case (e.g., the energy density is recovered from the energy-momentum tensor by double contraction with the velocity), are relatively simple, and, as already said, can be derived from kinetic theory. Yet, they are remarkably at odds with fundamental physical principles in that they are known to violate causality and are unstable [25,55]. Consequently, the Eckart and Landau and Lifshitz theories cannot be taken as viable theories of relativistic viscous fluids. In fact, a large class of first-order theories, of which the theories of Eckart and of Landau and Lifshitz are particular cases, are known to be acausal and unstable [25]. One naturally wonders what are the root causes of the failures of these theories, especially when at first sight they look very intuitive. We return to this point in Sec. II C.

A different approach for the construction of relativistic viscous fluid theories was taken by Israel and Stewart in a series of works [36,37,56–58], adapting ideas developed by Müller in the nonrelativistic setting [59]. The resulting theory is referred to Israel-Stewart or Müller-Israel-Stewart (MIS) theory, or sometimes simply Israel-Stewart theory. In the MIS theory, the energy momentum takes the form

$$T^{\mu\nu}_{\text{MIS}} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \pi^{\mu\nu}.$$

The quantities Π, $\pi^{\mu\nu}$, and $\mathcal{Q}^\mu$ represent the bulk viscosity, shear viscosity, and energy diffusion of the fluid, and are referred to as viscous fluxes. We see that $T^{\mu\nu}_{\text{Eckart}}$ corresponds to the choices where the bulk scalar $\Pi = -\zeta\nabla_\mu u^\mu$, the shear-stress tensor is given by $\pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$, and the energy diffusion reads $\mathcal{Q}^\mu = q^\mu \equiv -\kappa T(\Delta^\nu_\mu \nabla_\nu \ln T + u^\nu \nabla_\nu u_\mu)$. In the MIS theory, however, the viscous fluxes are taken to be new variables on the same par as the "ordinary" variables T, u^μ, etc., (see below). Because $\Pi, \pi^{\mu\nu}, \mathcal{Q}^\mu$ add to the number of variables, hence extending the state space, they are known as extended (thermodynamic) variables and theories that investigate extended variables are referred to as extended (thermodynamic) theories [60,61]. An important point to make (already alluded to earlier) is that one cannot dispense with a choice of hydrodynamic frame even in extended theories, since one still needs to make a definition of out-of-equilibrium temperature, flow velocity, and chemical potential.

At this point, it is convenient to make the following definition. The variables T, u^μ, μ and those derived from them via the first law of thermodynamics and a choice of equation of state are known as *hydrodynamic variables or fields.* In other words, the hydrodynamic variables are the "ordinary" fields already present in the case of an ideal fluid (although the physical interpretation of these variables is not precisely the same as in the ideal fluid case; as discussed, the meaning of, e.g., temperature is different in or out of equilibrium). In this language, we can say that the Eckart and Landau and Lifshitz theories involve only the hydrodynamic variables, whereas the MIS theory involves both hydrodynamic and extended fields. In addition, the gradient expansion is always an expansion in the hydrodynamic variables [62].

Because the MIS formalism introduces new variables in addition to the hydrodynamic fields, it also requires new equations of motion besides the standard conservation laws such as $\nabla_\mu T^{\mu\nu}_{\text{MIS}} = 0$. The desired equations are postulated to be relaxation-type equations whose precise form is chosen so that entropy production is non-negative—where the entropy current is also extended from its usual form used in ideal fluids to include the extended variables $\Pi, \pi^{\mu\nu}, \mathcal{Q}^\mu$. For example, Π satisfies

$$\tau_\Pi u^\mu \nabla_\mu \Pi + \Pi = -\zeta\nabla_\mu u^\mu - \frac{1}{2}\zeta T \Pi \nabla_\mu \left(\frac{\tau_\Pi}{\zeta T} u^\mu\right),$$

where τ_Π is a relaxation time. See, e.g., Ref. [2] for the full set of equations satisfied by $\Pi, \pi^{\mu\nu}, \mathcal{Q}^\mu$, the form of the entropy current including these fields, and the derivation of the equations of motion from the second law of thermodynamics [63].

The MIS theory enjoys the following good properties: the equations of motion are stable, thus satisfying property II, and their linearization about equilibrium states is causal, thus satisfying property I [24,64]. Also, it can, in certain limits, be derived from kinetic theory [2,57,58].

We next discuss three other theories of great interest that employ extended variables: Denicol-Niemi-Molnar-Rischke (DNMR), resumed Baier-Romatschke-Son-Starinets-Stephanov (rBRSSS), and anisotropic hydrodynamics (AHYDRO) theories. The DNMR theory is an effective theory derived from kinetic theory via an expansion in moments [65]. The moment expansion goes back to Grad in his work on nonrelativistic fluids [66,67]. Applying this formalism to the relativistic Boltzmann equations, together with a new power-counting scheme involving Knudsen and inverse Reynolds number expansions, DNMR arrived at a set of equations for the hydrodynamic fields and a set of extended variables Π, $\pi^{\mu\nu}$, and $\mathcal{Q}^{\mu}$ that represent the bulk viscosity, shear viscosity, and energy diffusion, similarly to the MIS equations. Also similar to the MIS equation is the fact that the equations satisfied by the viscous fluxes in the DNMR theory are relaxation-type equations. Despite their similarities, it is important to stress that the MIS and DNMR equations are not the same.

The DNMR theory enjoys many good properties. It is stable and its linearization about equilibrium states [68] is causal [26]. When only bulk viscosity is present, the DNMR theory is causal, locally well posed, and strongly hyperbolic; these properties hold with and without dynamical coupling to Einstein's equations [71]. When all viscous fluxes are present, but chemical potential is absent, the DNMR equations have recently been shown to be causal (again, with or without coupling to Einstein's equations) [72] (see Refs. [26,73,74] for related results under symmetry assumptions). Hence, property II holds in general for the DNMR equations; properties I, III, and IV hold if shear viscosity and heat conduction are absent (with or without dynamical coupling to Einstein's equations); and property I holds with all viscous fluxes present but in the absence of chemical potential [75] (with or without dynamical coupling to Einstein's equations). Most importantly, the DNMR theory has been very successful in phenomenological studies of the quark-gluon plasma, particularly in numerical simulations of its dynamical behavior; see, e.g., Refs. [5,76].

We now move to discuss the resumed Baier-Romatschke-Son-Starinets-Stephanov theory [77]. In order to do so, we need to start with the (plain, not resumed) BRSSS theory [77]. This is an effective theory obtained from the gradient expansion truncated at second order. As such, it involves only the hydrodynamic fields, and the equations of motion were chosen in Ref. [77] to be defined in the Landau frame. This effective theory-based approach was originally developed for conformal fluids in Ref. [77], and the same equations of motion for a conformal system were concurrently derived in Ref. [44] through the fluid-gravity correspondence, a powerful technique introduced in that work which was motivated by the holographic duality of string theory [78]. In order to address the issues with causality and stability, Baier *et al.* [77] proposed a MIS-like theory with transport coefficients that ensure its agreement with the gradient expansion at second order. In the context of Ref. [77], this approach provides a resummation of higher-order terms and the latter explains the differences found, for instance, between rBRSSS and DNMR. However, at the linearized level, this resummed BRSSS theory shares the same properties of DNMR. Furthermore, the techniques used in Ref. [72] can be adapted to establish causality for this theory in the nonlinear regime. The local well-posedness and hyperbolicity aspects of rBRSSS have not yet been established.

Because the MIS, DNMR, and rBRSSS theories share many properties, in particular, the use of extended variables that satisfy similar relaxation-type equations, and their linearizations about equilibrium agree, they are sometimes collectively referred to as Israel-Stewart or Müller-Israel-Stewart theories, Israel-Stewart-like or Müller-Israel-Stewart-like theories, or yet generalized Israel-Stewart or Müller-Israel-Stewart theories. They are sometimes also collectively referred to as second-order theories. While there is no harm in grouping these theories together in this fashion, especially if one is concerned only with their general qualitative behavior, it is important to note that when it comes to specific features, including properties I–IV, the exact form of the equations matters and, therefore, the differences among these theories become important.

The fourth extended theory we would like to briefly discuss is the anisotropic hydrodynamics theory [79–83]. The latter is, in principle, more general than most approaches as it investigates the problem of small deviations around a given anisotropic nonequilibrium state. Formally, this approach involves a resummation in both Knudsen and inverse Reynolds numbers, which may be interpreted as a generalization of DNMR's power-counting ideas [84]. The equations of motion, which are in practice derived using kinetic theory, can be approximated to give rise to a MIS-like theory. As such, causality and stability in the linearized regime follow from previous results. Nothing is known about causality in the nonlinear regime of this theory. The local well-posedness and hyperbolicity aspects of AHYDRO have not yet been established.

The above summary highlights how the use of extended variables has led to many successes in the study of relativistic viscous fluids. These accomplishments seem even more impressive when they are contrasted with the fact already mentioned that first-order theories (which do not employ extended variables) had been largely ruled out for decades due to instabilities and lack of causality [25,55].

Such successes nonetheless, it is important to keep in mind several actual or potential limitations of the extended theories discussed above, as we now discuss.

First of all, observe that none of the theories MIS, DNMR, rBRSSS, or AHYDRO is known to satisfy all the properties I–IV. To the extent that they satisfy some of these properties, this happens under *restrictive* assumptions. Indeed, in the case of the quark-gluon plasma it is abundantly clear that one needs to consider situations when all viscous fluxes are present and the chemical potential is nonzero [85] (and it is likely that this is also true in neutron star mergers [12,14]), in which case none of these theories is known to be causal and locally well posed. Moreover, while numerical simulations of the dynamics of the quark-gluon plasma based on the DNMR equations have been carried out for a long time [86–88], only recently, with the aforementioned causality results [71,72], one can determine regions in the parameter and state spaces for which causality holds or fails. When such constraints are taken into consideration, it is found that state-of-the-art numerical simulations of the quark-gluon plasma violate causality [89,90], especially at early times [90]. Although further research is required to find out the implications of such causality violations to our current understanding of those properties of the quark-gluon plasma that have been extracted from numerical simulations, such results should serve as a definite cautionary tale about running numerical simulations of relativistic viscous fluids whose causality properties are poorly understood. Furthermore, if causality violations can be a real issue in numerical simulations of the quark-gluon plasma, which are carried out in flat spacetime, the situation is even more precarious in simulations of general relativistic viscous fluids, such as in neutron star mergers. While some simulations have been implemented in this setting [11], they rely on a formulation for which the key properties I, III, and IV are not known to hold.

Another potential limitation of the extended theories discussed above is that they do not seem appropriate for describing shock waves [91–93]. This is a potentially important limitation given the preponderance of shock waves in fluid dynamics, which is aggravated by the recent discovery that solutions to MIS-like equations can become singular in finite time [94]. Additionally, MIS-like and AHYDRO theories are only expected to describe the transient regime of dilute gases as their derivation is most naturally understood within kinetic theory [36,65]. Therefore, their use in other types of systems, such as in strongly coupled relativistic fluids, is *a priori* not justified. In fact, it is known that MIS-like equations do not generally describe the complex transient regime of holographic strongly coupled gauge theories [95–97] (see Ref. [98] for the case of higher-derivative corrections). In this aspect, we anticipate that the causal and stable first-order theory developed here does not describe this transient regime either, despite satisfying properties I–IV. However, this is not an issue *per se* given that the description of such a far-from-equilibrium state is certainly beyond the regime of applicability of first-order hydrodynamics.

Finally, MIS-like theories lack the degree of universality expected to hold in hydrodynamics as the equations of motion themselves change depending on the derivation. For instance, the equations of motion in Ref. [77] have different terms than in Ref. [65], which is explained by the different power-counting scheme employed in those works. This situation should be contrasted with theories derived from the gradient expansion: although, of course, a plethora of different effective theories can be derived in the gradient expansion formalism, these different theories can always be viewed as particular cases, obtained via different frame choices, of the most general expansion truncated at a certain order. In fact, an approach of this type is employed in this paper; see Sec. II C.

Summarizing, despite its undeniable success in advancing our understanding of relativistic viscous fluids in general, and of the quark-gluon plasma in particular, MIS-like and AHYDRO theories still face many challenges, especially when it comes to settings where general relativity is involved. Thus, it is extremely important to also consider alternative theories of relativistic viscous fluids. This is especially the case when pursuing the study of viscous effects in neutron star mergers [12,14,99,100] and, as already mentioned, it is far from clear that the MIS-like and AHYDRO approach are the correct approaches for this setting.

In view of the above, it is not surprising that researchers have explored other theories of relativistic viscous fluids than those discussed so far. A natural place to start such an investigation is the gradient expansion, and the simplest possibility that includes viscous effects is that of first-order theories, i.e., effective fluid descriptions arising as a truncation of the gradient expansion at first order. On the other hand, since, as said, large classes of first-order theories are acausal and unstable, one might naturally wonder whether such an approach would be doomed to fail. In order to answer this, it is important to understand the assumptions involved. While it is true that the acausality and instability results [25,64] cover large classes of first-order theories, these results apply only to theories that satisfy

$$u_\mu u_\nu T^{\mu\nu} = \varepsilon, \tag{1}$$

i.e., only to frame choices that preserve the relation (1). In other words, the latter means that an observer moving with the fluid always sees the energy density as if it were in equilibrium, even for states where entropy is produced. Therefore, the construction of stable and causal first-order theories remains a distinct possibility as long as one avoids constitutive relations that imply (1). First-order theories for which (1) holds are often collectively referred to as the (relativistic) Navier-Stokes theory [65], although there is no universal agreement on the terminology [35].

The physical meaning of (1), as well as of not satisfying it, is discussed in Sec. II C. We also remark that the assumptions in Refs. [25,64] imply other special relations than (1). But here, for simplicity, we focus only on (1), since our goal is not to have a detailed discussion of the assumptions involved in those works but rather to illustrate how their conclusions apply only for a particular class of theories that employ very specific frame choices and, therefore, say nothing about first-order theories that employ other hydrodynamic frames. In other words, here the reader can take (1) as a placeholder for the class of frames that are assumed in the instability and acausality results [25,64]. Such a class of frames is far from exhaustive. Consequently, the results in Refs. [25,64] simply do not apply if different constitutive relations are used.

This motivated researchers to construct stable and causal first-order theories of viscous fluids. Important attempts in this direction go back to the first decade of this century [21,42,47,101]. The first more formal indication that causal and stable first-order theories could be constructed if the frame choice (1) is avoided is given in Refs. [102–104]. These works were also the first ones to carry out a systematic study of viscous shocks in relativistic theories, a topic that in fact seems to be one of the main goals in these references.

The first construction of a stable and causal first-order theory of viscous fluids was carried out by Bemfica *et al.* [32] for the case of conformal fluids (see also Ref. [105] for some of the mathematical details of Ref. [32]). These results hold with or without dynamical coupling to Einstein's equations. Although Ref. [32] was restricted to conformal fluids, it provided an unequivocal proof that first-order stable and causal theories are possible, provided that one avoids the frame choice (1). Soon thereafter, causal and stable first-order theories were obtained by Kovtun [34] and by Bemfica *et al.* [33] for the case of nonconformal fluids without a chemical potential [106]—although stability was obtained only with the help of a numerical investigation, so it might be more precise to say that stability was only strongly suggested and not established. The resulting first-order theory became known in the literature as the BDNK theory [35]. Its local well posedness and strong hyperbolicity was established in Refs. [107,108]. The stability and causality of the BDNK theory in the presence of a chemical potential was obtained in Ref. [35] (again, stability in this case was inferred only numerically). We also mention the closely related results [109,110]. Of course, all these results are obtained using frame choices different than (1). Perhaps not surprisingly, after these results, the community took a renewed interest in first-order theories. See, e.g., Refs. [51,98,111–119], and references therein. We remark that choices of frames other than (1) have been studied before BDNK in Refs. [42,101,102,120], but, as said, the first construction of a stable and causal first-order theory was done in Ref. [32] in the case of a conformal fluid. We return to the BDNK theory in Sec. II C. In what follows, we continue with our brief review of viscous theories.

Another first-order theory of interest is the Lichnerowicz theory [121], introduced in the 1950s but not investigated in detail until recently (see references that follow). The Lichnerowicz theory has been shown to be causal in the (very special) case of irrotational fluids [122] by the second author of this paper (see also Ref. [123]). While irrotationality is too strong of a constraint to be useful for most physical applications, Ref. [122] is of interest because it initiated the techniques that have since then been employed to study the causality of the BDNK theory, including the techniques employed in this work. We should also mention that the Lichnerowicz theory has found some interesting applications in the study of dissipative cosmological models [124–127].

Another formalism of importance in the study of viscous theories is that of divergence-type (DT) theories [128]. In this approach, all the conserved quantities describing the dynamics of the fluid are obtained from a single generating function χ which is a function of a dynamical set of variables $\zeta_A = (\zeta, \zeta_\mu, \zeta_{\mu\nu})$ (with $\zeta_{\mu\nu}$ trace-free and symmetric) representing the degrees of freedom of the fluid. For example, in the DT approach the energy-momentum tensor is obtained as

$$T^{\mu\nu}_{\rm DT} = \frac{\partial\chi}{\partial\zeta_\mu \partial\zeta_\nu}.$$

DT theories provide a far-reaching subject with many important contributions to the physics of fluids, kinetic theory, and out-of-equilibrium phenomena. Here, we limit ourselves to discuss DT theories with respect to properties I–IV. See Refs. [2,61,92,128–131] for further discussion of DT theories and Refs. [132–134] for applications of DT theories to the quark-gluon plasma.

All information of DT theories is contained in the generating function χ. Unfortunately, there is no prescription on how to construct χ, not to speak of how to construct a generating function that leads to a theory satisfying properties I–IV. In fact, we think it would be more accurate to consider the DT approach as a general formalism instead of a precisely defined theory or set of theories. That is because radically different theories, such as Eckart's and certain types of extended theories, can be cast in divergence-type by the choice of a suitable generating function [128].

Properties I–IV have been investigated in the context of DT theories in Ref. [128]. The authors constructed a DT theory that satisfies properties I–IV for states in equilibrium, i.e., when $\zeta_A = \zeta_A|_{\rm eq}$. Next, they argued that, by continuity, these properties will also hold for ζ_A sufficiently close to $\zeta_A|_{\rm eq}$. However, no estimate is obtained for how close to $\zeta_A|_{\rm eq}$ the state ζ_A needs to be. Thus, given *any* nonequilibrium state ζ_A, this continuity result does not provide any information on whether this specific system satisfies the desired properties I–IV. In particular, without a quantitative estimate on how small $\zeta_A - \zeta_A|_{\rm eq}$ needs to be, one does not know whether the states ζ_A for which properties I–IV hold include states of physical interest. It could in principle happen that this continuity argument only guarantees the desired properties in a neighborhood of $\zeta_A|_{\rm eq}$ that is orders of magnitude smaller than the size of any deviation from equilibrium that one typically considers in viscous fluid dynamics.

Another way of saying this is that the results in Ref. [128] are purely qualitative, not providing a quantitative assessment of their applicability to physical systems. This should be contrasted with the precise quantitative results we establish here (see Secs. IV–VI) and in the predecessor works [32,33,72], which are obtained by employing substantially more refined techniques than a general continuity argument. In Refs. [92,130–132,134], further results have been obtained, but they are all of the same qualitative nature as above, relying on precisely the same continuity argument. Thus, we believe that a fair assessment of DT theories is that they can in principle accommodate properties I–IV, but precise conditions ensuring that such properties hold—in particular, conditions that allow application to concrete physical problems—are yet unknown.

We finally briefly mention recent formulations of viscous fluids [135,136] inspired by Carter's formalism and the variational principle [112]. Such formulations address some of the properties I–IV but do not establish them in completeness.

Although the review here provided is not exhaustive, we believe that it suffices to get across the following main point, namely, despite intense work on the subject and many different proposals made in the last 80 years, one still does not have a theory of relativistic viscous fluids that incorporates all relevant viscous fluxes and chemical potential while satisfying all the properties I–IV. Constructing such a theory is the goal of the present paper.

C. Summary and discussion of our results

In this paper, we consider the BDNK theory with chemical potential and all relevant viscous fluxes, namely, bulk viscosity, shear viscosity, and heat conduction, and show that it satisfies all the properties I–IV, i.e., causality, stability, local well posedness, and strong hyperbolicity. Our results hold in the full nonlinear regime for the fluid equations in a fixed background or dynamically coupled to Einstein's equations. We work in $3+1$ dimensions and do not make any symmetry or simplifying assumptions. As explained in the previous section, this is the first time that a theory of relativistic viscous fluids with all these properties is constructed. In addition, all our results are mathematically rigorous and we provide a set of precise inequalities among scalar quantities (e.g., shear and bulk viscosity) that determine the regions in parameter and state space for which properties I–IV hold. Such inequalities are useful for numerical simulations as they allow us to check, at each time step, whether conditions for causality and stability are fulfilled.

The key conceptual ingredient that allows us to establish our results is the realization that the causality and stability properties of a theory are intrinsically tied to its hydrodynamic frame. This happens because different choices affect the properties of the corresponding partial differential equations (PDEs) that describe the evolution of the fluid. In particular, we avoid the frame choice (1), which in first-order theories leads to acausality and instability. The frame choice (1) has a natural intuitive appeal; namely, it states that the energy density measured by an observer moving with the fluid (i.e., in the fluid's local rest frame), $u_\mu u_\nu T^{\mu\nu}$, can be parametrized by a single scalar that can be identified with the energy density of the fluid in equilibrium [note that (1) holds for an ideal fluid]. It is not surprising, therefore, that Eckart and Landau and Lifshitz adopted frames satisfying (1). On the other hand, such a simplicity in the definition of the hydrodynamic fields out of equilibrium, while desirable, is by no means a fundamental property. The key idea underlying the BDNK theory is that one should let the fundamental principle of causality (and also of stability and local well posedness) dictate which frame choices (i.e., parametrizations of $T^{\mu\nu}$) are allowed, rather than choose a frame based on nonfundamental principles and only then investigate properties such as causality. In passing, we note that the MIS-like theories discussed in this section also adopt (1), although, as just said, other frame choices can be made. Different frames have been recently investigated in the context of extended theories in Refs. [137,138].

The idea of exploring different frame choices to construct a first-order theory that satisfies properties I–IV is not entirely new to this work. It was, in fact, the key idea employed in the earlier versions of the BDNK theory that have been showed to satisfy those properties in some particular cases (see Sec. II B). We next explain what the new aspects of this work are, but in order to do so, we need to first review some other key ideas employed in the earlier constructions of the BDNK theory.

Since we do not want to make premature frame choices, our first step is to consider the most general frame; i.e., we write down the most general expression for $T^{\mu\nu}$ (and also J^μ in the case of the present work since we here consider nonzero chemical potential) compatible with the gradient expansion truncated at first order; see Eqs. (5) and (6) for the precise expression. By considering the most general constitutive relations compatible with the symmetries of the problem as our starting point, we are in fact applying the basic tenets behind the construction of effective theories [48,139–141] to formulate hydrodynamics as a classical effective theory that describes the near equilibrium, long time, long wavelength behavior of many-body systems in terms of the same variables $\{T, \mu, u^\nu\}$ already present in equilibrium. For completeness, we remind the reader that an effective theory is constructed to capture the most general dynamics among low-energy degrees of freedom that is consistent with the assumed symmetries. When this procedure is done using an action principle, the action must include all possible fields consistent with the underlying symmetries up to a given operator dimension and the coefficients of this expansion can then be computed from the underlying microscopic theory. These coefficients are ultimately constrained by general physical principles such as unitarity, CPT (charge, parity, and time reversal) invariance, and vacuum stability. Analogously, in an effective theory formulation of relativistic viscous hydrodynamics, the equations of motion must take into account all the possible terms in the constitutive relations up to a given order in derivatives that describe deviations from equilibrium. The coefficients that appear in this expansion can then be computed from the underlying microscopic theory (using, for instance, linear response theory [48]), being ultimately constrained by general physical principles such as causality in the case of relativistic fluids [20] and also by the fact that the equilibrium state must be stable; i.e., small disturbances from equilibrium in an interacting (unitary) many-body system should decrease with time [142].

Observe that by considering the most general energy-momentum tensor at first order, we are allowing viscous corrections to the equilibrium energy density; i.e., one has

$$u_\mu u_\nu T^{\mu\nu} = \varepsilon + \partial(T, \mu).$$

[See Eq. (7) for the precise expression.] Even though this is in sharp contrast with (1), in hindsight it seems the natural thing to do. After all, it is standard to do precisely the same with the pressure, i.e., to split $\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu}$ into an "equilibrium" part and a "viscous part" (see Sec. II B) [143]. There is no reason not to follow a similar recipe for the energy density seen by a comoving observer.

We next investigate how causality constrains the constitutive relations. The idea that one should let causality determine which frames are allowed in a theory, while conceptually powerful, does not tell us how to in practice find the appropriate frames. Causality of a theory can be determined by computing its characteristics [144]. Roughly, the characteristics are hypersurfaces in spacetime that correspond to the propagation modes of a theory. For example, in the case of Einstein's equations, the characteristics are simply the light cones $g_{\mu\nu}v^\mu v^\nu = 0$. While in principle we can always compute the characteristics of a system of PDEs, in practice a brute-force calculation of the characteristics seems unattainable for a nonlinear system of PDEs as complex as the BDNK system. In order to be able to compute the characteristics, we take a cue from the system's underlying geometric properties. Inspired by structures found in the case of ideal fluids by Disconzi and Speck in Ref. [145], which need to be recovered in the ideal limit, we look for acoustical-metric-like structures. In addition, knowing what the characteristics of the system should be in some particular limit (e.g., in the conformal case that had already been treated) is also helpful to guide the calculations. In the case treated here, in particular, we already know what needs to be recovered in the limit of zero chemical potential. Finally, physical intuition also tells us what kinds of modes of propagation should be present in the system. In a nutshell, by relying on geometrical and physical intuition and an understanding of the causal properties of the theory in some particular limits, we can have a good educated guess for what the characteristics should look like. This allows us to look for a specific factorization of the characteristic determinant that points in that direction. This is the reason why, in our calculations, we group certain terms in certain ways, leading to expressions that can be managed in the end. Naturally, a brute-force approach would not be able to anticipate how one should group and factor terms in a way that would allow an explicit determination of the characteristics.

The next step is to carry out a diagonalization of the principal part of the equations of motion in order to establish strong hyperbolicity. We are able to do so because we have a precise understanding of the system's characteristics. Even so, in order to carry out the diagonalization, we need to write the system as a system of first-order PDEs (notice that $\nabla_\mu T^{\mu\nu} = 0$ is a system of second-order PDEs because $T^{\mu\nu}$ involves up to first derivatives of the hydrodynamic fields). In doing so, there is the risk of introducing spurious characteristics. For example, in the standard linear wave equation the characteristics are the light cones. However, when one writes it as a first-order system in the standard way, the resulting system has a spurious characteristic (it corresponds, in the language of eigenvalues that can be applied to first-order systems, to a zero eigenvalue). While the presence of spurious characteristics *per se* is not an obstacle to diagonalization, the more of them there are, the more likely there will be obstacles to the diagonalization. Thus, we seek to choose as variables for our first-order system quantities that have direct physical or geometrical meaning, so that the roots of the resulting characteristic polynomial resemble as closely as possible the ones of the original system. Of course, this does not guarantee diagonalizability. We still need to carry out some work mostly technical in nature to assure that the system is diagonalizable. But mutilating the equations upon rewriting them as first order by introducing new, fake features is likely to only make the technical work harder or even insurmountable.

With diagonalization at hand, we can proceed to establish local well posedness. The basic idea is that once the system is diagonalized, one can rely on techniques of diagonal systems of PDEs. There is a catch, though. The diagonalization of the system is at the level of the so-called principal symbol (i.e., it is a purely algebraic procedure that does not deal directly with differential operators). In order to apply it to the actual system of PDEs, one needs to introduce pseudo-differential operators, and the quasilinear nature of the equations causes further complications as we need to deal with pseudo-differential operators with limited smoothness. While there are results available in the literature for such situations (see, e.g., Ref. [146]), we have not found a result that could be directly applied to our case. Thus, Bemfica and co-workers developed the necessary tools in Refs. [107,108] with applications to the BDNK equations with zero chemical potential in mind. From these techniques and the diagonalization, local well posedness follows.

Finally, let us address stability. For this, one needs to find the roots of the polynomial determining the Fourier modes of the perturbations. More precisely, only the sign of the roots is relevant. Since the corresponding polynomial is of high order, there is little hope of determining its roots exactly, and even the analysis of the sign of the roots is very challenging. Moreover, differently than what happens to the causality analysis, geometrical intuition is not of much help here because the Fourier modes are not covariant quantities. Because of these difficulties, in previous works the stability of the BDNK equations was not determined rigorously, being obtained numerically or only in the homogeneous Lorentz boosted frame [33,35]. Because of a new result demonstrated in this paper, this limitation is eliminated, as we discuss below.

We are now ready to discuss specific novelties of the present work. While we continue to employ the ideas described above and in fact improve on them, especially with respect to some of the technical aspects that are more challenging for the complete system here considered, we want to highlight what are the truly new aspects introduced in this work. First, we are able to completely and rigorously determine the stability of the system. For this, we rely on a new stability theorem, which roughly says that stability in the fluid's local rest frame (which can in general be determined because in this case the polynomial for the modes simplifies considerably) implies stability in any Lorentz boosted frame provided that the system is causal and strong hyperbolic; see Sec. VI for the precise assumptions and statement of the theorem. The theorem thus establishes a close relationship between causality and stability. While connections between causality and stability have been discussed before, see Refs. [24,26] and references therein, these results focused on specific theories, thus making unclear whether they were due to the specific form of the equations of motion or if they were examples of a yet undiscovered connection between causality and stability as general physical principles. Our theorem, in contrast, is a general theorem that can be applied to many different systems, showing that the relationship between causality and stability runs deeper and is not a feature of specific systems. In fact, we obtain stability of the BDNK system by showing that it satisfies the assumptions of the general theorem.

Interestingly, recently, a related theorem was proven in Ref. [147], albeit using entirely different methods. The results in Ref. [147] also provide further physical intuition on the relationship between causality and stability, showing that lack of causality allows that dissipation in one Lorentz frame be viewed as "antidissipation" (i.e., dissipation running "backward in time") in another Lorentz frame. We also note the related work, Ref. [69]. Combined, our paper and the works of Refs. [69,147] provide a comprehensive picture of the relationship between causality and stability, an idea that was hinted at several times before in the literature (see above references) but that had eluded the community until now.

We now discuss strong hyperbolicity. While strong hyperbolicity has been obtained for the BDNK theory before in the absence of a chemical potential [33,107,108], the introduction of a chemical potential causes new severe difficulties and the approach used in the case without chemical potential does not seem to work. Indeed, in Refs. [33,107,108], the choice of variables to write the system as first order was based primarily on their physical interpretation. For example, the viscous correction to the equilibrium energy density was one of the variables chosen. As just said, a similar approach does not work here. While it is often a good idea to consider variables with a physical meaning, the first-order reduction we seek to establish itself does not need to carry much physical meaning, so an approach employing easily identifiable physical variables might not bear any fruit. The first-order system does carry, however, some intrinsic *geometric* properties, such as natural decompositions in the directions parallel and perpendicular to u^μ or the fact that the characteristics of the original system are preserved by the reduction to first order. Thus, a choice of geometric variables seems more appropriate. That is what we have done, considering new variables that involve several tensorial decompositions of the original variables. This has the extra advantage that several tensorial and geometric properties of the fields can be used to carry out the difficult calculations needed to diagonalize the system. Yet another advantage is that while the previous physical choice of variables was specific to the form of the BDNK equations, the geometric approach is much more general and, thus, can be adapted to other theories in that similar tensorial decompositions hold for several fluids equations. Therefore, a second novel aspect of this work is a new framework to investigate strong hyperbolicity in relativistic fluids. We remark that once the diagonalization is carried out, we can rely on the techniques developed in Refs. [107,108] to establish local well posedness. Thus, while local well posedness is probably the most technical and mathematical aspect of our results, we were able to rely more on previous techniques than any other of the results we obtain here.

In addition, it should by no means be overlooked that, although the proof of causality provided here follows similar ideas as in our earlier work [33], the fact that we are now considering the full set of equations makes the analysis much more difficult. Thus, a third novelty of our work is a substantial improvement of the techniques previously employed to analyze causality. From our causality analysis, it follows that the characteristics of the BDNK theory are the flow lines, sound waves, the so-called second sound, corresponding to the propagation of temperature perturbations [24], and shear waves (plus heat diffusion). In addition, when coupling to Einstein's equations is considered, we find another set of characteristics corresponding to gravitational waves.

Finally, as already stressed many times, the main end product of this paper is itself a major novelty, namely, the first construction of a viscous theory containing all relevant fields and satisfying properties I–IV. We accomplish so by building and expanding on several previous ideas and also by introducing a series of novel ideas, as described above.

Having discussed the new aspects of our work, we move on to discuss how they combine with other aspects of the BDNK theory to provide a promising theoretical tool for the study of general relativistic viscous phenomena. We begin by pointing out that the BDNK theory has been shown to be derivable from kinetic theory and holographic arguments [32,33,148]. While derivation from kinetic theory by itself is not guarantee that a theory is physically meaningful since the coarse-grain procedure might introduce nonphysical features—indeed, recall that the Eckart and Landau-Lifshitz theories are derivable from kinetic theory—it is reassuring to establish this connection with a microscopic theory. As shown in Ref. [148], the derivation of BDNK theory from holography can be done in the context of the fluid-gravity correspondence [44] by carefully taking into account the presence of zero modes of the corresponding differential operators in the holographic bulk.

Next, we should point out that, contrary to MIS-like theories, the BDNK theory is capable of handling shocks. By this, we mean that Rankine-Hugoniot-type conditions can in principle be obtained for the BDNK theory simply due to the fact that the BDNK equations are written as the conservation laws $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu J^\mu = 0$. Aside from this simple observation, viscous shocks have been recently studied for the BDNK theory in the case of a conformal fluid using numerical methods in Ref. [149], while mathematically rigorously properties were established in Ref. [150].

At this point, we need to explain the role of shocks in the BDNK theory. Since the BDNK theory is an effective theory truncated at first order in the gradient expansion, it is expected to be valid when gradients are not very large, which is precisely the opposite of shocks. In order to explain what we mean by a description of shocks in the BDNK formalism, let us consider for a moment an ideal fluid. In this case, one also is assuming that gradients are small. Alternatively, one may also see this as the limit where microscopic length scales are much smaller than the length scales associated with the gradients. However, shocks are known to develop in solutions of ideal hydrodynamics, and the study of shocks is indeed an important topic within the community. To what extent such shocks are accurate depictions of the state of the physical system is a legitimate question. Nevertheless, once we have decided to study shocks in the context of ideal hydrodynamics, the formalism allows us to do so in that the equations of motion of ideal fluids can accommodate weak solutions (also known as distributional solutions) using the Rankine-Hugoniot conditions [23]. The same situation happens with BDNK: the formalism in principle allows for the study of shocks. Whether or not such solutions are physical, or accurate in the sense that the results would change significantly if the formalism was extended to second order, is an important question that is beyond the scope of our paper. However, the point we are making is that we can, in principle, study shock solutions in the BDNK theory.

In other words, while the derivation of BDNK theory rests on the assumption of small gradients, one might try to apply it to situations where in principle gradients are not small (like shocks), just like it was done before in the context of ideal fluids. Although this seems inconsistent, it is precisely what it is done when one employs the equations of ideal fluids to the study of shocks. Moreover, it is also the case that MIS-like theories are often applied to situations where gradients are not so small; see, e.g., Refs. [84,90,151–155]. It is an intriguing, almost philosophical, question why one can sometimes still obtain meaningful results in such cases, even though shocks are formally beyond the regime of validity of any known approach to viscous fluids—an important question, however, that is beyond our scope here.

We now discuss another aspect of importance in viscous theories, which is entropy production. Naturally, one needs the second law of thermodynamics to be satisfied; i.e., entropy production for physically realizable states of the system must be non-negative. Before addressing this point in the BDNK theory, however, some important points need to be highlighted. Strictly speaking, there is no universally understood expression for the entropy of a given system out of equilibrium, aside from the one given by the Boltzmann equation. Thus, while it is useful to define an out-of-equilibrium entropy (which must, of course, reduce to the definition of equilibrium entropy in the absence of dissipation), we need to keep in mind that such a definition is not fundamental or even unique. Moreover, the requirement that entropy production be non-negative on-shell unconditionally, i.e., to all orders in gradients, is certainly too stringent. In fact, since a fluid description is an effective description, it has a certain limit of applicability. Therefore, one should require that entropy production be non-negative only within the regime of validity of the theory (which is constructed within a certain approximation scheme). This point was stressed in Ref. [102] and discussed in detail in Ref. [34]. In fact, enforcing non-negative entropy production even in the presence of any size of gradients was part of the Eckart and Landau and Lifshitz theories, but the resulting theory is unstable and acausal, as seen, showing that this requirement by itself is not guaranteed to lead to sensible theories in the context of the gradient expansion. Non-negative entropy production to all gradients is also a guiding principle in the construction of the MIS theory, but so far properties I, III, and IV remain open for it. On the other hand, the DNMR equations, that are stable, causal (in the absence of chemical potential), and are extensively used in numerical simulations of the quark-gluon plasma, do not have entropy production non-negative to all orders in Knudsen and inverse Reynolds numbers, but they should have non-negative entropy production within the limit of validity of the theory [65]. The same is true for the BDNK theory, as pointed out in Ref. [34] and shown in Sec. III A. A thorough discussion of the role of entropy in viscous theories can be found in Ref. [51].

We finally comment on the ability of the BDNK theory to describe realistic physical systems. In order to go beyond theoretical aspects and make connection with experiments, one needs to carry out realistic numerical simulations of the BDNK equations. Not surprisingly, given how recent the theory is, such investigations are at an initial stage, but the results so far have been encouraging. In Ref. [149], the authors carry out numerical simulations of the BDNK theory in $1+1$ dimensions in the case of a conformal fluid and compare the results with simulations of MIS (rBRSSS) equations in the same setting. They found that for small values of the coefficient of shear viscosity, BDNK and MIS provide essentially the same evolution, but their dynamics differ for larger viscosity values. Given that small viscosity is one of the main regimes of interest of both theories (higher-order corrections might become relevant in both theories if viscosity is not small), this shows that at least in this test case the BDNK theory reproduces the well-studied and considerably successful behavior of MIS theory. In addition, the BDNK theory also reproduces well-known behavior considering Bjorken [156] and Gubser [157–159] flows, including the presence of a hydrodynamic attractor [32]. Further numerical studies of BDNK theory can be found in Refs. [160,161].

We also stress the obvious point that being a causal, stable, and locally well-posed theory are themselves fundamental properties that need to be satisfied as a prerequisite for describing actual physical phenomena. Thus, while on the one hand a theory possessing these properties is only of formal interest if it is not connected to experiments, on the other hand, a theory that has some phenomenological success but violates, say, causality, cannot be taken as an accurate description of real relativistic physical phenomena. In this regard, we once more remark that, in view of the results presented in this paper, the BDNK theory is currently the only theory that satisfies the fundamental requirements I–III and the additional property IV when all viscous contributions and chemical potential are incorporated, including in the case when dynamical coupling to Einstein's equations is considered.

III. GENERALIZED NAVIER-STOKES THEORY

We consider a general-relativistic fluid described by an energy-momentum tensor $T^{\mu\nu}$ and a timelike conserved current J^μ associated with a global $U(1)$ charge that we take to represent baryon number. In our approach, the equations of relativistic fluid dynamics are given by the conservation laws,

$$\nabla_\mu J^\mu = 0 \quad \text{and} \quad \nabla_\mu T^{\mu\nu} = 0, \tag{2}$$

which are dynamically coupled to Einstein's field equations:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3}$$

For the sake of completeness, we begin by recalling the case of a fluid in local equilibrium [2]. In this limit, one uses the following expressions in the conservation laws:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} \quad \text{and} \quad J^\mu = n u^\mu, \tag{4}$$

where ε is the equilibrium energy density, n is the equilibrium baryon density, $P = P(\varepsilon, n)$ is the thermodynamical pressure defined by the equation of state, u^μ is a normalized timelike vector (i.e., $u_\mu u^\mu = -1$) called the flow velocity, and $\Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is a projector onto the space orthogonal to u^μ. The thermodynamical quantities in equilibrium are connected via the first law of thermodynamics $\varepsilon + P = Ts + \mu n$, where T is the temperature, s is the equilibrium entropy density, and μ is the chemical potential associated with the conserved baryon charge. We note that $u^\mu \nabla_\mu \varepsilon = 0$ and $u^\mu \nabla_\mu n = 0$ in global equilibrium. These are much stronger constraints on the dynamical variables than in the case of local equilibrium where, e.g., only the combination $u^\mu \nabla_\mu \varepsilon + (\varepsilon + P)\nabla_\mu u^\mu$ vanishes. In local equilibrium, both $u_\mu T^{\mu\nu}$ and J^ν are proportional to u^ν and, thus, the flow velocity may be defined using either quantity [2].

The system of equations (2) and (3) for an ideal fluid [defined by Eq. (4)] is causal in the full nonlinear regime.

Furthermore, given suitably defined initial data for the dynamical variables, solutions for the nonlinear problem exist and are unique. The latter properties establish that the equations of motion of ideal relativistic fluid dynamics are locally well posed in general relativity [16,17].

Let us now consider the effects of dissipation. Without any loss of generality, one may decompose the current and the energy-momentum tensor in terms of an arbitrary future-directed unit timelike vector u^μ as follows [48]:

$$J^\mu = \mathcal{N} u^\mu + \mathcal{J}^\mu, \tag{5}$$

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu + \mathcal{T}^{\mu\nu}, \tag{6}$$

where $\mathcal{N} = -u_\mu J^\mu$, $\mathcal{E} = u_\mu u_\nu T^{\mu\nu}$, and $\mathcal{P} = \Delta_{\mu\nu} T^{\mu\nu}/3$ are Lorentz scalars while the vectors $\mathcal{J}^\nu = \Delta^\nu_\mu J^\mu$, $\mathcal{Q}^\nu = -u_\mu T^{\mu\lambda}\Delta^\nu_\lambda$, and the traceless symmetric tensor $\mathcal{T}^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}$, with $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})$, are all transverse to u_ν. Observe that this decomposition is purely algebraic and simply expresses the fact that a vector and a symmetric two-tensor can be decomposed relatively to a future-directed unit timelike vector. The physical content of the theory is prescribed by relating the several components in this decomposition to physical observables, which will then evolve [162] according to Eqs. (5) and (6).

The general decomposition in Eqs. (5) and (6) expresses $\{J^\mu, T^{\mu\nu}\}$ in terms of 17 variables $\{\mathcal{E}, \mathcal{N}, \mathcal{P}, u^\mu, \mathcal{J}^\mu, \mathcal{Q}^\mu, \mathcal{T}^{\mu\nu}\}$, and the conservation laws in Eq. (2) give five equations of motion for these variables. Therefore, additional assumptions must be made to properly define the evolution of the fluid. As mentioned before, the NS theory, including the standard approach in Refs. [15,52], assumes that $\mathcal{E} = \varepsilon$ and $\mathcal{N} = n$. The same assumption is usually made in the MIS theory [36], though different prescriptions can be easily defined in the context of kinetic theory [46,65,163]. A further constraint is usually imposed on the transverse vectors, i.e., either $\mathcal{J}^\mu = 0$ or $\mathcal{Q}^\mu = 0$ throughout the evolution. For instance, the former gives $J^\mu = n u^\mu$ and $T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + u^\mu \mathcal{Q}^\nu + u^\nu \mathcal{Q}^\mu + \mathcal{T}^{\mu\nu}$, where Π is the bulk viscous pressure (in equilibrium, $\Pi = 0$, $\mathcal{Q}^\nu = 0$, and $\mathcal{T}^{\mu\nu} = 0$). In this case, in an extended variable approach such as MIS [36], Π, $\mathcal{Q}^\nu$, and $\mathcal{T}^{\mu\nu}$ obey additional equations of motion that must be specified and solved together with the conservation laws, whereas in the NS approach these quantities are expressed in terms of u^μ, ε, and its derivatives.

In this paper, we investigate the problem of viscous fluids in general relativity using the BDNK formulation of relativistic fluid dynamics. See Secs. II B and II C for a detailed discussion of the origins of the BDNK theory and the conceptual framework that it entails. As explained in those sections, the starting point in the formulation of the BDNK theory is the most general expression for the energy-momentum tensor and the baryon current at first order.

In practice, the most general expressions for the *constitutive relations* that define the quantities in Eqs. (5) and (6), truncated to first order in derivatives, are (following the notation in Ref. [34])

$$\mathcal{E} = \varepsilon + \varepsilon_1 \frac{u^\alpha \nabla_\alpha T}{T} + \varepsilon_2 \nabla_\alpha u^\alpha + \varepsilon_3 u^\alpha \nabla_\alpha (\mu/T), \tag{7a}$$

$$\mathcal{P} = P + \pi_1 \frac{u^\alpha \nabla_\alpha T}{T} + \pi_2 \nabla_\alpha u^\alpha + \pi_3 u^\alpha \nabla_\alpha (\mu/T), \tag{7b}$$

$$\mathcal{N} = n + \nu_1 \frac{u^\alpha \nabla_\alpha T}{T} + \nu_2 \nabla_\alpha u^\alpha + \nu_3 u^\alpha \nabla_\alpha (\mu/T), \tag{7c}$$

$$\mathcal{Q}^\mu = \theta_1 \frac{\Delta^{\mu\nu}\nabla_\nu T}{T} + \theta_2 u^\alpha \nabla_\alpha u^\mu + \theta_3 \Delta^{\mu\nu}\nabla_\nu (\mu/T), \tag{7d}$$

$$\mathcal{J}^\mu = \gamma_1 \frac{\Delta^{\mu\nu}\nabla_\nu T}{T} + \gamma_2 u^\alpha \nabla_\alpha u^\mu + \gamma_3 \Delta^{\mu\nu}\nabla_\nu (\mu/T), \tag{7e}$$

$$\mathcal{T}^{\mu\nu} = -2\eta\sigma^{\mu\nu}, \tag{7f}$$

where $\sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta}\nabla_\alpha u_\beta$ is the shear tensor. The transport parameters $\{\varepsilon_i, \pi_i, \theta_i, \nu_i, \gamma_i\}$ and the shear viscosity η are functions of T and μ. Thermodynamic consistency of the equilibrium state (i.e., that ε, P, and n have the standard interpretations of equilibrium quantities connected via well-known thermodynamic relations) imposes that $\gamma_1 = \gamma_2$ and $\theta_1 = \theta_2$ [34]. The final equations of motion for $\{T, \mu, u^\alpha\}$, which are of second order in derivatives, are found by substituting the expressions above in the conservation laws. In the language of Sec. II A, expressions (7) for Eqs. (5) and (6) correspond to the most general choice of a hydrodynamic frame for a first-order theory. As stressed in Ref. [34], it is of course impossible to not choose a hydrodynamic frame since the latter actually defines the meaning of the variables $\{T, \mu, u^\mu\}$ out of equilibrium (see Sec. II A for details).

In fact, in the regime of validity of the first-order theory, one may shift $\{T, \mu, u^\mu\}$ by adding terms that are of first order in derivatives, shifting also the transport parameters $\{\varepsilon_i, \pi_i, \theta_i, \nu_i, \gamma_i\}$, without formally changing the physical content of $T^{\mu\nu}$ and J^μ [34]. However, there are combinations of the transport parameters that remain invariant under these field redefinitions. In fact, the shear viscosity η and the combination of coefficients that give the bulk viscosity ζ and charge conductivity σ are invariant under first-order field redefinitions, as explained in Ref. [34]. Additional constraints among the transport parameters appear when the underlying theory displays conformal invariance, as discussed in detail in Ref. [32] at $\mu = 0$, and at finite chemical potential in Refs. [34,35] (see also Ref. [110]).

Hoult and Kovtun [35] investigated Eq. (7) at nonzero chemical potential using a class of hydrodynamic frames where $\varepsilon_3 = \pi_3 = \theta_3 = 0$. This corresponds to the case where there are nonequilibrium corrections to both the conserved current and the heat flux. This choice is useful when considering relativistic fluids where the net baryon density is not very large, as in high-energy heavy-ion collisions. Conditions for causality were derived and limiting cases were studied that strongly indicated that this choice of hydrodynamic frame is stable against small disturbances around equilibrium. Further studies are needed to better understand the nonlinear features of its solutions (well posedness) and also the stability properties of this class of hydrodynamic frames at nonzero baryon density in a wider class of equilibrium states.

In this paper, we consider another class of hydrodynamic frames that we believe can be more naturally implemented in simulations of the baryon-rich matter formed in neutron star mergers or in low-energy heavy-ion collisions. Our choice for the hydrodynamic frame is closer to Eckart's as we define the flow velocity using the baryon current; i.e., $J^\mu = nu^\mu$ holds throughout the evolution ($\gamma_i = \nu_i = 0$). Clearly, this limits the domain of applicability of the theory to problems where there are many more baryons than antibaryons so the net baryon charge is large.

In this case, it is more convenient to use ε and n as dynamical variables instead of T and μ/T because the most general expressions for the Lorentz scalar contributions to the constitutive relations involve only linear combinations of $u^\mu\nabla_\mu\varepsilon$ and $\nabla_\mu u^\mu$, given that current conservation implies that the replacement $u^\lambda\nabla_\lambda n = -n\nabla_\lambda u^\lambda$ is valid. For simplicity, we choose to parametrize the out-of-equilibrium corrections to the scalars as follows [we note that $\theta_1 = \theta_2$ and $\gamma_1 = \gamma_2$, and in practice, 8 out of the 14 parameters in Eq. (7) can be set using first-order field redefinitions [34], so one is then left with η, ζ, σ, and three other parameters]:

$$\mathcal{E} = \varepsilon + \tau_\varepsilon[u^\lambda\nabla_\lambda\varepsilon + (\varepsilon+P)\nabla_\lambda u^\lambda], \tag{8a}$$

$$\mathcal{P} = P - \zeta\nabla_\lambda u^\lambda + \tau_P[u^\lambda\nabla_\lambda\varepsilon + (\varepsilon+P)\nabla_\lambda u^\lambda], \tag{8b}$$

where τ_ε and τ_P have dimensions of a relaxation time and ζ is the bulk viscosity transport coefficient. When evaluated on the solutions of the equations of motion, one can see that these quantities assume their standard form as in Eckart's theory up to second order in derivatives because $\mathcal{E} \sim \varepsilon + \mathcal{O}(\partial^2)$ and $\mathcal{P} = P - \zeta\nabla_\mu u^\mu + \mathcal{O}(\partial^2)$ on shell (we follow traditional terminology where a given quantity is said to be on shell when it is evaluated using the solutions to the equations of motion).

In fact, we remind the reader that in Eckart's theory [52] the energy-momentum tensor is given by $T_{\mu\nu} = \varepsilon u_\mu u_\nu + (P - \zeta\nabla_\lambda u^\lambda)\Delta_{\mu\nu} - 2\eta\sigma_{\mu\nu} + u_\mu\mathcal{Q}_\nu + u_\nu\mathcal{Q}_\mu$, with heat flux $\mathcal{Q}_\mu = -\kappa T(u^\lambda\nabla_\lambda u_\mu + \Delta_\mu^\lambda\nabla_\lambda T/T)$, where $\kappa = (\varepsilon+P)^2\sigma/(n^2T)$ is the thermal conductivity coefficient. However, as remarked in Ref. [34], in the domain of validity of the first-order theory one may rewrite the Eckart expression for the heat flux as $\mathcal{Q}_\nu = \sigma T[(\varepsilon+P)/n]\Delta_\nu^\lambda\nabla_\lambda(\mu/T)$ plus second-order terms. This is done by noticing that $(\varepsilon+P)u^\lambda\nabla_\lambda u^\mu + \Delta^{\mu\lambda}\nabla_\lambda P = 0 + \mathcal{O}(\partial^2)$ on shell, which implies that one may write, using the standard thermodynamic relation $[(dP)/(\varepsilon+P)] = [(dT)/T] + [(nT)/(\varepsilon+P)]\times d(\mu/T)$,

$$u^\lambda\nabla_\lambda u^\alpha + \frac{\Delta^{\alpha\lambda}\nabla_\lambda T}{T} = -\frac{nT}{\varepsilon+P}\Delta^{\alpha\lambda}\nabla_\lambda(\mu/T) + \mathcal{O}(\partial^2). \tag{9}$$

Therefore, one can always choose the coefficients such that the heat flux $\mathcal{Q}^\mu$ has the same physical content of Eckart's theory plus terms that are of second order on shell. We use this to write this quantity as

$$\mathcal{Q}_\nu = \sigma T\frac{(\varepsilon+P)}{n}\Delta_\nu^\lambda\nabla_\lambda(\mu/T) + \tau_Q[(\varepsilon+P)u^\lambda\nabla_\lambda u_\nu + \Delta_\nu^\lambda\nabla_\lambda P], \tag{10}$$

where τ_Q has dimensions of a relaxation time.

In this work, we make the following choice for the constitutive relations that give the energy-momentum tensor and the baryon current:

$$J^\mu = nu^\mu, \tag{11a}$$

$$T^{\mu\nu} = (\varepsilon+\mathcal{A})u^\mu u^\nu + (P+\Pi)\Delta^{\mu\nu} - 2\eta\sigma^{\mu\nu} + u^\mu\mathcal{Q}^\nu + u^\nu\mathcal{Q}^\mu, \tag{11b}$$

$$\mathcal{A} = \tau_\varepsilon[u^\lambda\nabla_\lambda\varepsilon + (\varepsilon+P)\nabla_\lambda u^\lambda], \tag{11c}$$

$$\Pi = -\zeta\nabla_\lambda u^\lambda + \tau_P[u^\lambda\nabla_\lambda\varepsilon + (\varepsilon+P)\nabla_\lambda u^\lambda], \tag{11d}$$

$$\mathcal{Q}^\nu = \tau_Q(\varepsilon+P)u^\lambda\nabla_\lambda u^\nu + \beta_\varepsilon\Delta^{\nu\lambda}\nabla_\lambda\varepsilon + \beta_n\Delta^{\nu\lambda}\nabla_\lambda n, \tag{11e}$$

where

$$\beta_\varepsilon = \tau_Q\left(\frac{\partial P}{\partial\varepsilon}\right)_n + \frac{\sigma T(\varepsilon+P)}{n}\left(\frac{\partial(\mu/T)}{\partial\varepsilon}\right)_n, \tag{12a}$$

$$\beta_n = \tau_Q\left(\frac{\partial P}{\partial n}\right)_\varepsilon + \frac{\sigma T(\varepsilon+P)}{n}\left(\frac{\partial(\mu/T)}{\partial n}\right)_\varepsilon, \tag{12b}$$

and τ_ε, τ_P, and τ_Q quantify the magnitude of second-order corrections to the out-of-equilibrium contributions to the energy-momentum tensor given by the energy density correction $\mathcal{A}$, the bulk viscous pressure Π, and the heat flux $\mathcal{Q}^\mu$. In other words, Eqs. (11) and (12) correspond to the frame we consider in this work; thus they provide a definition of what we mean by the nonequilibrium hydrodynamic fields.

The reason for considering the constitutive relations (11) and (12) is that they lead to a theory satisfying properties I–IV, as we show below. We refer the reader to Sec. II C for a discussion of the ideas and techniques that led to the particular choice of Eqs. (11) and (12).

The equations of motion for the fluid variables are obtained from the conservation laws and they can be written explicitly as

$$u^\lambda \nabla_\lambda n + n \nabla_\lambda u^\lambda = 0, \tag{13a}$$

$$u^\lambda \nabla_\lambda \varepsilon + (\varepsilon + P)\nabla_\lambda u^\lambda = -u^\lambda \nabla_\lambda \mathcal{A} - (\mathcal{A} + \Pi)\nabla_\lambda u^\lambda - \nabla_\mu \mathcal{Q}^\mu - \mathcal{Q}^\mu u^\lambda \nabla_\lambda u_\mu + 2\eta \sigma_{\mu\nu}\sigma^{\mu\nu}, \tag{13b}$$

$$(\varepsilon + P)u^\nu \nabla_\nu u^\beta + \Delta^{\beta\lambda}\nabla_\lambda P = -(\mathcal{A} + \Pi)u^\nu \nabla_\nu u^\beta - \Delta^{\beta\lambda}\nabla_\lambda \Pi + \Delta^\beta_\lambda \nabla_\mu (2\eta \sigma^{\mu\lambda}) - u^\lambda \nabla_\lambda \mathcal{Q}^\beta - \frac{4}{3}\nabla_\lambda u^\lambda \mathcal{Q}^\beta - \mathcal{Q}_\mu \sigma^{\mu\beta} - \mathcal{Q}_\mu \omega^{\mu\beta}, \tag{13c}$$

where $\omega_{\mu\nu} = \frac{1}{2}(\Delta^\lambda_\mu \nabla_\lambda u_\nu - \Delta^\lambda_\nu \nabla_\lambda u_\mu)$ is the kinematic vorticity tensor [2]. The equations above show that, on shell, $\mathcal{A} \sim 0 + \mathcal{O}(\partial^2)$, $\Pi \sim -\zeta \nabla_\mu u^\mu + \mathcal{O}(\partial^2)$, and $\mathcal{Q}_\nu = \sigma T[(\varepsilon + P)/n]\Delta^\lambda_\nu \nabla_\lambda (\mu/T) + \mathcal{O}(\partial^2)$. Equations (11)–(13) define a causal and stable generalization of Eckart's theory that is fully compatible with general relativity, as we prove in the next sections. We remark that when one neglects the effects of a conserved current altogether, the theory reduces to the case studied in Refs. [33,34]. For additional discussion about the case without a chemical potential, including far-from-equilibrium behavior and also the presence of analytical solutions, see Refs. [111,116,117].

A. Entropy production

It is instructive to investigate how the second law of thermodynamics is obeyed in this general first-order approach. This was discussed in detail by Kovtun in Ref. [34] and, more recently, by other authors in Ref. [51].

The standard covariant definition of the entropy current based on the first law of thermodynamics $T\mathbb{S}^\mu = Pu^\mu - u_\nu T^{\nu\mu} - \mu J^\mu$ [36], together with Eq. (11), can be used to show that the entropy density measured by a comoving observer is given by

$$-u_\mu \mathbb{S}^\mu = s + \frac{\mathcal{A}}{T}. \tag{14}$$

Note that in our system one finds that $\mathcal{A} = 0 + \mathcal{O}(\partial^2)$ on shell. Furthermore, using Eqs. (11) and (13) one finds that the divergence of the entropy current is given by

$$\nabla_\mu \mathbb{S}^\mu = \frac{2\eta \sigma_{\mu\nu}\sigma^{\mu\nu}}{T} - \frac{\Pi}{T}\nabla_\mu u^\mu + \frac{n}{\varepsilon + P}\mathcal{Q}^\nu \Delta^\lambda_\nu \nabla_\lambda (\mu/T) - \frac{\mathcal{Q}^\nu}{T}\left[u^\lambda \nabla_\lambda u_\nu + \frac{\Delta^\lambda_\nu \nabla_\lambda P}{\varepsilon + P}\right] - \frac{\mathcal{A}}{T}\frac{u^\lambda \nabla_\lambda T}{T}. \tag{15}$$

It is crucial to note [34] that in a first-order approach $\nabla_\mu \mathbb{S}^\mu$ can only be correctly determined up to second order in derivatives [recall that in this argument terms such as $\nabla_\mu \nabla_\nu \phi$ and $(\nabla_\mu \phi)(\nabla_\nu \phi)$, for any field ϕ, count as second-order terms; see Sec. II A]. This means that not all the terms in Eq. (15) actually contribute to this expression at second order. For instance, when evaluating Eq. (15) *on shell* one must keep in mind that the last two terms in Eq. (15) are already at least of third order and must, thus, be dropped. A similar argument can be used to show that the term $\Pi \nabla_\mu u^\mu = -\zeta(\nabla_\mu u^\mu)^2 + \mathcal{O}(\partial^3)$. Therefore, one can see that

$$\nabla_\mu \mathbb{S}^\mu = \frac{2\eta \sigma_{\mu\nu}\sigma^{\mu\nu}}{T} + \frac{\zeta(\nabla_\mu u^\mu)^2}{T} + \sigma T[\Delta^\lambda_\nu \nabla_\lambda (\mu/T)][\Delta^{\nu\alpha}\nabla_\alpha (\mu/T)] + \mathcal{O}(\partial^3), \tag{16}$$

which is non-negative when $\eta, \zeta, \sigma \geq 0$. Hence, there are no violations of the second law of thermodynamics in the domain of validity of the first-order theory—higher-order derivative terms $\mathcal{O}(\partial^3)$ in the entropy production can only be understood by considering terms of higher order in derivatives in the constitutive relations in $T^{\mu\nu}$ and J^μ, which is beyond the scope of the first-order approach.

IV. CAUSALITY

In order to determine the conditions under which causality holds in this theory, we need to understand the system's characteristics. Our system is a mixed first-second-order system of PDEs. While the principal part and characteristics of systems of this form can be investigated using Leray's theory [21,105,164], here it is simpler to transform our equations into a system where all equations are of second order. We thus apply $u^\mu \nabla_\mu$ on Eq. (13a). In this case, the conservation laws (2) coupled to Einstein's equations (3) written in harmonic gauge, $g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$, read

$$u^\beta u^\alpha \partial^2_{\alpha\beta} n + n \delta^\alpha_\nu u^\beta \partial^2_{\alpha\beta} u^\nu + \tilde{\mathcal{B}}_1(n, u, g)\partial^2 g = \mathcal{B}_1(\partial n, \partial u, \partial g), \tag{17a}$$

$$(\tau_\varepsilon u^\alpha u^\beta + \beta_\varepsilon \Delta^{\alpha\beta})\partial^2_{\alpha\beta}\varepsilon + \beta_n \Delta^{\alpha\beta}\partial^2_{\alpha\beta} n + \rho(\tau_\varepsilon + \tau_Q)u^{(\alpha}\delta^{\beta)}_\nu \partial^2_{\alpha\beta} u^\nu + \tilde{\mathcal{B}}_2(\varepsilon, n, u, g)\partial^2 g = \mathcal{B}_2(\partial\varepsilon, \partial n, \partial u, \partial g), \tag{17b}$$

$$(\beta_\varepsilon + \tau_P)u^{(\alpha}\Delta^{\beta)\mu}\partial^2_{\alpha\beta}\varepsilon + \beta_n u^{(\alpha}\Delta^{\beta)\mu}\partial^2_{\alpha\beta} n + \mathcal{C}^{\mu\alpha\beta}_\nu \partial^2_{\alpha\beta} u^\nu + \tilde{\mathcal{B}}^\mu_3(\varepsilon, n, u, g)\partial^2 g = \mathcal{B}^\mu_3(\partial\varepsilon, \partial n, \partial u, \partial g), \tag{17c}$$

$$g^{\alpha\beta}\partial^2_{\alpha\beta} g^{\mu\nu} = \mathcal{B}^{\mu\nu}_4(\partial\varepsilon, \partial n, \partial u, \partial g), \tag{17d}$$

where $\partial^2_{\alpha\beta} = \partial_\alpha \partial_\beta$ (using standard partial derivatives), $\rho = (\varepsilon + P)$, and $A_{(\alpha}B_{\beta)} = (A_\alpha A_\beta + A_\beta B_\alpha)/2$. The remaining notation is as follows. We use $\partial^\ell \phi$ to indicate that a term depends on at most ℓ derivatives of ϕ. A term of the form $\mathcal{B}(\partial^{\ell_1}\phi_1, \ldots, \partial^{\ell_k}\phi_k)\partial^{\ell}\phi_i$, $i \in \{1, \ldots, k\}$, indicates an expression that is linear in $\partial^{\ell}\phi_i$ with coefficients depending on at most ℓ_1 derivatives of $\phi_1, \ldots, \ell_k$ derivatives of ϕ_k. For example, the term $(u^\mu\partial_\mu\varepsilon + \partial_\mu u^\mu)g^{\alpha\beta}\partial^2_{\alpha\beta}g_{\gamma\delta}$ would be written as $\mathcal{B}(\partial\varepsilon, \partial u, g)\partial^2 g$ (a term of this form is not present in our system; we write it here only for illustration). The terms $\tilde{\mathcal{B}}$ above are top order in derivatives of g and thus belong to the principal part, although, as we will see, their explicit form is not needed for our argument, whereas the $\mathcal{B}$ terms are lower order and do not contribute to the principal part. We have also defined

$$\mathcal{C}^{\mu\alpha\beta}_{\nu} = \left(\tau_P\rho - \zeta - \frac{\eta}{3}\right)\Delta^{\mu(\alpha}\delta^{\beta)}_{\nu} + (\rho\tau_Q u^\alpha u^\beta - \eta\Delta^{\alpha\beta})\delta^\mu_\nu. \quad (18)$$

We notice that by taking $u^\mu\nabla_\mu$ of Eq. (13a) we are not introducing new characteristics in the system. This can be viewed from the characteristic determinant computed below which contains an overall factor of $u^\mu\xi_\mu$ to a power greater than one. Theorem I below establishes necessary and sufficient conditions for causality to hold in our system of equations. We show that the assumptions of Theorem I are not empty in Sec. VII A. Throughout this paper, we use the following definition for the speed of sound c_s:

$$c_s^2 = \left(\frac{\partial P}{\partial\varepsilon}\right)_{\bar{s}} = \left(\frac{\partial P}{\partial\varepsilon}\right)_n + \frac{n}{\rho}\left(\frac{\partial P}{\partial n}\right)_\varepsilon, \quad (19)$$

where $\bar{s}$ is the equilibrium entropy per particle. Also, we define

$$\kappa_s = \frac{\rho^2 T}{n}\left[\frac{\partial(\mu/T)}{\partial\varepsilon}\right]_{\bar{s}} = \frac{\rho^2 T}{n}\left[\frac{\partial(\mu/T)}{\partial\varepsilon}\right]_n + T\rho\left[\frac{\partial(\mu/T)}{\partial n}\right]_\varepsilon. \quad (20)$$

Theorem I.—Let $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ be a solution to Eqs. (3) and (13), with $u^\mu u_\mu = -1$, defined in a globally hyperbolic spacetime $(M, g_{\alpha\beta})$. Assume that Assumption 1

(A1) $\rho = \varepsilon + P, \tau_\varepsilon, \tau_Q, \tau_P > 0$ and $\eta, \zeta, \sigma \geq 0$.

Then, causality holds for $(\varepsilon, n, u^\mu, g_{\alpha\beta})$ if, and only if, the following conditions are satisfied:

$$\rho\tau_Q > \eta, \quad (21a)$$

$$\left[\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q\right]^2 \geq 4\rho\tau_\varepsilon\tau_Q\left[\tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right)\right] \geq 0, \quad (21b)$$

$$2\rho\tau_\varepsilon\tau_Q > \tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q \geq 0, \quad (21c)$$

$$\rho\tau_\varepsilon\tau_Q + \sigma\kappa_s\tau_P > \tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) + \rho\tau_P\tau_Q(1 - c_s^2) + \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right). \quad (21d)$$

The same result holds true for Eqs. (13) if the metric is not dynamical.

Proof.—The proof can be reduced to a computation of the characteristics of Eq. (17) [164]. Technical details are found in Appendix A.

V. STRONG HYPERBOLICITY AND LOCAL WELL POSEDNESS

In this section, we investigate the initial-value problem for Eqs. (3) and (13). The goal is to show that the system is causal and locally well posed under very general conditions. First, we briefly discuss the initial data required to solve the system of equations. Then, we rewrite our system as a first-order system. We show that this first-order system is diagonalizable in the sense of Proposition I. This means, in particular, that the system is *strong hyperbolic* according to the usual definition of the term, as in, e.g., Refs. [2,23]. The importance of having strongly hyperbolic equations is due to its implications for the initial-value problem. As already mentioned, one is generally interested in evolution equations that are locally well posed [165]. For equations with constant coefficients, local well posedness is equivalent to strong hyperbolicity [166]. For nonconstant coefficients and nonlinear systems, such an equivalence does not hold [167–169]. However, there remains a close connection between strong hyperbolicity and local well posedness. For most reasonable systems, once diagonalizability is available, one can use known techniques to derive energy estimates which, in turn, can be used to prove local well posedness; see Sec. II C for more discussion on the techniques involved. This is precisely the case for our system of equations. Even though our equations consist of a system of second-order PDEs, we can use the diagonalized system of first-order equations to derive energy estimates. Once these estimates are available, we use a standard approximation argument as in Refs. [17,170] to obtain local well posedness (see Theorem II).

A. Initial data

Equations (13) are second order in ε, n, and u^μ. Thus, initial data along a noncharacteristic hypersurface consist of the values of ε, n, u^μ and their first-order time derivatives. Clearly, the initial u^μ has to satisfy $u^\mu u_\mu = -1$. Also, it is important to note that Eq. (13a) is first order and, thus, the initial data cannot be arbitrary but must satisfy a compatibility condition ensuring that Eq. (13a) holds at $t = 0$. Therefore, one can use Eq. (13a) to write the time derivative of n in terms of the time derivative of u^μ

(this feature would also appear in Navier-Stokes theory in the Eckart hydrodynamic frame).

A natural choice to determine the initial conditions for the matter sector is to set an initial state that is within the regime of validity of the first-order theory and closely reproduces Eckart's theory. First, one can directly extract n and u^μ from J^μ at the initial spacelike hypersurface. Then, one sets the nonequilibrium correction to the energy density $\mathcal{A}$ in Eq. (11) to zero in the initial state, so then the initial value for ε equals $T^{\mu\nu}u_\mu u_\nu$ and the first-order time derivative of ε is defined in terms of the first-order time derivative of the flow velocity (plus spatial derivatives that are known in the initial state). Clearly, $\mathcal{A}$ will be different than zero later during the actual evolution, and its value can be used to check if the simulations remain within the regime of validity of the first-order approach (i.e., $|\mathcal{A}|/\varepsilon$ must remain less than unity). Finally, the time derivative of the flow velocity can be set by imposing that the second-order on-shell term $(\varepsilon + P)u^\lambda \nabla_\lambda u^\nu + \Delta^{\nu\lambda}\nabla_\lambda P$ vanishes. Hence, one can obtain the time derivative of the flow velocity and all the other required initial data in the regime of validity of the first-order approach, emulating Eckart's theory as much as possible.

We recall that the initial data for the gravitational sector has to further satisfy the well-known Einstein constraint equations. We briefly make some comments on this in Sec. VIII.

B. Diagonalization and eigenvectors

In this section, we write Eqs. (3) and (13) as a first-order system, as discussed above. For this, we begin defining the variables $V = u^\alpha\partial_\alpha\varepsilon$, $\mathcal{V}^\mu = \Delta^{\mu\alpha}\partial_\alpha\varepsilon$, $W = u^\alpha\partial_\alpha n$, $\mathcal{W}^\mu = \Delta^{\mu\alpha}\partial_\alpha n$, $S^\mu = u^\alpha\nabla_\alpha u^\mu$, $\mathcal{S}^\nu_\lambda = \Delta^\alpha_\lambda \nabla_\alpha u^\nu$, $F_{\mu\nu} = u^\alpha\partial_\alpha g_{\mu\nu}$, and $\mathcal{F}^\lambda_{\mu\nu} = \Delta^{\lambda\alpha}\partial_\alpha g_{\mu\nu}$. Then, the equations of motion can be cast as

$$\tau_\varepsilon u^\alpha\partial_\alpha V + \tau_Q\rho\partial_\nu S^\nu + \tau_\varepsilon\rho u^\alpha\partial_\alpha \mathcal{S}^\nu_\nu + \beta_\varepsilon\partial_\nu\mathcal{V}^\nu + \beta_n\partial_\nu\mathcal{W}^\nu = r_1, \tag{22a}$$

$$\tau_P\Delta^{\mu\alpha}\partial_\alpha V + \tau_Q\rho u^\alpha\partial_\alpha S^\mu + \beta_\varepsilon u^\alpha\partial_\alpha\mathcal{V}^\mu + \beta_n u^\alpha\partial_\alpha\mathcal{W}^\mu + \eta\Pi^{\mu\lambda\alpha}_\nu\partial_\alpha\mathcal{S}^\nu_\lambda = r^\mu_2, \tag{22b}$$

$$u^\alpha\partial_\alpha\mathcal{V}^\mu - \Delta^{\mu\alpha}\partial_\alpha V = r^\mu_3, \tag{22c}$$

$$u^\alpha\partial_\alpha\mathcal{W}^\mu + n\Delta^{\mu\alpha}\partial_\alpha\mathcal{S}^\nu_\nu = r^\mu_4, \tag{22d}$$

$$u^\alpha\partial_\alpha\mathcal{S}^\nu_\lambda - \Delta^\alpha_\lambda\partial_\alpha S^\nu - \mathcal{X}^{\nu A\alpha}_\lambda\partial_\alpha F_A - \mathcal{Y}^{\nu A\alpha}_{\lambda\delta}\partial_\alpha\mathcal{F}^\delta_A = r^\nu_{5\lambda}, \tag{22e}$$

$$u^\alpha\partial_\alpha F_A - \Delta^\alpha_\delta\mathcal{F}^\delta_A = r_{6A}, \tag{22f}$$

$$u^\alpha\partial_\alpha\mathcal{F}^\delta_A - \Delta^{\delta\alpha}\partial_\alpha F_A = r^\delta_{7A}, \tag{22g}$$

$$u^\alpha\partial_\alpha\varepsilon = r_8, \tag{22h}$$

$$u^\alpha\partial_\alpha n = r_9, \tag{22i}$$

$$u^\alpha\partial_\alpha u^\mu = r^\mu_{10}, \tag{22j}$$

$$u^\alpha\partial_\alpha g_A = r_{11A}, \tag{22k}$$

where the r's are functions of the fields $\varepsilon, u^\nu, \ldots, \mathcal{F}^\lambda_{\mu\nu}$ but not its derivatives and $A = \sigma\beta$ for $\sigma \geq \beta$; i.e., A takes the 10 independent values 00,01,02,03,11,12,13,22,23,33 with repeated index A summing from 00 to 33,

$$\Pi^{\mu\lambda\alpha}_\nu = -\eta(\Delta^{\mu\lambda}\delta^\alpha_\nu + \Delta^{\alpha\lambda}\delta^\mu_\nu) + \left(\rho\tau_P - \zeta + \frac{2\eta}{3}\right)\Delta^{\mu\alpha}\delta^\lambda_\nu, \tag{23a}$$

$$\mathcal{X}^{\nu A(=\sigma\beta)\alpha}_\lambda = \frac{1}{2}[g^{\nu(\sigma}\Delta^{\beta)}_\lambda u^\alpha - u^{(\sigma}\Delta^{\beta)}_\lambda g^{\nu\alpha} - u^{(\sigma}\Delta^{\beta)\nu}\Delta^\alpha_\lambda](2-\delta_A), \tag{23b}$$

$$\mathcal{Y}^{\nu A(=\sigma\beta)\alpha}_{\lambda\delta} = \frac{1}{2}u^{(\sigma}u^{\beta)}\Delta^\alpha_\lambda\delta^\nu_\delta(2-\delta_A). \tag{23c}$$

By δ_A we mean the Kronecker delta in the sense that when $A = \sigma\beta$, then $\delta_A = \delta_{\sigma\delta}$, which equals one when $\sigma = \beta$ and zero otherwise. Also, the terms r may be functions of the 95 variables. Equations (22) were obtained as follows. Equations (22a) and (22b) come from the conservation law $\nabla_\nu T^{\mu\nu} = 0$ when projected into the directions parallel and perpendicular to u^ν, respectively. Equations (22c), (22d), (22e), and (22g) correspond, respectively, to the identities $\nabla_\alpha\nabla_\beta\varepsilon - \nabla_\beta\nabla_\alpha\varepsilon = 0$, $\nabla_\alpha\nabla_\beta n - \nabla_\beta\nabla_\alpha n = 0$, $\nabla_\alpha\nabla_\beta u^\nu - \nabla_\beta\nabla_\alpha u^\nu = R^\nu_{\alpha\beta\sigma}u^\sigma = (\partial_\alpha\Gamma^\nu_{\beta\sigma} - \partial_\beta\Gamma^\nu_{\alpha\sigma})u^\sigma +$ terms of order zero in derivatives, and $\partial_\alpha\partial_\beta g_{\mu\nu} - \partial_\beta\partial_\alpha g_{\mu\nu} = 0$, all contracted with $u^\alpha\Delta^\beta_\lambda$. Equations (22f) is the Einstein equation in the harmonic gauge, i.e., $g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} =$ terms of lower order in derivatives, while Eqs. (22h)–(22k) are the definitions of V, W (also using the identity $u^\alpha\nabla_\alpha n + n\nabla_\alpha u^\alpha = W + n\mathcal{S}^\alpha_\alpha = 0$ to eliminate W thoroughly), S^μ, and F_A, respectively. We may now define the 95×1 column vectors Ψ and $\mathfrak{B}$ as

$$\Psi = \begin{bmatrix} \psi_m \\ \psi_g \\ \psi_d \end{bmatrix}, \tag{24}$$

and $\mathfrak{B} = (r_1, \ldots, r_{11A})^T$, where $\psi_m = (V, S^\nu, \mathcal{V}^\nu, \mathcal{W}^\nu, \mathcal{S}^\nu_0, \mathcal{S}^\nu_1, \mathcal{S}^\nu_2, \mathcal{S}^\nu_3)^T \in \mathbb{R}^{29}$, $\psi_g = (F_A, \mathcal{F}^0_A, \mathcal{F}^1_A, \mathcal{F}^2_A, \mathcal{F}^3_A)^T \in \mathbb{R}^{50}$, and $\psi_d = (\varepsilon, n, u^\nu, g_A)^T \in \mathbb{R}^{16}$, to write the quasilinear first-order system (22) in matrix form as

$$\mathfrak{A}^\alpha\partial_\alpha\Psi = \mathfrak{B}, \tag{25}$$

where, here, $\mathfrak{A}^\alpha = \mathbb{A}^\alpha \oplus u^\alpha I_{16}$ ($\oplus$ being the direct sum). The matrix $\mathbb{A}^\alpha$ is split in the following way:

$$\mathbb{A}^\alpha = \begin{bmatrix} \mathbb{A}_m^\alpha & -L^\alpha \\ 0_{50\times 29} & \mathbb{A}_g^\alpha \end{bmatrix}, \tag{26}$$

where

$$\mathbb{A}_m^\alpha = \begin{bmatrix} \tau_\varepsilon u^\alpha & \rho\tau_Q\delta_\nu^\alpha & \beta_\varepsilon\delta_\nu^\alpha & \beta_n\delta_\nu^\alpha & \rho\tau_\varepsilon u^\alpha\delta_\nu^0 & \rho\tau_\varepsilon u^\alpha\delta_\nu^1 & \rho\tau_\varepsilon u^\alpha\delta_\nu^2 & \rho\tau_\varepsilon u^\alpha\delta_\nu^3 \\ \tau_P\Delta^{\mu\alpha} & \rho\tau_Q u^\alpha\delta_\nu^\mu & \beta_\varepsilon u^\alpha\delta_\nu^\mu & \beta_n u^\alpha\delta_\nu^\mu & \Pi_\nu^{\mu 0\alpha} & \Pi_\nu^{\mu 1\alpha} & \Pi_\nu^{\mu 2\alpha} & \Pi_\nu^{\mu 3\alpha} \\ -\Delta^{\mu\alpha} & 0_{4\times 4} & u^\alpha\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & 0_{4\times 4} & 0_{4\times 4} & u^\alpha\delta_\nu^\mu & n\Delta^{\mu\alpha}\delta_\nu^0 & n\Delta^{\mu\alpha}\delta_\nu^1 & n\Delta^{\mu\alpha}\delta_\nu^2 & n\Delta^{\mu\alpha}\delta_\nu^3 \\ 0_{4\times 1} & -\Delta_0^\alpha\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & u^\alpha\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & -\Delta_1^\alpha\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & u^\alpha\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 1} & -\Delta_2^\alpha\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & u^\alpha\delta_\nu^\mu & 0_{4\times 4} \\ 0_{4\times 1} & -\Delta_3^\alpha\delta_\nu^\mu & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & 0_{4\times 4} & u^\alpha\delta_\nu^\mu \end{bmatrix}, \tag{27}$$

while

$$\mathbb{A}_g^\alpha = \begin{bmatrix} u^\alpha I_{10} & -\Delta_0^\alpha I_{10} & -\Delta_1^\alpha I_{10} & -\Delta_2^\alpha I_{10} & -\Delta_3^\alpha I_{10} \\ -\Delta^{0\alpha} I_{10} & u^\alpha I_{10} & 0_{10\times 10} & 0_{10\times 10} & 0_{10\times 10} \\ -\Delta^{1\alpha} I_{10} & 0_{10\times 10} & u^\alpha I_{10} & 0_{10\times 10} & 0_{10\times 10} \\ -\Delta^{2\alpha} I_{10} & 0_{10\times 10} & 0_{10\times 10} & u^\alpha I_{10} & 0_{10\times 10} \\ -\Delta^{3\alpha} I_{10} & 0_{10\times 10} & 0_{10\times 10} & 0_{10\times 10} & u^\alpha I_{10} \end{bmatrix} \tag{28}$$

and

$$L^\alpha = \begin{bmatrix} 0_{1\times 10} & 0_{1\times 10} & 0_{1\times 10} & 0_{1\times 10} & 0_{1\times 10} \\ 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} \\ 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} \\ 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} & 0_{4\times 10} \\ \mathcal{X}_0^{\mu A\alpha} & \mathcal{Y}_{00}^{\mu A\alpha} & \mathcal{Y}_{01}^{\mu A\alpha} & \mathcal{Y}_{02}^{\mu A\alpha} & \mathcal{Y}_{03}^{\mu A\alpha} \\ \mathcal{X}_1^{\mu A\alpha} & \mathcal{Y}_{10}^{\mu A\alpha} & \mathcal{Y}_{11}^{\mu A\alpha} & \mathcal{Y}_{12}^{\mu A\alpha} & \mathcal{Y}_{13}^{\mu A\alpha} \\ \mathcal{X}_2^{\mu A\alpha} & \mathcal{Y}_{20}^{\mu A\alpha} & \mathcal{Y}_{21}^{\mu A\alpha} & \mathcal{Y}_{22}^{\mu A\alpha} & \mathcal{Y}_{23}^{\mu A\alpha} \\ \mathcal{X}_3^{\mu A\alpha} & \mathcal{Y}_{30}^{\mu A\alpha} & \mathcal{Y}_{31}^{\mu A\alpha} & \mathcal{Y}_{32}^{\mu A\alpha} & \mathcal{Y}_{33}^{\mu A\alpha} \end{bmatrix}. \tag{29}$$

We are now ready to establish that, when written as a first-order system as above, the equations of motion are strongly hyperbolic. In Sec. VII A, we show that the assumptions of Proposition I are not empty.

Proposition I.—Consider the system (22). Assume that (A1) with $\eta > 0$ holds and that Eq. (21) in Theorem I holds in strict form, i.e., with $>$ instead of $\geq$. Let ξ be a timelike covector. Then, (i) $\det(\mathfrak{A}^\alpha\xi_\alpha) \neq 0$, and (ii) for any spacelike vector ζ, the eigenvalue problem $(\zeta_\alpha + \Lambda\xi_\alpha)\mathfrak{A}^\alpha R = 0$ has only real eigenvalues Λ and a complete set of right eigenvectors R.

Proof.—The proof of this proposition is very lengthy and we refer the interested reader to check all the details and the proof presented in Appendix B.

C. Local well posedness

In this section, we establish the local existence and uniqueness of solutions to the nonlinear equations of motion in Eqs. (3) and (13).

We begin by noticing that Eq. (13) used the normalization $u^\mu u_\mu = -1$ to project the divergence of $T_{\mu\nu}$ and J^μ onto the directions parallel and orthogonal to u^μ. In order to show that the condition $u^\mu u_\mu = -1$ is propagated by the flow, it is more convenient to work directly with Eqs. (2) and (3). In order to complete the system, we differentiate $u^\mu u_\mu = -1$ twice in the u^μ direction:

$$u^\beta\nabla_\beta[u^\alpha\nabla_\alpha(u^\alpha u_\alpha)] = 0. \tag{30}$$

We also differentiate $\nabla_\mu J^\mu = 0$ once, as in Sec. IV:

$$u^\mu\nabla_\mu(\nabla_\nu J^\nu) = 0. \tag{31}$$

Observe that Eqs. (30) and (31) imply that $u^\mu u_\mu = -1$ and $\nabla_\mu J^\mu = 0$ hold at later times if these hold at the initial time.

The main result of this section can be found below.

Theorem II.—Let $(\Sigma, \mathring{g}_{\alpha\beta}, \hat{\kappa}_{\alpha\beta}, \mathring{\varepsilon}, \hat{\varepsilon}, \mathring{n}, \hat{n}, \mathring{u}^\alpha, \hat{u}^\alpha)$ be an initial-data set for the system composed of Einstein's equations (2) and $\nabla_\mu J^\mu = 0$, where $T_{\alpha\beta}$ and J^μ are given in Eq. (11). Assume that $\mathring{u}^\mu\mathring{u}_\mu = -1$, $\mathring{n} > 0$ [171], and that $\nabla_\mu J^\mu = 0$ holds for the initial data. Assume (A1) with $\eta > 0$ and suppose that Eqs. (21) of Theorem I hold in strict form and that the transport coefficients are analytic functions of their arguments. Finally, assume that $\mathring{g}_{\alpha\beta}, \mathring{\varepsilon}, \mathring{n}, \mathring{u}^\alpha \in H^N(\Sigma)$ and that $\hat{\kappa}_{\alpha\beta}, \hat{\varepsilon}, \hat{n}, \hat{u}^\alpha \in H^{N-1}(\Sigma)$, $N \geq 5$, where H^N is the Sobolev space. Then, there exists a globally hyperbolic development of the initial data. This globally hyperbolic development is unique if taken to be the maximum globally hyperbolic development of the initial data.

Proof.—The proof is found in Appendix C.

VI. NEW THEOREM ABOUT LINEAR STABILITY

Any ordinary fluid must be stable against small deviations from the thermodynamic equilibrium state [15]. (We only consider systems such that the equilibrium state is unique and has a finite correlation length. Therefore, in principle, our discussion does not apply to systems where the correlation length in equilibrium can become arbitrarily large, such as at a critical point.) We recall that in equilibrium $\beta_\mu = u_\mu/T$ must be a Killing vector, i.e., $\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0$, and also $\nabla_\alpha(\mu/T) = 0$ [36,172,173]. In Minkowski spacetime, nonrotating equilibrium corresponds to a class of states with constant T and μ and background flow velocity $u^\mu = \gamma(1, \mathbf{v})$ defined by a constant subluminal three-velocity $\mathbf{v}$, where $\gamma = 1/\sqrt{1-\mathbf{v}^2}$. (In this paper, we neglect the constant thermal vorticity term; see Ref. [172] for a nice discussion of its physical content and consequences.) In the local rest frame $\mathbf{v} = 0$ and the background flow is simply $u^\mu = (1,0,0,0)$. In a stable theory, small disturbances from the general equilibrium state $T \to T + \delta T(t, \mathbf{x})$, $\mu \to \mu + \delta\mu(t, \mathbf{x})$, and $u^\mu \to u^\mu + \delta u^\mu(t, \mathbf{x})$ (with $u_\mu \delta u^\mu = 0$) lead to small variations in the energy-momentum tensor and current, $\delta T^{\mu\nu}(t, \mathbf{x})$ and $\delta J^\mu(t, \mathbf{x})$, which decay with time.

The standard theories from Eckart and Landau-Lifshitz are unstable, as shown by Hiscock and Lindblom many years ago [25]. This instability appears because such theories possess exponentially growing, hence *unstable*, nonhydrodynamic modes, which spoil linear stability around equilibrium even at vanishing wave number. (The frequency of a hydrodynamic mode, such as a sound wave, vanishes in a spatially uniform state. On the other hand, a nonhydrodynamic mode correspond to a collective excitation that possesses nonzero frequency even at zero wave number.) For Landau-Lifshitz theory at zero chemical potential, this instability is only observed when considering a general equilibrium state with nonzero $\mathbf{v}$ [25,26,73], while in the case of Eckart the instability already appears even when $\mathbf{v} = 0$. The lack of causality in these approaches implies that it is not sufficient to investigate only the static $\mathbf{v} = 0$ case in order to determine the stability properties of a general equilibrium state where $\mathbf{v} \neq 0$, even though such states are in principle connected via a simple Lorentz transformation.

The necessity to investigate the stability properties of general equilibrium states where $\mathbf{v} \neq 0$ makes linear stability analyses of viscous hydrodynamic theories very complicated. Already in the local rest frame, finding whether the linear modes of the system are stable requires determining the sign of the imaginary part of the roots of a high-order polynomial, which becomes a daunting task when $\mathbf{v} \neq 0$ (see Refs. [35,70] for recent examples of how complicated a $\mathbf{v} \neq 0$ analysis can become in BDNK and MIS theory, respectively).

We prove below a new theorem that gives sufficient conditions for causal fluid dynamic equations to be linearly stable against disturbances of a general nonrotating equilibrium state with arbitrary background velocity. In this case, proving stability for the local rest frame implies stability in any other frame (note that the word frame here is used in the standard context of special relativity, i.e., to refer to an inertial observer, and has nothing to do with the concept of a hydrodynamic frame discussed in previous sections, which concerned the definition of hydrodynamic variables out of equilibrium) connected to the local rest frame via a Lorentz transformation. This general feature is expected to hold in any interacting relativistic system; i.e., no issues should appear if one simply observes a given system in another inertial frame. We then use this theorem in Sec. VII to find conditions under which the hydrodynamic theory presented here is stable. We remark that our results can be used to establish stability at nonzero $\mathbf{v} \neq 0$ in other theories as well, e.g., MIS, as long as the conditions discussed below are fulfilled.

A. Transforming a second-order system of linear differential equations into a first-order one

We begin by showing how one may convert a system of linear second-order PDEs into a first order one, as this is needed for the theory discussed in this paper. Let the system of linearized second-order PDEs be given by

$$\sum_b \mathfrak{M}(\partial)^a_b \delta\psi^b(X) = \mathfrak{N}(\partial\delta\psi)^a, \tag{32}$$

where a and b run from 1 to n, $\mathfrak{M}(\partial)^b_a$ are differential linear operators of order 2, $\mathfrak{N}(\partial\Psi)$ are linear terms containing derivatives of the perturbed fields $\delta\Psi$ up to order 1, and $\delta\psi^1(X), \ldots, \delta\psi^n(X)$ are the perturbed fields (for instance, $\delta\varepsilon$, δn, etc.). We suppose that Eq. (32) arises from the conservation laws $-u_\alpha \partial_\beta \delta T^{\alpha\beta} = 0$, $\Delta^\mu_\alpha \partial_\beta \delta T^{\alpha\beta} = 0$, and $\partial_\alpha \delta J^\alpha = -u_\beta u^\alpha \partial_\alpha \delta J^\beta + \Delta^{\alpha\beta}\partial_\alpha \delta J_\beta = 0$, where the first two come from $\partial_\alpha \delta T^{\alpha\beta} = 0$, while the last equation appears only when J^μ is included. In this manner, the derivatives in the equations of motion in Eq. (32) shall always appear as combinations of $u^\alpha \partial_\alpha$ and $\Delta^{\alpha\beta}\partial_\beta$ only. Thus, if the system in Eq. (32) has one or more second-order equations, it can be rewritten as a first-order system in the $N \equiv 5n$ new variables $\delta\bar{\psi}^a(X) = u^\alpha \partial_\alpha \psi^a(X)$ and $\delta\tilde{\psi}^a_\mu(X) = \Delta^\nu_\mu \partial_\nu \psi^a(X)$. These definitions automatically lead (32) to n first-order linear equations. One then needs to supplement those with the $4n$ dynamical equations that are missing. By means of the identity $\partial_\alpha \partial_\beta \psi^a(X) - \partial_\beta \partial_\alpha \psi^a(X) = 0$, one may find the extra $4n$ dynamical equations

$u^\alpha\partial_\alpha\delta\tilde{\psi}^a_\mu(X) - \Delta^\alpha_\mu\partial_\alpha\delta\bar{\psi}^a(X) = 0$, giving the needed $5n$ first-order dynamical equations, as required. In matrix form it becomes

$$\mathbb{A}^\alpha\partial_\alpha\delta\Psi(X) + \mathbb{B}\delta\Psi(X) = 0, \tag{33}$$

where $\mathbb{A}^\alpha$ and $\mathbb{B}$ are $N\times N$ constant real matrices and $\delta\Psi(X)$ is a $N\times 1$ column vector with entries $\delta\bar{\psi}^1, \delta\tilde{\psi}^1_\nu, \ldots, \delta\bar{\psi}^n, \delta\tilde{\psi}^n_\nu$. This ends the procedure. However, if one of the equations in Eq. (32) is already of first order but contains variables that have second-order derivative in other equations, then one can eliminate this equation by using it as a constraint to eliminate one of the variables. For example, consider the case of the ideal current $J^\mu = nu^\mu$. In this case, the conservation equation $\partial_\alpha J^\alpha = 0$ becomes $u^\alpha\partial_\alpha\delta n(X) + n\partial_\alpha\delta u^\alpha(X) = 0$. If $T^{\mu\nu}$ has shear or bulk contributions, for example, then the other equations must have second-order derivatives of δu^μ. Thus, one must write $\partial_\alpha\delta J^\alpha = 0$ as $\delta\bar{\psi} + n\delta\tilde{\psi}^\mu_\mu = 0$, where $\delta\tilde{\psi}^\mu_\nu = \Delta^\alpha_\nu\partial_\alpha u^\mu$ and $\delta\bar{\psi} = u^\alpha\partial_\alpha n$. This is a zeroth-order equation in the new variables and, therefore, is just a constraint. One may use this constraint in order to eliminate the variable $\delta\bar{\psi}$ in the other dynamical equations. Then, in this case one ends up with $5n-1$ dynamical equations for the $5n-1$ fields.

Finally, we remark that other approaches to viscous relativistic fluids, such as MIS, are already written in the format (33) in the linearized regime so the procedure to reduce the order of the equations of motion described above is not needed and one can move directly to the part below.

B. New linear stability theorem

To study linear stability, let us expand the perturbed fields in the Fourier modes $K^\mu = (i\Gamma, k^i)$ by substituting $\delta\Psi(X) \to \exp(iK_\mu X^\mu)\delta\Psi(K) = \exp(\Gamma t + ik_i x^i)\delta\Psi(K)$ in Eq. (33). The result is

$$iK_\mu\mathbb{A}^\mu\delta\Psi(K) + \mathbb{B}\delta\Psi(K) = 0. \tag{34}$$

Since K^μ appears, as aforementioned, as combinations of $-u^\alpha K_\alpha = \gamma(i\Gamma - k_i v^i)$ and $\Delta^{\mu\nu}K_\mu K_\nu = (u^\mu K_\mu)^2 + \Gamma^2 + k^2$, where $k^2 = k_i k^i$, then the direction of k^i is not relevant once one keeps v^i arbitrary. Thus, we may write $K^\mu = -n^\mu n_\nu K^\nu + \zeta^\mu\zeta_\nu K^\nu$, where n_μ is timelike and ζ_μ is spacelike, with $n^\mu n_\mu = -1$, $n_\mu\zeta^\mu = 0$, and $\zeta_\mu\zeta^\mu = 1$ [for example, it is common to choose $K^\mu = (K^0, k, 0, 0)$ so that n_μ and ζ_ν are $(-1,0,0,0)$ and (0,1,0,0), respectively]. In this case we define $\Omega = n_\alpha K^\alpha$ and $\kappa = \zeta_\alpha K^\alpha$ such that $K^\mu = -\Omega n^\mu + \kappa\zeta^\mu$ [70]. Then, Eq. (34) can be written as

$$i\Omega(-n_\alpha\mathbb{A}^\alpha)\delta\Psi(K) = -i\kappa\zeta_\alpha\mathbb{A}^\alpha\delta\Psi(K) - \mathbb{B}\delta\Psi(K). \tag{35}$$

The general form of the covectors n and ζ is $n_\alpha = \gamma_n(-1, c^i)$ for any c^i such that $0 \le c^i c_i < 1$ and where $\gamma_n = 1/\sqrt{1 - c^i c_i} \ge 1$, and $\zeta_\alpha = \gamma_\zeta(-\hat{d}^j c_j, \hat{d}^i) \ge 1$, where $\hat{d}^i\hat{d}_i = 1$ for an arbitrary unitary $\hat{d}^i$ and $\gamma_\zeta = 1/\sqrt{1 - (\hat{d}^i c_i)^2} \ge 1$. From the Cauchy-Schwarz inequality $(\hat{d}^i c_i)^2 \le |c^i|^2$ (here, $|c^i| = \sqrt{c^i c_i}$), then one obtains that

$$\gamma_n \ge \gamma_\zeta. \tag{36}$$

Stability demands that the perturbed modes $\Gamma = \Gamma(k^i)$ are such that $\Gamma_R \le 0$. Now, consider the eigenvalue problem,

$$(\Lambda n_\alpha + \zeta_\alpha)\mathbb{A}^\alpha\mathfrak{r} = 0, \tag{37}$$

where here Λ is the eigenvalue associated with the right eigenvector $\mathfrak{r}$.

Proposition II.—If Eq. (33) is causal, then the eigenvalues Λ are real and lie in the range $[-1, 1]$. Furthermore, $\det(n_\alpha\mathbb{A}^\alpha) \ne 0$.

Proof.—Causality demands that the roots of $Q(\xi) = \det(\xi_\alpha\mathbb{A}^\alpha) = 0$ are such that (i) $\xi_0 = \xi_0(\xi_i) \in \mathbb{R}$ and that (ii) the curves ξ_0 lie outside or over the light cone. In other words, $\xi^\alpha\xi_\alpha \ge 0$. If one writes $\xi_\alpha = \Lambda n_\alpha + \zeta_\alpha$, where n and ζ are real, then condition (i) means that Λ is real. On the other hand, since n and ζ are orthonormal, then condition (ii) means that $\xi_\alpha\xi^\alpha = -\Lambda^2 + 1 \ge 0$, which demands that $\Lambda^2 \le 1$, i.e., $\Lambda \in [-1, 1]$. Now, since $Q(\xi) = 0$ if and only if ξ is spacelike or lightlike, this means that $\det(n_\alpha\mathbb{A}^\alpha) \ne 0$. ■

Theorem III.—Let Eq. (37) have a set of N linearly independent (LI) real eigenvectors $\{\mathfrak{r}_1, \ldots, \mathfrak{r}_N\}$. If Eq. (33) is causal and stable in the local rest frame $\mathcal{O}$, then it is also stable in any other Lorentz frame $\mathcal{O}'$ connected to $\mathcal{O}$ by a Lorentz transformation.

Proof.—The details of the proof are found in Appendix D. However, we summarize some steps here. Note that causality enables us to invert the matrix $(-n_\alpha\mathbb{A}^\alpha)$. Then, it is possible to rewrite Eq. (35) as

$$\begin{aligned} &i\Omega\delta\Psi(K)^\dagger(R^T)^{-1}R^{-1}\delta\Psi(K) \\ &\quad = -i\kappa\delta\Psi(K)^\dagger(R^T)^{-1}R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}(\zeta_\alpha\mathbb{A}^\alpha)\delta\Psi(K) \\ &\qquad - \delta\Psi(K)^\dagger(R^T)^{-1}R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}\delta\Psi(K), \end{aligned} \tag{38}$$

where the dagger stands for the matrix transpose and complex conjugate operations altogether, T stands for matrix transpose operation, while R is the square matrix that diagonalizes $(-n_\alpha\mathbb{A}^\alpha)^{-1}(\zeta_\alpha\mathbb{A}^\alpha)$, since Eq. (37) has a complete set of real eigenvectors in $\mathbb{R}^n$ with only real eigenvalues. Then, we can expand $\delta\Psi(K)$ in terms of these eigenvectors. In the proof, it is shown that $\delta\Psi(K)^\dagger(R^T)^{-1}R^{-1}\delta\Psi(K)$ and $\delta\Psi(K)^\dagger(R^T)^{-1}R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}(\zeta_\alpha\mathbb{A}^\alpha)\delta\Psi(K)$ are real for any Lorentz frame. After some work, we demonstrate that, under the theorem's statements, stability reduces to the condition that the term $\delta\Psi(K)^\dagger(R^T)^{-1}R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}\delta\Psi(K)$ must be greater than or equal to zero. Since this is proven to be a scalar under Lorentz boosts, it can be computed in any frame. Thus, this implies that if the theory is stable in the LRF and obeys the other conditions of the theorem, it is stable in any other Lorentz frame.

We note that this result implies that the original system of linearized second-order PDEs in Eq. (32) is stable under the stated assumptions.

1. Applying the stability theorem to a toy model

To illustrate the application of the stability theorem, consider the simple model described by the fields ϕ and ψ^μ that obey the first-order dynamical linear equations:

$$u^\alpha\partial_\alpha\phi - \alpha\Delta^\alpha_\nu\partial_\alpha\psi^\nu + \lambda\phi = 0, \tag{39a}$$

$$u^\alpha\partial_\alpha\psi^\mu - \beta\Delta^{\mu\alpha}\partial_\alpha\phi = 0. \tag{39b}$$

We consider the case where u^μ is constant [$u^\mu = \gamma(1, v^i)$, with $\gamma = 1/\sqrt{1-v^2}$ and $v^2 = v^i v_i < 1$] as done in the stability theorem of the last section. If we write Eq. (39) in matrix form as

$$\mathbb{A}^\alpha\partial_\alpha\Psi(X) + \mathbb{B}\Psi(X) = 0, \tag{40}$$

where $\Psi(X) = (\phi, \psi^\nu)$ is a 5×1 column vector,

$$\mathbb{B} = \begin{bmatrix} \lambda & 0_{1\times4} \\ 0_{4\times1} & 0_{4\times4} \end{bmatrix}$$

and

$$\mathbb{A}^\alpha = \begin{bmatrix} u^\alpha & -\alpha\Delta^\alpha_\nu \\ -\beta\Delta^{\mu\alpha} & u^\alpha\delta^\mu_\nu \end{bmatrix} \tag{41}$$

are 5×5 matrices, the propagation modes $\omega = \omega(k^i)$ are obtained by means of the Fourier transform $\Psi(X) \to e^{iK_\mu X^\mu}\tilde{\Psi}(K)$, where $K^\mu = (\omega, k^i)$, and are the roots of $\det[iK_\alpha\mathbb{A}^\alpha + \mathbb{B}] = 0$. Let us write $\omega = i\Gamma$. Then, stability requires that $\mathrm{Re}(\Gamma) \le 0$. In the local rest frame, these equations are $\Gamma = 0$ and $\Gamma^2 + \lambda\Gamma + \alpha\beta k^2 = 0$, where $k^2 = k_i k^i$. Then, stability in the LRF implies the conditions

$$\alpha\beta \ge 0, \tag{42a}$$

$$\lambda \ge 0. \tag{42b}$$

As for the boosted frame obtained by the Lorentz transform $\Gamma \to \gamma(\Gamma + iv^i k_i)$ and $k^2 \to \Gamma^2 + k^2 - \gamma^2(\Gamma + iv^i k_i)^2$, the first root is $\Gamma = -iv^i k_i$, which is stable, while the remaining two roots demand (after a long but straightforward computation)

$$\lambda \ge 0, \tag{43a}$$

$$0 \le \alpha\beta \le 1. \tag{43b}$$

To verify stability via the stability theorem proven in this paper, we must verify conditions where Eq. (39) is causal and if the matrix $\Phi_\alpha\mathbb{A}^\alpha$ (with $\Phi_\alpha = \Lambda n_\alpha + \zeta_\alpha$, n and ζ are the unitary timelike and spacelike covectors defined in the text) has a complete set of eigenvectors in $\mathbb{R}^5$. Proposition I guarantees that if Eq. (39) is causal, then $\Lambda \in \mathbb{R}$. In order to study causality, we compute the characteristics ξ_α of the system, which reduces to the roots of $\det(\mathbb{A}^\alpha\xi_\alpha) = (u^\alpha\xi_\alpha)^3[(u^\beta\xi_\beta)^2 - \alpha\beta\Delta^{\mu\nu}\xi_\mu\xi_\nu] = 0$. Causal roots must be real and obey $\xi_\mu\xi^\mu \ge 0$, which gives the conditions $0 \le \alpha\beta \le 1$. These conditions, together with stability in the LRF, coincide with the conditions obtained by means of the above direct calculation. However, if we did not know, *a priori*, the conditions for stability in any frame (which is the case when considering higher-order polynomials for the modes), we would still have to obtain the eigenvectors of

$$\Phi_\alpha\mathbb{A}^\alpha = \begin{bmatrix} u^\alpha\Phi_\alpha & -\alpha\Delta^\alpha_\nu\Phi_\alpha \\ -\beta\Delta^{\mu\alpha}\Phi_\alpha & u^\alpha\Phi_\alpha\delta^\mu_\nu \end{bmatrix}. \tag{44}$$

We can do it firstly by obtaining the eigenvalues Λ, which may be easily obtained by changing $\xi_\alpha \to \Phi_\alpha$ in the computation of the characteristics. With that result one obtains the eigenvalue $\Lambda^{(1)}$ that is the root of $u^\alpha\Phi^{(1)}_\alpha = 0$ with multiplicity 3 and the eigenvalue $\Lambda^{(2)}_\pm$, which give the two roots of $(u^\beta\Phi^{(2)}_{\pm\beta})^2 - \alpha\beta\Delta^{\mu\nu}\Phi^{(2)}_{\pm\mu}\Phi^{(2)}_{\pm\nu} = 0$. The corresponding eigenvectors are as follow.

(i) For $u^\alpha\Phi^{(1)}_\alpha = 0$, the system $\Phi^{(1)}_\alpha\mathbb{A}^\alpha r^{(1)}_a = 0$ has as eigenvectors the three linearly independent vectors given by

$$r^{(1)}_a = \begin{bmatrix} 0 \\ w^\nu_a \end{bmatrix}, \tag{45}$$

where $\{w^\nu_a\}^3_{a=1}$ is a set of three linearly independent vectors orthogonal to the vector $\Delta^{\mu\alpha}\Phi^{(1)}_\alpha$.

(ii) For $(u^\beta\Phi^{(2)}_{\pm\beta})^2 - \alpha\beta\Delta^{\mu\nu}\Phi^{(2)}_{\pm\mu}\Phi^{(2)}_{\pm\nu} = 0$, we assume $\alpha\beta \ne 0$ and obtain the two eigenvectors,

$$r^{(2)}_\pm = \begin{bmatrix} u^\alpha\Phi^{(2)}_{\pm\alpha} \\ \beta\Delta^{\nu\alpha}\Phi^{(2)}_{\pm\alpha} \end{bmatrix}. \tag{46}$$

[Note that in the special case $\alpha\beta = 0$, the root $u^\alpha\Psi_\alpha = 0$ is the only root with multiplicity 5. We end up with two distinct situations: first, if $\alpha \ne 0$ or $\beta \ne 0$ with $\alpha\beta = 0$, then one obtains four LI eigenvectors as can be seen from Eqs. (40) and (41). On the other hand, if $\alpha = \beta = 0$, then the system is already diagonal and the theorem applies directly.] Thus, Eq. (46) completes the remaining two linearly independent eigenvectors since $\Lambda_\pm$ are distinct eigenvalues, giving the five LI eigenvectors. Then, the stability theorem states that the system is stable if $\lambda \geq 0$ and $0 < \alpha\beta \leq 1$ or if $\lambda \geq 0$ and $\alpha = \beta = 0$. Note that there is a slight difference from the condition obtained from the direct calculation. To wit, it does not include the case $\alpha\beta = 0$ with α or β different from zero. The conclusion is that stability in any frame does not necessarily imply strong hyperbolicity. However, strong hyperbolicity plus causality plus stability in the LRF implies stability in any boosted frame. In other words, stability may occur outside the conditions imposed by the theorem.

2. Applying the stability theorem to the MIS system

As another example of the usefulness of Theorem III, let us briefly comment how it can be used to recover the stability conditions of the MIS equations [24] in the presence of bulk viscosity. More precisely, we take the MIS-like equations studied in Ref. [71] where only bulk viscous effects have been considered. In that case, it was proven that there exist conditions such that the system of PDEs is nonlinearly causal and symmetric hyperbolic; hence the principal part of the equations is diagonalizable. The linear version of such equations forms a system that is also symmetric hyperbolic and the conditions for stability needed for the application of Theorem III can be shown to agree with those found in Ref. [24] for the case where only bulk viscosity is present.

VII. CONDITIONS FOR LINEAR STABILITY

We now apply the theorem proved in the last section to determine conditions that ensure the stability of the hydrodynamic theory proposed in this paper. Let us first define

$$D \equiv \rho c_s^2(\tau_\varepsilon + \tau_Q) + \zeta + \frac{4\eta}{3} + \sigma\kappa_\varepsilon \tag{47}$$

and

$$\begin{aligned} E &\equiv \sigma[p'_\varepsilon \kappa_s - c_s^2 \kappa_\varepsilon] \\ &= \sigma T\rho\left[\left(\frac{\partial P}{\partial \varepsilon}\right)_n \left(\frac{\partial(\mu/T)}{\partial n}\right)_\varepsilon - \left(\frac{\partial P}{\partial n}\right)_\varepsilon \left(\frac{\partial(\mu/T)}{\partial \varepsilon}\right)_n\right], \end{aligned} \tag{48}$$

where $\kappa_s = (T\rho^2/n)[\partial(\mu/T)/\partial\varepsilon]_{\bar{s}} = \kappa_\varepsilon + \kappa_n$, $\kappa_\varepsilon = (T\rho^2/n)\times[\partial(\mu/T)/\partial\varepsilon]_n$, $\kappa_n = (T\rho)[\partial(\mu/T)/\partial n]_\varepsilon$, and $p'_\varepsilon = (\partial P/\partial\varepsilon)_n$. Standard thermodynamic identities imply that $p'_\varepsilon\kappa_s - c_s^2\kappa_\varepsilon > 0$, then $E \geq 0$ from (A1). By assuming the Cowling approximation [174] with $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ and $\delta g_{\mu\nu} = 0$, we find that the system described by Eq. (13) is linearly stable if it is causal within the strict form of the inequalities in Eq. (21) together with the additional restriction $\eta > 0$ in (A1) and

$$(\tau_\varepsilon + \tau_Q)|B| \geq \tau_\varepsilon\tau_Q D \geq \rho c_s^2 \tau_\varepsilon\tau_Q(\tau_\varepsilon + \tau_Q), \tag{49a}$$

$$\begin{aligned} &(\tau_\varepsilon + \tau_Q)|B|D + \rho\tau_\varepsilon\tau_Q(\tau_\varepsilon + \tau_Q)E \\ &\quad > \tau_\varepsilon\tau_Q D^2 + \rho(\tau_\varepsilon + \tau_Q)^2 C, \end{aligned} \tag{49b}$$

$$c_s^2 D - E \geq \rho c_s^4(\tau_\varepsilon + \tau_Q), \tag{49c}$$

$$\begin{aligned} &(\tau_\varepsilon + \tau_Q)[|B|(c_s^2 D - 2E) + 2c_s^2\rho\tau_\varepsilon\tau_Q E + CD] \\ &\quad > 2c_s^2\rho(\tau_\varepsilon + \tau_Q)^2 C + \tau_\varepsilon\tau_Q D(c_s^2 D - E), \end{aligned} \tag{49d}$$

$$\begin{aligned} &|B|D[C(\tau_\varepsilon + \tau_Q) + E\tau_\varepsilon\tau_Q] + 2\rho\tau_\varepsilon\tau_Q(\tau_\varepsilon + \tau_Q)CE \\ &\quad > \rho C^2(\tau_\varepsilon + \tau_Q)^2 + \tau_\varepsilon\tau_Q(CD^2 + \rho\tau_\varepsilon\tau_Q E^2) \\ &\quad + B^2 E(\tau_\varepsilon + \tau_Q), \end{aligned} \tag{49e}$$

where B and C are given by

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q, \tag{50a}$$

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right), \tag{50b}$$

as in Eq. (A5), with $|B| = -B > 0$ from Eq. (21c) in the strict form.

To prove the statement above, as before we may expand the perturbations $\delta\Psi = (\delta\varepsilon, \delta u^\mu, \delta n)$ in Fourier modes by means of the substitution $\delta\Psi(X) \to \exp[T(\Gamma t + k_i x^i)]\delta\Psi(K)$, where $K^\mu = (i\Gamma, k^i)$ is dimensionless due to the introduction of background temperature T in the exponent. We begin by proving stability in the local rest frame, where the modes are the roots of the shear and sound polynomials,

$$\text{shear channel:}\ \bar{\tau}_Q\Gamma^2 + \bar{\eta}k^2 + \Gamma = 0, \tag{51a}$$

$$\text{sound channel:}\ a_0\Gamma^5 + a_1\Gamma^4 + a_2\Gamma^3 + a_3\Gamma^2 + a_4\Gamma + a_5 = 0, \tag{51b}$$

where $k^2 = k^i k_i$ and

$$a_0 = \bar{\tau}_\varepsilon\bar{\tau}_Q, \tag{52a}$$

$$a_1 = \bar{\tau}_\varepsilon + \bar{\tau}_Q, \tag{52b}$$

$$a_2 = 1 + k^2|\bar{B}|, \tag{52c}$$

$$a_3 = k^2\bar{D}, \tag{52d}$$

$$a_4 = c_s^2 k^2 + k^4\bar{C}, \tag{52e}$$

$$a_5 = k^4\bar{E}. \tag{52f}$$

We defined the dimensionless quantities $\bar{\tau}_Q = T\tau_Q$, $\bar{\tau}_\varepsilon = T\tau_\varepsilon$, $\bar{\eta} = T\eta/\rho$, $\bar{B} = (T^2/\rho)B$, $\bar{C} = (T^2/\rho)C$, $\bar{D} = (T/\rho)D$, and $\bar{E} = (T/\rho)E$. From the second inequality in Eq. (21c) in its strict form one obtains that $\bar{B} < 0$ (see the definition of a_2). The analysis of stability in the LRF goes as follows.

Shear stability conditions.—The second-order polynomial (51a) has two roots with $\Gamma_R \leq 0$ only if $\tau_Q > 0$ and $\eta \geq 0$, which is in accordance with assumption (A1). One can see that τ_Q clearly acts as a relaxation time (the same role is played by the shear relaxation time coefficient τ_π present in MIS theory) for the shear channel, which ensures causality. In fact, the condition $\tau_Q > 0$ is clear since the leading contribution to the nonhydrodynamic frequency in this channel goes as $1/\tau_Q$ at zero wave number.

Sound stability conditions.—As for the sound channel in the rest frame, by means of the Routh-Hurwitz criterion [175], the necessary and sufficient conditions for $\Gamma_R < 0$ are (i) a_0, $a_1 > 0$, (ii) $a_1a_2 - a_0a_3 > 0$, (iii) $a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5) > 0$, (iv) $(a_1a_4 - a_0a_5)[a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5)] - a_5(a_1a_2 - a_0a_3)^2 > 0$, and (v) $a_5 > 0$. Condition (i) is already satisfied from (A1). Condition (ii) corresponds to the first inequality in Eq. (49a), while (iii) is the second inequality in Eqs. (49a) and (49b). Condition (iv) corresponds to Eqs. (49c)–(49e). Given that $E \geq 0$, thus, when $E = 0$ and (i)–(iv) are observed, then $\Gamma_R \leq 0$, which is in accordance with stability. Also, if $k = 0$, then $\Gamma_R \leq 0$ (three zero roots and two negative roots) because $a_0, a_1, a_2 > 0$ from (A1). Hence, the system is linearly stable in the local rest frame.

We remark that our system displays three types of hydrodynamic modes and three nonhydrodynamic modes. In the small k expansion that typically defines the linearized hydrodynamic regime, our shear channel gives a diffusive hydrodynamic mode with (real) frequency $\omega(k) = -ik^2\eta/(\varepsilon + P) + \cdots$, while in the sound channel one finds proper sound waves with $\omega(k) = \pm c_s k - ik^2\Gamma_s/2 + \cdots$ and also a heat diffusion mode with $\omega(k) = -iDk^2 + \cdots$, where $D \sim \sigma$, and $\Gamma_s = \Gamma_s(\eta, \zeta, \sigma)$ just as in Eckart theory (see Ref. [35] for their detailed expressions). Therefore, our theory has the same physical content of Eckart's theory in the hydrodynamic regime. On the other hand, the shear channel has a nonhydrodynamic mode with frequency given by $\omega(k) = -i/\tau_Q + \cdots$, while the sound channel has two nonhydrodynamic modes with frequency $\omega(k) = -i/\tau_\varepsilon + \cdots$ and $\omega(k) = -i/\tau_Q + \cdots$ in the low k limit. These nonhydrodynamic modes parametrize the UV behavior of the system in a way that ensures causality and stability, making sure that the theory is well defined (though, of course, not accurate) even outside the typical domain of validity of hydrodynamics.

The complete proof of linear stability demands an analysis of the linearized system around an equilibrium state at nonzero velocity. In this regard, we shall use the results presented in Sec. VI B. We first write the system in Eq. (13) as a first-order linear system of PDEs. Then, since we already have proven causality and also linear stability in the LRF, it remains to be shown that the first-order counterpart of Eq. (13) is diagonalizable in the sense of Eq. (D2). This is done below.

First-order system.—Following Sec. VI A, we may define $\delta V = u^\alpha\partial_\alpha\delta\varepsilon$, $\delta\mathcal{V}^\mu = \Delta^{\mu\alpha}\partial_\alpha\delta\varepsilon$, $\delta W = u^\alpha\partial_\alpha\delta n$, $\delta\mathcal{W}^\mu = \Delta^{\mu\alpha}\partial_\alpha\delta W$, $\delta S^\mu = u^\alpha\partial_\alpha\delta u^\mu$, $\delta\mathcal{S}^\nu_\lambda = \Delta^\alpha_\lambda\partial_\alpha\delta u^\nu$. Since the current is ideal, i.e., $J^\mu = nu^\nu$, then the linearized conservation equation $\partial_\mu\delta J^\mu = \delta W + n\delta\mathcal{S}^\nu_\nu = 0$ enables us to eliminate δW from the new system of equations. Hence, the first-order equations become

$$\tau_\varepsilon u^\alpha\partial_\alpha\delta V + \rho\tau_Q\partial_\alpha\delta S^\alpha + \beta_\varepsilon\partial_\alpha\delta\mathcal{V}^\alpha + \beta_n\partial_\alpha\delta\mathcal{W}^\alpha + \rho\tau_\varepsilon u^\alpha\partial_\alpha\delta\mathcal{S}^\nu_\nu + \delta V + \rho\delta\mathcal{S}^\nu_\nu = 0, \tag{53a}$$

$$\tau_P\Delta^{\mu\alpha}\partial_\alpha\delta V + \rho\tau_Q u^\alpha\partial_\alpha\delta S^\mu + \beta_\varepsilon u^\alpha\partial_\alpha\delta\mathcal{V}^\mu + \beta_n u^\alpha\partial_\alpha\delta\mathcal{W}^\mu + \Pi^{\mu\lambda\alpha}_\nu\partial_\nu\delta\mathcal{S}^\nu_\lambda + p'_\varepsilon\delta\mathcal{V}^\mu + p'_n\delta\mathcal{W}^\mu + \rho\delta S^\mu = 0, \tag{53b}$$

$$u^\alpha\partial_\alpha\delta\mathcal{V}^\mu - \Delta^{\mu\alpha}\partial_\alpha\delta V = 0, \tag{53c}$$

$$u^\alpha\partial_\alpha\delta\mathcal{W}^\mu + n\Delta^{\mu\alpha}\partial_\alpha\delta\mathcal{S}^\nu_\nu = 0, \tag{53d}$$

$$u^\alpha\partial_\alpha\delta\mathcal{S}^\mu_\lambda - \Delta^\alpha_\lambda\partial_\alpha\delta S^\mu = 0, \tag{53e}$$

where $p'_n = (\partial P/\partial n)_\varepsilon$ and

$$\Pi^{\mu\lambda\alpha}_\nu = -\eta(\Delta^{\mu\lambda}\delta^\alpha_\nu + \Delta^{\lambda\alpha}\delta^\mu_\nu) + \left(\rho\tau_P - \zeta + \frac{2\eta}{3}\right)\Delta^{\mu\alpha}\delta^\lambda_\nu. \tag{54}$$

The supplemental equations (53c)–(53e) come from the identities $\partial_\alpha\partial_\beta\delta\varepsilon - \partial_\beta\partial_\alpha\delta\varepsilon = 0$, $\partial_\alpha\partial_\beta\delta n - \partial_\beta\partial_\alpha\delta n = 0$, and $\partial_\alpha\partial_\beta\delta u^\mu - \partial_\beta\partial_\alpha\delta u^\mu = 0$, respectively, when contracted with $u^\alpha\Delta^{\beta\lambda}$. In particular, in Eq. (53d) we have substituted $\delta W = -n\delta\mathcal{S}^\nu_\nu$ that comes from the conservation equation of J^μ. Then, we may write Eq. (53) in matrix form, $\mathbb{A}^\alpha\partial_\alpha\delta\Psi(X) + \mathbb{B}\Psi(X) = 0$, were $\delta\Psi(X)$ is the 29×1 column matrix with entries $\delta V, \delta S^\nu, \delta\mathcal{V}^\nu, \delta\mathcal{W}^\nu, \delta\mathcal{S}^\nu_0$, $\delta\mathcal{S}^\nu_1, \delta\mathcal{S}^\nu_2, \delta\mathcal{S}^\nu_3$,

$$\mathbb{A}^\alpha = \begin{bmatrix} \tau_\varepsilon u^\alpha & \rho\tau_Q\delta^\alpha_\nu & \beta_\varepsilon\delta^\alpha_\nu & \beta_n\delta^\alpha_\nu & \rho\tau_\varepsilon u^\alpha\delta^0_\nu & \rho\tau_\varepsilon u^\alpha\delta^1_\nu & \rho\tau_\varepsilon u^\alpha\delta^2_\nu & \rho\tau_\varepsilon u^\alpha\delta^3_\nu \\ \tau_P\Delta^{\mu\alpha} & \rho\tau_Q u^\alpha\delta^\mu_\nu & \beta_\varepsilon u^\alpha\delta^\mu_\nu & \beta_n u^\alpha\delta^\mu_\nu & \Pi^{\mu0\alpha}_\nu & \Pi^{\mu1\alpha}_\nu & \Pi^{\mu2\alpha}_\nu & \Pi^{\mu3\alpha}_\nu \\ -\Delta^{\mu\alpha} & 0_{4\times4} & u^\alpha\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta^\mu_\nu & n\Delta^{\mu\alpha}\delta^0_\nu & n\Delta^{\mu\alpha}\delta^1_\nu & n\Delta^{\mu\alpha}\delta^2_\nu & n\Delta^{\mu\alpha}\delta^3_\nu \\ 0_{4\times1} & -\Delta^\alpha_0\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & -\Delta^\alpha_1\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & -\Delta^\alpha_2\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta^\mu_\nu & 0_{4\times4} \\ 0_{4\times1} & -\Delta^\alpha_3\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & u^\alpha\delta^\mu_\nu \end{bmatrix}, \tag{55}$$

and

$$\mathbb{B} = \begin{bmatrix} 1 & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & \rho\delta^0_\nu & \rho\delta^1_\nu & \rho\delta^1_\nu & \rho\delta^3_\nu \\ 0_{4\times1} & \rho\delta^\mu_\nu & p'_\varepsilon\delta^\mu_\nu & p'_n\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \end{bmatrix}. \tag{56}$$

We must now obtain the eigenvectors of Eq. (37). However, note that $\mathbb{A}^\alpha$ above is exactly the same as the matrix $\mathbb{A}^\alpha_m$ in Eq. (27) with the difference that now the coefficients of $\mathbb{A}^\alpha$ are constants. We have already proven in Sec. V that the matrix $\mathbb{A}^\alpha_m$ in Eq. (37) has real eigenvalues and a complete set of eigenvectors in $\mathbb{R}^{29}$. The same solution is true for $\mathbb{A}^\alpha$ in Eq. (37) if we change $\xi_\alpha \to n_\alpha$ (and also $\mathbb{A}^\alpha_m \to \mathbb{A}^\alpha$) in the results for the matter sector in Sec. V. Thus, the 29×29 matrix $(-n_\alpha\mathbb{A}^\alpha)\zeta_\beta\mathbb{A}^\beta$ is diagonalizable, completing the requirements from Theorem III. This shows that the theory is linearly stable in any other reference frame $\mathcal{O}'$ connected via a Lorentz transformation. Therefore, one then obtains that our set of linearized second-order PDEs is stable in any equilibrium state.

A. Fulfilling the causality, local well posedness, and linear stability conditions

We now give a simple example that illustrates that the set of linear stability conditions (and consequently, causality and local well posedness, since those are part of the linear stability conditions) is not empty. Let us analyze the case where $\tau_Q = \tau_\varepsilon$ and $\tau_P = c_s^2\tau_\varepsilon$, assuming an equation of state $P = P(\varepsilon)$, with $c_s^2 = p'_\varepsilon = 1/2$. Also, assume that $\zeta + 4\eta/3 > 0$ (their specific values are not relevant as far as they are positive and $\eta > 0$ for the sake of the stability and well-posedness theorems). Then, one may easily verify that the causality conditions (21) hold in their strict form, as required, and that the remaining conditions (49) are also observed when $\rho\tau_\varepsilon = 8(\zeta + 4\eta/3)$, $\kappa_\varepsilon = \kappa_s/2 = 1/4$, and in the three different situations, namely, $\sigma/(\zeta + 4\eta/3) = 0$, $1/4$, and 1.

VIII. CONCLUSIONS AND OUTLOOK

In this work, we presented the first generalization of relativistic Navier-Stokes theory that simultaneously satisfies the following properties: the system, with or without coupling to Einstein's equations, is causal, strongly hyperbolic, and locally well posed (see the content of Theorems I and II); equilibrium states in flat spacetime are stable (consequence of Theorem III); all dissipative contributions (shear viscosity, bulk viscosity, and heat conductivity) are included; and finally the effects from nonzero baryon number are also taken into account. All of the above hold without any simplifying symmetry assumptions and are mathematically rigorously established. In addition, entropy production is non-negative in the regime of validity of this effective theory.

This is accomplished in a natural way using a generalized Navier-Stokes theory containing only the original hydrodynamic variables, which is different than other approaches where the space of variables is extended (such as in Müller-Israel-Stewart theory). However, it is important to remark that the meaning of the hydrodynamic variables in our work is different than in standard approaches, such as Refs. [15,52]. In fact, in the context of the formalism put forward by Bemfica *et al.* [32,33] and Kovtun [34], our formulation uses a definition for the hydrodynamic variables (i.e., our choice of hydrodynamic frame) that is not standard as there are nonzero out-of-equilibrium corrections to the energy density and there is energy and heat diffusion even at zero baryon density. Despite these necessary differences (imposed by causality and stability), the theory still provides the simplest causal and strongly hyperbolic generalization of Eckart's original theory [52], sharing the same physical properties in the hydrodynamic regime (for instance, both theories have the same spectrum of hydrodynamic modes). However, differently than Eckart's approach, our formulation is fully compatible with the postulates of general relativity, and its physical content in dynamical settings can be readily investigated using numerical relativity simulations. In fact, we hope that the framework presented here will provide the starting point for future systematic studies of viscous phenomena in the presence of strong gravitational fields, such as in neutron star mergers.

Motivated by the task of establishing stability of general equilibrium states in flat spacetime, in this work we also proved a new general result (see Theorem III) concerning the stability of relativistic fluids. In fact, we found conditions that causal relativistic fluids should satisfy such that stability around the static equilibrium state directly implies stability in any other equilibrium state at nonzero background velocity. Theorem III is very general and its regime of applicability goes beyond BDNK theories and it could also be relevant when investigating the stability properties of other sets of linear equations of motion as well. In this regard, see the discussion in Sec. II B, and see also Secs VI B 1 and VI B 2 for further examples of the applicability of Theorem III.

Our generalized Navier-Stokes theory can be used to understand how matter in general relativity starts to deviate from equilibrium. An immediate application is in the modeling of viscous effects in neutron star mergers. Our approach can be useful in simulations that aim at determining the fate of the hypermassive remnant formed after the merger of neutron stars, hopefully leading to a better quantitative understanding of their evolution and eventual gravitational collapse toward a black hole. Differently than any other approach in the literature, the new features displayed by our formulation and its strongly hyperbolic character make it a suitable candidate to be used in such simulations. This will be especially relevant also when considering how viscous effects may modify the gravitational wave signals emitted soon after the merger [12,14]. In this regard, we remark that previous simulations performed in Ref. [11] employed a formulation of relativistic viscous hydrodynamics where the key properties studied here (causality, strong hyperbolicity, and local well posedness) are not known to hold in the nonlinear regime.

Our work is applicable in the case of baryon-rich matter, such as that formed in neutron star mergers or in low-energy heavy-ion collisions. The latter include the experimental efforts in the beam energy scan program at RHIC [176], the STAR fixed-target program [176], the HADES experiment at GSI [177], the future FAIR facility at GSI [178], and also NICA [179]. For a discussion of viscous effects in low-energy heavy-ion collisions at nonzero density, see Refs. [85,113,180]. High-energy heavy-ion collisions, such as those studied at the LHC, involve a different regime than the one considered here where the net baryon number can be very small and, thus, that case is better understood using a different formulation such as the one proposed in Ref. [35], also in the context of the BDNK formalism.

In our approach, we only take into account first-order derivative corrections to the dynamics. Therefore, the domain of validity of our theory is currently limited by the size of such deviations. Hence, further work is needed to extend our analysis, incorporating higher-order derivative corrections, to get a better understanding of what happens as the system gets farther and farther from equilibrium. In this context, it would be interesting to extend our equations to include second-order corrections and consider also, more generally, the large order behavior of the gradient expansion in an arbitrary hydrodynamic frame. The latter will be different than most approaches to the gradient expansion since in BDNK the constitutive relations contain time derivatives even in the local rest frame of the fluid. This essential difference has important consequences in a kinetic theory formulation; see the original references [32,33]. The large order behavior of the relativistic gradient series has been recently the focus of several works [84,181–194], and it would be interesting to extend such analyses to include the type of theories investigated here.

There are a number of ways in which our work could be extended or improved. First, it would be useful to obtain a better qualitative understanding of why some hydrodynamic frames (such as the Landau-Lifshitz frame or the Eckart frame) are not compatible with causality and stability in the BDNK approach, given that the situation is different in other formulations. In fact, the Landau frame seems to display no significant issues in the case of MIS-like theories in the nonlinear regime at least at zero chemical potential, as demonstrated in Ref. [72]. Perhaps a more in-depth investigation of how BDNK emerges in kinetic theory, going beyond the original work done in Refs. [32,33], can be useful in this regard (see also the recent work [148]). Also, it would be interesting to use the BDNK approach to investigate causality and stability in more exotic cases, such as in relativistic superfluids. Furthermore, the inclusion of electromagnetic field effects in the dynamics of relativistic viscous fluids can also be of particular relevance, especially in the context of neutron star mergers [195] and high-energy heavy-ion collisions [196]. This problem has been recently investigated using other formulations of viscous fluid dynamics, see for instance Refs. [197–202], and also most recently in the BDNK approach in Ref. [203]. Consistent modeling of relativistic viscous fluid dynamics coupled to electromagnetic fields can also be relevant to determine the importance of dissipative processes in the dynamics and radiative properties of slowly accreting black holes, as discussed in Ref. [197].

Further work needs to be done to understand the global in-time features of solutions of relativistic viscous fluid dynamics. For instance, one may investigate the presence of shocks, which is a topic widely investigated in the context of ideal fluids [21,145,204–206] and was done in Ref. [94] for the MIS theory (see Sec. II B for further discussion on shocks). The importance of hydrodynamic shocks has been recognized both in an astrophysical setting [197] as well as in the study of jets in the quark-gluon plasma [207–219]. We also remark that one task that we have not done here was the construction of initial data for the full Einstein plus fluid system by solving the Einstein constraint equations. We believe that standard arguments to handle the constraints [21] will be applicable in our case. This will be investigated in detail in a future work.

We believe our work will also be relevant to give insight into the physics of turbulent fluids embedded in general relativity. The fact that the equations of motion of the viscous fluid must be hyperbolic in relativity stands in sharp contrast to the parabolic nature of the nonrelativistic Navier-Stokes equations, usually employed in studies of turbulence. Recent works in Refs. [18,220] tackled the problem of turbulence in the relativistic regime and our formulation may be very useful in this regard, as it provides a simple strongly hyperbolic generalization of Eckart's theory that is fully compatible with general relativity.

In summary, in this paper we propose a new solution to the question initiated by Eckart in 1940 concerning the motion of viscous fluids in relativity. Our approach is rooted in well-known physical principles and solid mathematics, displays a number of desired properties, and extends the state of the art of the field in a number of ways. Potential applications of the formalism presented here spread across a numbers of areas, including astrophysics, nuclear physics, cosmology, and mathematical physics. This work establishes for the first time a common unifying framework, from heavy-ion collisions to neutron stars, that can be used to discover the novel properties displayed by ultradense baryonic matter as it evolves in spacetime.

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APPENDIX A: PROOF OF THEOREM I

We only consider the 10 independent components of the metric and, thus, this system of equations can be written in terms of a 16×1 column vector $\Psi = (\varepsilon, n, u^\nu, g_{\mu\nu})$, and its equation of motion in Eq. (17) can be expressed in matrix form as $\mathfrak{M}(\partial)\Psi = \mathfrak{N}$, where $\mathfrak{N}$ contains the $\mathcal{B}$ terms that do not enter in the principal part. The matrix $\mathfrak{M}(\partial)$ is given by

$$\mathfrak{M}(\partial) = \begin{bmatrix} \mathbb{M}(\partial) & \mathfrak{b}(\partial) \\ 0_{6\times 10} & I_{10} g^{\alpha\beta}\partial^2_{\alpha\beta} \end{bmatrix}, \tag{A1}$$

where the 6×10 matrix $\mathfrak{b}(\partial)$ contains the $\tilde{\mathcal{B}}$ terms and

$$\mathbb{M}(\partial) = \begin{bmatrix} 0 & u^\alpha u^\beta & n\delta_\nu^{(\alpha} u^{\beta)} \\ (\tau_\varepsilon u^\alpha u^\beta + \beta_\varepsilon \Delta^{\alpha\beta}) & \beta_n \Delta^{\alpha\beta} & \rho(\tau_\varepsilon + \tau_Q) u^{(\alpha}\delta_\nu^{\beta)} \\ (\beta_\varepsilon + \tau_P) u^{(\alpha}\Delta^{\beta)\mu} & \beta_n u^{(\alpha}\Delta^{\beta)\mu} & C_\nu^{\mu\alpha\beta} \end{bmatrix} \times \partial^2_{\alpha\beta}. \tag{A2}$$

The system's characteristics are obtained by replacing $\partial_\alpha \to \xi_\alpha$ and determining the roots of $\det[\mathfrak{M}(\xi)] = 0$. The system is causal when the solutions for $\xi_\alpha = (\xi_0(\xi_i), \xi_i)$ are such that condition 1 (Cond-1) ξ_α is real and condition 2 (Cond-2) $\xi_\mu \xi^\mu \geq 0$ [21]. It is easy to see that $\det[\mathfrak{M}(\xi)] = (\xi_\alpha \xi^\alpha)^{10} \det[\mathbb{M}(\xi)]$. The roots associated with the vanishing of the overall factor $(\xi_\alpha\xi^\alpha)^{10} = 0$ coming from the gravitational sector are clearly causal. The remaining roots come from $\det[\mathbb{M}(\xi)] = 0$, which we will investigate next.

We first define $b \equiv u^\alpha \xi_\alpha$ and $v^\alpha \equiv \Delta^{\alpha\beta}\xi_\beta$, which gives $\xi_\alpha = -b u_\alpha + v_\alpha$ and $\xi_\alpha \xi^\alpha = -b^2 + v\cdot v$, where $v\cdot v = \Delta^{\alpha\beta}\xi_\alpha\xi_\beta$. We proceed by also defining the tensor

$$\mathfrak{D}^\mu_\nu = C_\nu^{\mu\alpha\beta}\xi_\alpha\xi_\beta = \left(\tau_P\rho - \zeta - \frac{\eta}{3}\right) v^\mu \xi_\nu + [\rho\tau_Q b^2 - \eta(v\cdot v)]\delta^\mu_\nu, \tag{A3}$$

which gives

$$\det[\mathbb{A}(\xi)] = \det\begin{bmatrix} 0 & b^2 & nb\xi_\nu \\ \tau_\varepsilon b^2 + \beta_\varepsilon(v\cdot v) & \beta_n(v\cdot v) & \rho(\tau_\varepsilon+\tau_Q)bv_\nu \\ (\beta_\varepsilon+\tau_P)bv^\mu & \beta_n b v^\mu & \mathfrak{D}^\mu_\nu \end{bmatrix}$$
$$= -b^2[\rho\tau_Q b^2 - \eta(v\cdot v)]^3 \times [Ab^4 + Bb^2(v\cdot v) + C(v\cdot v)^2] \tag{A4a}$$
$$= -\rho^4\tau_Q^4\tau_\varepsilon(u^\alpha\xi_\alpha)^2 \prod_{a=1,\pm}[(u^\alpha\xi_\alpha)^2 - c_a\Delta^{\alpha\beta}\xi_\alpha\xi_\beta]^{n_a}, \tag{A4b}$$

where, to shorten notation in Eq. (A4a) we defined

$$A \equiv \rho\tau_\varepsilon\tau_Q, \tag{A5a}$$

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q, \tag{A5b}$$

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right), \tag{A5c}$$

and used the fact that $\beta_\varepsilon + n\beta_n/\rho = \tau_Q c_s^2 + \sigma\kappa_s/\rho$. In Eq. (A4a) it becomes evident that assumption (A1) guarantees that $v^\mu \neq 0$, eliminating one of the possible acausal roots. From Eqs. (A4a)–(A4b) we defined $n_1 = 3$, $n_\pm = 1$, $c_1 = [\eta/(\rho\tau_s)]$, and $c_\pm = [(-B \pm \sqrt{B^2 - 4AC})/2A]$. Note that since $\xi^\alpha\xi_\alpha = -b^2 + (v\cdot v)$, the roots in Eq. (A4b) can be cast as $b^2 = c_a(v\cdot v)$. Then, (Cond-1) demands that $c_a \in \mathbb{R}$ together with $c_a \geq 0$ and (Cond-2) that $c_a < 1$ for causality [221], which comes from the fact that the root $b^2 = c_a v\cdot v$ must obey $\xi_\mu\xi^\mu = -b^2 + v\cdot v = (1-c_a)v\cdot v > 0$. Thus, causality is ensured if $0 \leq c_a < 1$ in the matter sector. Clearly, the root $b = u^\alpha\xi_\alpha = 0$ is causal. Also, the six roots related to c_1 are causal when Eq. (21a) is observed. As for the roots $c_\pm$, they are real if $B^2 - 4AC \geq 0$, i.e., if the first inequality in Eq. (21b) holds. On the other hand, $c_\pm \geq 0$ is obtained whenever $c_- \geq 0$, which is guaranteed if $-B \geq 0$ [second inequality in condition (21c)] together with $C \geq 0$ [second inequality of Eq. (21a)], while $c_\pm < 1$ is ensured if $c_+ < 1$, which demands that $2A + B > 0$ [first inequality in condition (21c)] and $A + B + C > 0$ [condition (21d)]. ■

We observe that, although we employed the harmonic gauge to calculate the system's characteristics, the causality established in Theorem I does not depend on any gauge choices. This follows from well-known properties of Einstein's equations [22] and the geometric invariance of the characteristics [144]. See the end of Sec. V C for further comments in this direction.

The analysis above and the conditions we obtained for causality are valid in the full nonlinear regime of the theory. However, we remark in passing that the principal part concerning only the fluid equations would have exactly the same structure if one were to linearize the fluid dynamic equations about equilibrium with nonzero flow in Minkowski spacetime. This is a generic feature of the BDNK approach (at least, when truncated at first order), i.e., the analysis of the system's characteristics, and thus of its causality properties, is formally the same in the nonlinear regime and in the linearization about a generic equilibrium state. This is not, however, a general feature of hydrodynamic models as it does not hold in MIS-like theories. In fact, as discussed at length in Refs. [71,72], in MIS the thermodynamic fluxes explicitly enter in the calculation of the characteristics, but they are not present in the linear analysis.

APPENDIX B: PROOF OF PROPOSITION I

To prove (i) we may compute the determinant $\det(\xi_\alpha\mathfrak{A}^\alpha) = \det(\xi_\alpha\mathbb{A}_m^\alpha)\det(\xi_\alpha\mathbb{A}_g^\alpha)(u^\alpha\xi_\alpha)^{16}$. Note that $u^\alpha\xi_\alpha \neq 0$ if ξ is timelike. We must then look into the matter and gravity sector in what follows. We again define $b = u^\alpha\xi_\alpha$ and $v^\mu = \Delta^{\mu\alpha}\xi_\alpha$, $v\cdot v = \Delta^{\mu\nu}\xi_\mu\xi_\nu$, and introduce

$$\Xi^\mu_\nu = v_\lambda \Pi^{\mu\lambda\alpha}_\nu \xi_\alpha = -\eta(v\cdot v)\delta^\mu_\nu - \eta v^\mu\xi_\nu + \left(\rho\tau_P - \zeta + \frac{2\eta}{3}\right)v^\mu v_\nu \tag{B1}$$

to obtain

$$\det(\xi_\alpha\mathbb{A}_m^\alpha) = \det\begin{bmatrix}
\tau_\varepsilon b & \rho\tau_Q\xi_\nu & \beta_\varepsilon\xi_\nu & \beta_n\xi_\nu & \rho\tau_\varepsilon b\delta^0_\nu & \rho\tau_\varepsilon b\delta^1_\nu & \rho\tau_\varepsilon b\delta^2_\nu & \rho\tau_\varepsilon b\delta^3_\nu \\
\tau_P v^\mu & \rho\tau_Q b\delta^\mu_\nu & \beta_\varepsilon b\delta^\mu_\nu & \beta_n b\delta^\mu_\nu & \Pi^{\mu 0\alpha}_\nu\xi_\alpha & \Pi^{\mu 1\alpha}_\nu\xi_\alpha & \Pi^{\mu 2\alpha}_\nu\xi_\alpha & \Pi^{\mu 3\alpha}_\nu\xi_\alpha \\
-v^\mu & 0_{4\times4} & b\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & 0_{4\times4} & 0_{4\times4} & b\delta^\mu_\nu & nv^\mu\delta^0_\nu & nv^\mu\delta^1_\nu & nv^\mu\delta^2_\mu & nv^\mu\delta^3_\nu \\
0_{4\times1} & -v_0\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & b\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & -v_1\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & b\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} \\
0_{4\times1} & -v_2\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & b\delta^\mu_\nu & 0_{4\times4} \\
0_{4\times1} & -v_3\delta^\mu_\nu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & b\delta^\mu_\nu
\end{bmatrix}$$

$$= b^{19}\det\begin{bmatrix}
\tau_\varepsilon b^2 + \beta_\varepsilon(v\cdot v) & b^2(\rho\tau_Q\xi_\nu + \rho\tau_\varepsilon v_\nu) - n\beta_n(v\cdot v)v_\nu \\
(\tau_P + \beta_\varepsilon)v^\mu & \rho\tau_Q b^2\delta^\mu_\nu + \Xi^\mu_\nu - n\beta_n v^\mu v_\nu
\end{bmatrix}$$

$$= b^{19}[\rho\tau_Q b^2 - \eta(v\cdot v)]^3[Ab^4 + Bb^2(v\cdot v) + C(v\cdot v)^2]$$

$$= \rho^4\tau_Q^4\tau_\varepsilon b^{19}\prod_{a=1,\pm}[b^2 - c_a(v\cdot v)]^{n_a}, \tag{B2}$$

where, as we have obtained in Eqs. (A4) and (A5), and in the text below it,

$$A \equiv \rho\tau_\varepsilon\tau_Q, \tag{B3a}$$

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q, \tag{B3b}$$

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right), \tag{B3c}$$

$n_1 = 3$, $n_\pm = 1$, $c_1 = [\eta/(\rho\tau_s)]$, and $c_\pm = [(-B \pm \sqrt{B^2 - 4AC})/2A]$. It is worth mentioning that the assumptions of Proposition I guarantee that $0 < c_1, c_\pm < 1$. Under assumptions (A1), $\eta > 0$, and conditions (21) in the strict form, then one obtains that $\det(\xi_\alpha A^\alpha_m) = 0$ only if $0 \le c_a < 1$ (with the equality holding only in the case $a = 0$); i.e., the equation $b_a^2 - c_a(v_a \cdot v_a) = 0$ gives $\xi_{a,\alpha}$ such that $\xi_{a,\alpha}\xi_a^\alpha = -b_a^2 + v_a \cdot v_a = (1 - c_a) v_a \cdot v_a > 0$. Thus, if ξ is timelike, then (i) is guaranteed for the matter sector as well. As for the gravity sector, one obtains that

$$\det(\xi_\alpha \mathbb{A}^\alpha_g) = \det \begin{bmatrix} bI_{10} & -v_0 I_{10} & -v_1 I_{10} & -v_2 I_{10} & -v_3 I_{10} \\ -v^0 I_{10} & bI_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -v^1 I_{10} & 0_{10\times10} & bI_{10} & 0_{10\times10} & 0_{10\times10} \\ -v^2 I_{10} & 0_{10\times10} & 0_{10\times10} & bI_{10} & 0_{10\times10} \\ -v^3 I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & bI_{10} \end{bmatrix}$$
$$= \frac{1}{b^{10}} \det \begin{bmatrix} (b^2 - v^\nu v_\nu) I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -v^0 I_{10} & bI_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -v^1 I_{10} & 0_{10\times10} & bI_{10} & 0_{10\times10} & 0_{10\times10} \\ -v^2 I_{10} & 0_{10\times10} & 0_{10\times10} & bI_{10} & 0_{10\times10} \\ -v^3 I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & bI_{10} \end{bmatrix}$$
$$= (u^\alpha \xi_\alpha)^{30} (\xi_\alpha \xi^\alpha)^{10}. \tag{B4}$$

Again, note that if ξ is timelike, then $\det(\xi_\alpha \mathbb{A}^\alpha_g) \neq 0$. This completes the proof of (i).

As for (ii), let us define $\phi_\alpha = \zeta_\alpha + \Lambda \xi_\alpha$ and make the changes $\xi \to \phi$ in the determinant calculations above. Then, the eigenvalues Λ are obtained from the roots of $\det(\phi_\alpha \mathfrak{A}^\alpha) = \det(\phi_\alpha \mathbb{A}^\alpha_m) \det(\phi_\alpha \mathbb{A}^\alpha_g)(u^\alpha \phi_\alpha)^{16} = 0$. Note that the general form of the equations implies that the roots $\phi_\alpha = -u_\alpha u^\beta \phi_\beta + \Delta_\alpha^\beta \phi_\beta$ obey

$$(u^\alpha \phi_\alpha)^2 - \beta \Delta^{\alpha\beta} \phi_\alpha \phi_\beta = 0, \tag{B5}$$

where, from causality, in any of the above cases we have that $0 \le \beta \le 1$. Then, for each β, the eigenvalues Λ are

$$\Lambda = \frac{\beta(\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta) - (u^\alpha\xi_\alpha)(u^\alpha\zeta_\alpha) \pm \sqrt{\mathcal{Z}}}{(u^\alpha\xi_\alpha)^2 - \beta\Delta^{\alpha\beta}\xi_\alpha\xi_\beta}, \tag{B6}$$

where, since $\xi_\alpha\xi^\alpha < 0$, then $(u^\alpha\xi_\alpha)^2 - \beta\Delta^{\alpha\beta}\xi_\alpha\xi_\beta > 0$ because $0 \le \beta \le 1$ and

$$\begin{aligned} \mathcal{Z} &= \beta\{\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta(u^\mu\xi_\mu)^2 + \Delta^{\alpha\beta}\xi_\alpha\xi_\beta(u^\mu\zeta_\mu)^2 - 2(u^\alpha\xi_\alpha)(u^\beta\zeta_\beta)\Delta^{\mu\nu}\xi_\mu\zeta_\nu - \beta[(\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta)(\Delta^{\mu\nu}\xi_\mu\xi_\nu) - (\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta)^2]\} \\ &> \beta[\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta(u^\mu\xi_\mu)^2 + \Delta^{\alpha\beta}\xi_\alpha\xi_\beta(u^\mu\zeta_\mu)^2 - 2(u^\alpha\xi_\alpha)(u^\beta\zeta_\beta)\Delta^{\mu\nu}\xi_\mu\zeta_\nu - (\Delta^{\alpha\beta}\zeta_\alpha\zeta_\beta)(\Delta^{\mu\nu}\xi_\mu\xi_\nu) + (\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta)^2] \\ &= \beta\{(-\xi^\alpha\xi_\alpha)(\zeta^\beta\zeta_\beta) + [(u^\alpha\xi_\alpha)(u^\beta\zeta_\beta) - \Delta^{\alpha\beta}\xi_\alpha\zeta_\beta]^2\} > 0. \end{aligned} \tag{B7}$$

In the operations above we used the fact that $0 \le \beta \le 1$, $(\Delta^{\alpha\beta}\xi_\alpha\zeta_\beta)^2 \le (\Delta^{\alpha\beta}\xi_\alpha\xi_\beta)(\Delta^{\mu\nu}\zeta_\mu\zeta_\nu)$ from the Cauchy-Schwarz inequality and that ξ is timelike and ζ spacelike. Thus, causality guarantees reality of the eigenvalues.

Now we turn to the problem of completeness of the set of eigenvectors. We begin by counting the linearly independent eigenvectors of $\phi^{(m)}_{a,\alpha}\mathbb{A}^\alpha_m$, where $\phi^{(m)}_{a,\alpha} = \zeta_\alpha + \Lambda^{(m)}_a \xi_\alpha$ and $\Lambda^{(m)}_a$ are the eigenvalues of the matter sector and are obtained by means of Eq. (B6) in the cases $\beta = c_0 = 0$ when $a = 0$ and $\beta = c_a$ when $a = 1, \pm$. Let us define an arbitrary vector,

$$r^{(m)} = \begin{bmatrix} F \\ G^\nu \\ H^\mu \\ I^\mu \\ J_0^\nu \\ J_1^\nu \\ J_2^\nu \\ J_3^\nu \end{bmatrix}. \tag{B8}$$

Then, for each of the eigenvalues $\Lambda_a^{(m)}$, $a = 0, 1, \pm$, we must verify how many of the 29 variables in the vector (B8) are free parameters under the equation $\phi_{a,\alpha}^{(m)} \mathbb{A}_m^\alpha r_a^{(m)} = 0$. In fact, this is the dimension of the null space of the matrix $\phi_{a,\alpha}^{(m)} \mathbb{A}_m^\alpha$ and corresponds to the number of linearly independent eigenvectors of $\Lambda_a^{(m)}$. The eigenvectors are the following.

(i) $\Lambda_0^{(m)}$: This root has multiplicity 19. The eigenvector that obeys $\phi_{0,\alpha}^{(m)} \mathbb{A}^\alpha r_0^{(m)} = 0$ is

$$r_0^{(m)} = \begin{bmatrix} 0 \\ 0_{4\times1} \\ H^\mu \\ I^\mu \\ J_0^\nu \\ J_1^\nu \\ J_2^\nu \\ J_3^\nu \end{bmatrix}, \tag{B9}$$

where only 19 out of the 24 components $H^\mu, I^\mu, J_\lambda^\nu$ are free variables because of the $1+1+3$ constraints $\beta_\varepsilon \phi_{0,\nu}^{(m)} H^\nu + \beta_n \phi_{0,\nu}^{(m)} I^\nu = 0$, $J_\lambda^\lambda = 0$, and $\Delta^{\mu\lambda}\phi_{0,\nu}^{(m)} J_\lambda^\nu + \Delta^{\lambda\beta}\phi_{0,\beta}^{(m)} J_\lambda^\mu = 0$ (note that the last four equations are not all independent since the contraction with u_μ is identically zero, resulting in three independent constraints). Thus, the multiplicity of Λ_0 equals the number of LI eigenvectors, i.e., 19.

(ii) $\Lambda_1^{(m)\pm}$: In this case each of the two eigenvalues have multiplicity 3 since $n_1 = 3$ in Eq. (B2) (note that since we assumed here that $\eta > 0$, then $c_1 \neq 0$ and, thus, $c_1 \neq c_0$ and the eigenvalues are different from the case $c_0 = 0$). We may perform some elementary row operations over the linear system $\phi_{1,\alpha}^{(m)} \mathbb{A}^\alpha r_1^{(m)} = 0$ to obtain, by imposing $b^2 - c_1(v\cdot v) = 0$ (remember that $b = u^\alpha \phi_\alpha$ and $v^\alpha = \Delta^{\alpha\beta}\phi_\beta$ after the change $\xi \to \phi$),

$$\begin{bmatrix} \tau_\varepsilon b^2 + \beta_\varepsilon(v\cdot v) & b\rho\tau_Q\phi_\nu + b\rho\tau_\varepsilon v_\nu - \frac{n\beta_n(v\cdot v)}{b} v_\nu & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} & 0_{1\times4} \\ 0_{4\times1} & \mathcal{K}_\nu v^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ -v^\mu & 0_{4\times4} & b\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & \frac{n v^\mu v_\nu}{b} & 0_{4\times4} & b\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & -v_0\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & b\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & -v_1\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & b\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} \\ 0_{4\times1} & -v_2\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & b\delta_\nu^\mu & 0_{4\times4} \\ 0_{4\times1} & -v_3\delta_\nu^\mu & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & 0_{4\times4} & b\delta_\nu^\mu \end{bmatrix} r_1^{(m)} = 0, \tag{B10}$$

where

$$\mathcal{K}_\nu = \left[-\eta\xi_\nu + \left(\rho\tau_P - \zeta + \frac{2\eta}{3} - n\beta_n\right)v_\nu\right] \times [\tau_\varepsilon b^2 + \beta_\varepsilon(v\cdot v)] - (\tau_P + \beta_\varepsilon)[b^2\rho\tau_Q\xi_\nu + b^2\rho\tau_\varepsilon v_\nu - n\beta_n(v\cdot v)v_\nu]. \tag{B11}$$

This enables us to find the eigenvectors,

$$^{\pm}r_1^{(m)} = \begin{bmatrix} F_\pm \\ G_\pm^\nu \\ H_\pm^\nu \\ I_\pm^\nu \\ ^{\pm}J_0^\nu \\ ^{\pm}J_1^\nu \\ ^{\pm}J_2^\nu \\ ^{\pm}J_3^\nu \end{bmatrix}, \tag{B12}$$

where, from the $29 + 29 = 58$ components of the above eigenvectors (29 for $\Lambda_1^{(m)+}$ and 29 $\Lambda_1^{(m)-}$ cases), they are subjected to the following $26 + 26$ constraints: $1 + 1 = 2$ constraints,

$$[\tau_\varepsilon b_\pm^2 + \beta_\varepsilon(v_\pm\cdot v_\pm)]F_\pm + b_\pm\rho\tau_Q\, {}^{\pm}\phi_{1,\nu}^{(m)} G^\nu + b_\pm\rho\tau_\varepsilon v_\nu^\pm G^\nu - \frac{n\beta_n(v_\pm\cdot v_\pm)}{b_\pm} v_\nu^\pm G_\pm^\nu = 0,$$

$1 + 1 = 2$ constraints $\mathcal{K}_\nu^\pm G_\pm^\nu = 0$, $4 + 4 = 8$ constraints $b_\pm H_\pm^\mu = v_\pm^\mu F_\pm$, $4 + 4 = 8$ constraints $n v_\pm^\mu v_\nu^\pm G^\nu + b_\pm^2 I_\pm^\mu = 0$, and the $16 + 16 = 32$ constraints $b_\pm\, {}^{\pm}J_{\pm\lambda}^\mu = v_\lambda^\pm G_\pm^\mu$, where ${}^{\pm}\phi_{1,\alpha}^{(m)} = {}^{\pm}\Lambda_1^{(m)}\xi_\alpha + \zeta_\alpha$ and $b^\pm$ and $v_\pm^\alpha$ are defined in terms of ${}^{\pm}\phi_{1,\nu}^{(m)}$. Hence, there is a total of $3 + 3 = 6$ free parameters. Once again, the degeneracy equals the number of LI eigenvectors.

(iii) $(\Lambda_\pm)^\pm$: Since there is no degeneracy in these four last eigenvalues and they are distinct from the others because $c_\pm \neq 0$ in the strict form of the inequalities in Eq. (21) and different among them, then one has four LI eigenvectors.

Thus, the system has $19+6+4=29$ LI eigenvectors. Therefore, there is a complete set in $\mathbb{R}^{29}$, namely, $\{r_b^{(m)}\}_{b=1}^{29}$ such that $\phi_a^{(m)}\mathbb{A}_m^\alpha r_b^{(m)}=0$. Hence, we can use the 29 linearly independent set $\mathcal{S}^{(m)}=\{R_b^{(m)}\}_{b=1}^{29}$ to verify that

$$R_b^{(m)}=\begin{bmatrix} r_b^{(m)} \\ 0_{66\times 1}\end{bmatrix} \tag{B13}$$

obeys $(\zeta_\alpha+\Lambda_a^{(m)}\xi_\alpha)\mathfrak{A}^\alpha R_b^{(m)}=0$.

Now, before we discuss the gravity sector $\{F_A,\mathcal{F}_A^\delta\}$, let us look at the sector containing the original fields ε, n, u^ν, and $g_{\mu\nu}$. In this case, let us define

$$R^{(d)}=\begin{bmatrix} 0_{79\times 1} \\ r^{(d)}\end{bmatrix}, \tag{B14}$$

where $r^{(d)}$ is a 16×1 column vector. Then, $(\zeta_\alpha+\Lambda_a^{(d)}\xi_\alpha)\mathfrak{A}^\alpha R_a^{(d)}=0$ reduces to the eigenvalue problem $u^\alpha\phi_\alpha^{(d)}I_{16}r^{(d)}=0$ whose eigenvalues are $u^\alpha\phi_\alpha^{(d)}=0$, i.e., $\Lambda^{(d)}=\zeta_\alpha u^\alpha/\xi_\alpha u^\alpha$. Thus, the eigenvectors may be any basis of $\mathbb{R}^{16}$. Let $\{r_a^{(d)}\}_{a=1}^{16}$ be a basis of $\mathbb{R}^{16}$. Then, the set $\mathcal{S}^{(d)}=\{R_a^{(d)}\}_{a=1}^{16}$ is a linearly independent set of 16 eigenvectors of $\phi_\alpha^{(d)}\mathfrak{A}^\alpha$.

To finalize the eigenvector counting we have to analyze the sector containing F_A and $\mathcal{F}_A^\delta$. In this case, let us define

$$R^{(g)}=\begin{bmatrix} w \\ r^{(g)} \\ 0_{16\times 1}\end{bmatrix}, \tag{B15}$$

where w is some 29×1 columns vector while $r^{(g)}$ is a 50×1 columns vector. The eigenvalues of this sector are in Eq. (B4) and are given by $\Lambda_0^{(g)}=u^\alpha\zeta_\alpha/u^\beta\xi_\beta$, coming from $u^\alpha\phi_{0,\alpha}^{(g)}=0$ (here $\phi_{a,\alpha}^{(g)}=\zeta_\alpha+\Lambda_a^{(g)}\xi_\alpha$) with multiplicity 30 and corresponding to $\beta=0$, and the two roots ${}^\pm\Lambda_1^{(g)}$ with multiplicity 10 each coming from ${}^\pm\phi_{1,\alpha}^{(g)}{}^\pm\phi_1^{(g)\alpha}=-[u^\alpha{}^\pm\phi_{1,\alpha}^{(g)}]^2+\Delta^{\alpha\beta}{}^\pm\phi_{1,\alpha}^{(g)}{}^\pm\phi_{1,\beta}^{(g)}=0$, which corresponds to $\beta=1$, i.e., gravitational waves moving at the speed of light. Then, the eigenvalue problem $\phi_{a,\alpha}^{(g)}\mathfrak{A}^\alpha R_a^{(g)}=0$ reduces to the two equations:

$$\phi_{a,\alpha}^{(g)}\mathbb{A}_m^\alpha w_a=L^\alpha r_a^{(g)}, \tag{B16a}$$

$$\phi_{a,\alpha}^{(g)}\mathbb{A}_g^\alpha r_a^{(g)}=0. \tag{B16b}$$

For the eigenvalues ${}^\pm\Lambda_1^{(g)}$, one obtains that $\det[{}^\pm\phi_{1,\alpha}^{(g)}\mathbb{A}_m^\alpha]\neq 0$ because the root $\beta=1$ has been eliminated from the matter sector (remember that $c_a<1$). Thus, there exists a solution of Eq. (B16a) for each $r_a^{(g)}$ in Eq. (B16b). One needs to count the number of linearly independent $r_1^{(g)}$ for $\Lambda_1^{(g)}$, i.e., the number of vectors in the basis of the kernel of $\phi_{1,\alpha}^{(g)}\mathbb{A}_g^\alpha$. In this case, after some elementary row operations [look at the second equality in Eq. (B4) after setting $b^2=v\cdot v$] one obtains that

$${}^\pm\phi_{1,\alpha}^{(g)}\mathbb{A}_g^\alpha\sim\begin{bmatrix} 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -\Delta^{0\alpha}{}^\pm\phi_{1,\alpha}^{(g)}I_{10} & (u^\alpha{}^\pm\phi_{1,\alpha}^{(g)})I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} \\ -\Delta^{1\alpha}{}^\pm\phi_{1,\alpha}^{(g)}I_{10} & 0_{10\times10} & (u^\alpha{}^\pm\phi_{1,\alpha}^{(g)})I_{10} & 0_{10\times10} & 0_{10\times10} \\ -\Delta^{2\alpha}{}^\pm\phi_{1,\alpha}^{(g)}I_{10} & 0_{10\times10} & 0_{10\times10} & (u^\alpha{}^\pm\phi_{1,\alpha}^{(g)})I_{10} & 0_{10\times10} \\ -\Delta^{3\alpha}{}^\pm\phi_{1,\alpha}^{(g)}I_{10} & 0_{10\times10} & 0_{10\times10} & 0_{10\times10} & (u^\alpha{}^\pm\phi_{1,\alpha}^{(g)})I_{10}\end{bmatrix}, \tag{B17}$$

which has 40 pivots and 10 independent variables (corresponding to the variables associated to the first 10 columns). Thus, there are 10 linearly independent vectors for each eigenvalue ${}^\pm\Lambda_1^{(g)}$; i.e., there is a set $\{{}^-r_{1,b}^{(g)},{}^+r_{1,b}^{(g)}\}_{b=1}^{10}$ of 20 linearly independent vectors with corresponding $w_{1,b}^\pm=[{}^\pm\phi_{1,\alpha}^{(g)}\mathbb{A}_m^\alpha]^{-1}L^\alpha{}^\pm r_{1,b}^{(g)}$ coming from Eq. (B16a) such that $\mathcal{S}_1^{(g)}=\{{}^+R_{1,b}^{(g)},{}^-R_{1,b}^{(g)}\}_{b=1}^{10}$, where

$${}^\pm R_{1,b}^{(g)}=\begin{bmatrix} w_{1,b}^\pm \\ {}^\pm r_{1,b}^{(g)} \\ 0_{16\times 1}\end{bmatrix}$$

is a linearly independent set of 20 eigenvectors of $\phi_{1,\alpha}^{(g)}\mathfrak{A}^\alpha$.

As for the eigenvalue $\Lambda_0^{(g)}$, note that in this case $\det[\phi_{0,\alpha}^{(g)}\mathbb{A}_m^\alpha] = 0$ because $\beta = c_0 = 0$ is also a root of this equation. Thus, for every solution $r_a^{(g)}$ in Eq. (B16b), Eq. (B16a) can be either undetermined or have infinite solutions. However, for any two different solutions, say, w_a^1 and w_a^2 for one $r_a^{(g)}$, the difference between $R_a^{(g)1} - R_a^{(g)2}$ corresponds to a vector in the space spanned by $\mathcal{S}^{(m)}$, that lies in the kernel of $\phi_{0,\alpha}^{(g)}\mathbb{A}_m^\alpha$. Therefore, since we are counting the number of linearly independent eigenvectors, we must choose one particular solution w_a, if it exists, for each $r_a^{(g)}$. We begin by solving Eq. (B16b). Let $\{l_1^\mu = u^\mu, l_2^\mu, l_3^\mu\}$ be a set of linearly independent vectors that are orthogonal to $\phi_{0,\alpha}^{(g)} = \zeta_\alpha + \Lambda_0^{(g)}\xi_\alpha$, to wit, $l_c^\alpha\phi_{0,\alpha}^{(g)} = 0$ and $\{e_a\}_{a=1}^{10}$ be any basis of $\mathbb{R}^{10}$. Then, one may verify that the 30 linearly independent vectors

$$r_{0,ac}^{(g)} = \begin{bmatrix} 0_{10\times1} \\ l_c^0 e_a \\ l_c^1 e_a \\ l_c^2 e_a \\ l_c^3 e_a \end{bmatrix} \tag{B18}$$

satisfy $\phi_{0,\alpha}^{(g)}\mathbb{A}_g^\alpha r_{0,ac}^{(g)} = 0$. Now we must solve Eq. (B16a), where

$$\phi_{0,\alpha}^{(g)} L^\alpha r_{0,ac}^{(g)} = \begin{bmatrix} 0_{13\times1} \\ \phi_{0,\alpha}^{(g)}\mathcal{Y}_{0\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \\ \phi_{0,\alpha}^{(g)}\mathcal{Y}_{1\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \\ \phi_{0,\alpha}^{(g)}\mathcal{Y}_{2\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \\ \phi_{0,\alpha}^{(g)}\mathcal{Y}_{3\delta}^{\mu A\alpha} l_c^\delta (e_a)_A \end{bmatrix} = K_a \begin{bmatrix} 0_{13\times1} \\ \phi_{0,0}^{(g)} l_c^\mu \\ \phi_{0,1}^{(g)} l_c^\mu \\ \phi_{0,2}^{(g)} l_c^\mu \\ \phi_{0,3}^{(g)} l_c^\mu \end{bmatrix}, \tag{B19}$$

where we defined

$$K_a \equiv \frac{1}{2}\left[\sum_{\substack{\sigma,\beta \\ \sigma\leq\beta}} (2 - \delta_{\sigma\beta}) u^{(\sigma} u^{\beta)} (e_a)_{\sigma\beta}\right].$$

Let us look for the particular solution

$$w_{ac} = \begin{bmatrix} 0 \\ -\beta_\varepsilon y_{ac}^\nu \\ \rho\tau_Q y_{ac}^\nu \\ 0_{20\times1} \end{bmatrix}. \tag{B20}$$

Note that

$$\phi_{0,\alpha}^{(g)}\mathbb{A}_m^\alpha w_{ac} = \begin{bmatrix} 0_{13\times1} \\ \beta_\varepsilon \phi_{0,0}^{(g)} y_{ac}^\mu \\ \beta_\varepsilon \phi_{0,1}^{(g)} y_{ac}^\mu \\ \beta_\varepsilon \phi_{0,2}^{(g)} y_{ac}^\mu \\ \beta_\varepsilon \phi_{0,3}^{(g)} y_{ac}^\mu \end{bmatrix}, \tag{B21}$$

and then, by inserting Eqs. (B19) and (B21) into Eq. (B16a), one finds that

$$\beta_\varepsilon \phi_{0,\nu}^{(g)} y_{ac}^\mu = K_a \phi_{0,\nu}^{(g)} l_c^\mu. \tag{B22}$$

This leads to the solution $y_{ac}^\mu = K_a l_c^\mu / \beta_\varepsilon$ and, thus,

$$w_{ac} = \begin{bmatrix} 0 \\ -K_a l_c^\nu \\ \frac{\rho\tau_Q}{\beta_\varepsilon} K_a l_c^\nu \\ 0_{20\times1} \end{bmatrix}. \tag{B23}$$

As a consequence, the set $\mathcal{S}_0^{(g)} = \{R_{1,1}^{(g)}, R_{1,2}^{(g)}, R_{1,3}^{(g)}, \ldots, R_{10,1}^{(g)}, R_{10,2}^{(g)}, R_{10,3}^{(g)}\}$ with

$$R_{ac}^{(g)} = \begin{bmatrix} w_{ac} \\ r_{0,ac}^{(g)} \\ 0_{16\times1} \end{bmatrix}$$

is a linearly independent set of 30 eigenvectors of $\phi_{0,\alpha}^{(g)}\mathfrak{A}^\alpha$. Thus, $\mathfrak{S} = \mathcal{S}^{(m)} \cup \mathcal{S}^{(d)} \cup \mathcal{S}_1^{(g)} \cup \mathcal{S}_0^{(g)}$ contains a complete set of eigenvectors R of $\phi_\alpha \mathfrak{A}^\alpha R = 0$ in $\mathbb{R}^{95}$. This completes the proof. ■

We remark that the assumption that the inequalities hold in strict form is technical. If equality is allowed, then the multiplicity of the eigenvalues might change. This is because with equality one can have $c_a = 0$ for $a = 1$ or $\pm$ and thus the characteristics defined by $b^2 - c_a(v\cdot v) = 0$ can degenerate into the characteristics $b = 0$. Since the latter is already present in the system, the multiplicity of the characteristics would change. This does not mean that the system would not be diagonalizable. Nor does it imply that local well posedness, established in the next section, would fail [222]. However, a different proof would be needed to show diagonalization in the case $c_a = 0$ in the cases $a = 1$ or $\pm$. We believe that treating this very special case here would be a distraction from the main points of the paper. We also recall that already in the case of an ideal fluid, a different approach to local well posedness has to be employed when the characteristics degenerate [223].

APPENDIX C: PROOF OF THEOREM II

As usual in studies of the initial-value problem for Einstein's equations [22], we embed Σ into $\mathbb{R}\times\Sigma$ and work in harmonic coordinates in the neighborhood of a point. Observe that we already know the system to be causal under our assumptions, thus localization arguments are allowed.

The equations to be studied read

$$u^\alpha u^\beta \partial^2_{\alpha\beta} n + n u^\alpha \delta^\beta_\nu \partial^2_{\alpha\beta} u^\nu + \tilde{\mathcal{B}}_1(n,u,g)\partial^2 g = \mathcal{B}_1(\partial n, \partial u, \partial g), \tag{C1a}$$

$$u_\nu u^\alpha u^\beta \partial_\alpha \partial_\beta u^\nu + \tilde{\mathcal{B}}_2(n,\varepsilon,u,g)\partial^2 g = \mathcal{B}_2(\partial n, \partial\varepsilon, \partial u, \partial g), \tag{C1b}$$

$$\beta_n(u^\mu \Delta^{\alpha\beta} + \Delta^{\mu(\alpha}u^{\beta)})\partial_\alpha\partial_\beta n + \mathfrak{E}^{\mu\alpha\beta}\partial_\alpha\partial_\beta\varepsilon + \bar{\mathcal{C}}^{\mu\alpha\beta}_\nu \partial_\alpha\partial_\beta u^\nu + \tilde{\mathcal{B}}^\mu_3(n,\varepsilon,u,g)\partial^2 g = \mathcal{B}^\mu_3(\partial n,\partial\varepsilon,\partial u,\partial g), \tag{C1c}$$

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = \mathcal{B}_{4\mu\nu}(\partial n,\partial\varepsilon,\partial u,\partial g), \tag{C1d}$$

where

$$\bar{\mathcal{C}}^{\mu\alpha\beta}_\nu = \left(\tau_P\rho - \zeta - \frac{\eta}{3}\right)\Delta^{\mu(\alpha}\delta^{\beta)}_\nu - \eta\Delta^{\alpha\beta}\delta^\mu_\nu + \rho(\tau_\varepsilon+\tau_Q)u^\mu\Delta^{(\alpha}_\nu u^{\beta)} + \tau_Q\rho u^\alpha u^\beta \delta^\mu_\nu, \tag{C2a}$$

$$\mathfrak{E}^{\mu\alpha\beta} = u^\mu(\beta_\varepsilon\Delta^{\alpha\beta} + \tau_\varepsilon u^\alpha u^\beta) + (\beta_\varepsilon + \tau_P)\Delta^{\mu(\alpha}u^{\beta)}, \tag{C2b}$$

and the notation for the $\tilde{\mathcal{B}}$'s and $\mathcal{B}$'s follow the same construction as in Sec. IV.

We can write Eq. (C1) in matrix form as

$$\mathfrak{M}(\partial)\Psi = \mathfrak{N}(\partial\Psi), \tag{C3}$$

where $\Psi = (\varepsilon, n, u^\nu, g_{\mu\nu})^T$ is a 16×1 column vector (we count only the 10 independent $g_{\mu\nu}$), $\mathfrak{B}(\partial\Psi)$ is also a 16×1 column vector containing the $\mathfrak{N}$'s, i.e., the lower-order terms in derivatives of each equation, and

$$\mathfrak{M}(\partial) = \begin{bmatrix} \mathbb{M}(\partial) & \mathfrak{b}(\partial) \\ 0_{10\times 6} & g^{\alpha\beta}\partial_\alpha\partial_\beta I_{10} \end{bmatrix}. \tag{C4}$$

The 6×10 matrix $\mathfrak{b}(\partial)$ contains the terms $\tilde{\mathcal{B}}\partial^2 g$, while

$$\mathbb{M}(\partial) = \begin{bmatrix} 0 & u^\alpha u^\beta & n\delta^{(\alpha}_\nu u^{\beta)} \\ 0 & 0 & u_\nu u^\alpha u^\beta \\ \mathfrak{E}^{\mu\alpha\beta} & \beta_n(u^\mu\Delta^{\alpha\beta} + \Delta^{\mu(\alpha}u^{\beta)}) & \bar{\mathcal{C}}^{\mu\alpha\beta}_\nu \end{bmatrix}\partial^2_{\alpha\beta}. \tag{C5}$$

Let us compute the characteristic determinant of the system and its roots, i.e., $\det[\mathfrak{M}(\xi)] = \det[\mathbb{M}(\xi)](\xi^\alpha\xi_\alpha)^{10} = 0$, where the substitution $\partial\to\xi$ takes place. The pure gravity sector has the roots $\xi^\alpha\xi_\alpha = 0$. As for the matter sector, by again defining $b = u^\alpha\xi_\alpha$, $v^\mu = \Delta^{\mu\nu}\xi_\nu$, $v\cdot v = v^\mu v_\mu$, and

$$\begin{aligned}\tilde{\mathcal{C}}^\mu_\nu &= \bar{\mathcal{C}}^{\mu\alpha\beta}_\nu\xi_\alpha\xi_\beta \\ &= [\tau_Q\rho b^2 - \eta(v\cdot v)]\delta^\mu_\nu + \left(\tau_P\rho - \zeta - \frac{\eta}{3}\right)v^\mu\xi_\nu \\ &\quad + \rho(\tau_\varepsilon + \tau_Q)bu^\mu v_\nu,\end{aligned} \tag{C6a}$$

$$\mathfrak{D}^\mu_\nu = \left(\tau_P\rho - \zeta - \frac{\eta}{3} - n\beta_n\right)v^\mu\xi_\nu + [\tau_Q\rho b^2 - \eta(v\cdot v)]\delta^\mu_\nu, \tag{C6b}$$

$$\begin{aligned}\tilde{\mathfrak{E}}^\mu &= \mathfrak{E}^{\mu\alpha\beta}\xi_\alpha\xi_\beta \\ &= [\beta_\varepsilon(v\cdot v) + \tau_\varepsilon b^2]u^\mu + (\beta_\varepsilon + \tau_P)bv^\mu,\end{aligned} \tag{C6c}$$

where $\mathfrak{D}^\mu_\nu$ is the same as the one defined in Eq. (A3), we obtain that (by carrying out some elementary row operations)

$$\begin{aligned}\det[\mathbb{M}(\xi)] &= \det\begin{bmatrix} 0 & b^2 & nb\xi_\nu \\ 0 & 0 & b^2 u_\nu \\ \tilde{\mathfrak{E}}^\mu & \beta_n[u^\mu(v\cdot v) + bv^\mu] & \tilde{\mathcal{C}}^\mu_\nu \end{bmatrix} \\ &= \frac{b^3}{\tau_Q\rho b^2 - \eta(v\cdot v)} \\ &\quad\times\det\begin{bmatrix} 0 & b & n\xi_\nu \\ \tau_\varepsilon b^2 + \beta_\varepsilon(v\cdot v) & \beta_n(v\cdot v) & \rho(\tau_\varepsilon+\tau_Q)bv_\nu \\ (\beta_\varepsilon+\tau+P)bv^\mu & \beta_n bv^\mu & \mathfrak{D}^\mu_\nu \end{bmatrix}.\end{aligned} \tag{C7}$$

The last determinant is the same as the one obtained in Eq. (A4) and the result turns out to be

$$\begin{aligned}\det[\mathbb{M}(\xi)] &= -b^4[\rho\tau_Q b^2 - \eta(v\cdot v)]^2 \\ &\quad\times[Ab^4 + Bb^2(v\cdot v) + C(v\cdot v)^2] \\ &= -\rho^4\tau_Q^4\tau_\varepsilon(u^\alpha\xi_\alpha)^4\prod_{a=1,\pm}[(u^\alpha\xi_\alpha)^2 - c_a\Delta^{\alpha\beta}\xi_\alpha\xi_\beta]^{\tilde{n}_a},\end{aligned} \tag{C8}$$

where, as in Eqs. (A5),

$$A \equiv \rho\tau_\varepsilon\tau_Q, \tag{C9a}$$

$$B \equiv -\tau_\varepsilon\left(\rho c_s^2\tau_Q + \zeta + \frac{4\eta}{3} + \sigma\kappa_s\right) - \rho\tau_P\tau_Q, \tag{C9b}$$

$$C \equiv \tau_P(\rho c_s^2\tau_Q + \sigma\kappa_s) - \beta_\varepsilon\left(\zeta + \frac{4\eta}{3}\right), \tag{C9c}$$

while $c_1 = [\eta/(\rho\tau_s)]$ and $c_\pm = [(-B\pm\sqrt{B^2-4AC})/2A]$, while $\tilde{n}_1 = 2$ and $\tilde{n}_\pm = 1$. Note that the characteristics are still the same as in Sec. IV, as expected, although the multiplicity of the roots changed (and there was no reason for the multiplicities to be the same). We conclude that the characteristic determinant of the system is a product of strictly hyperbolic polynomials. We verify at once that the system is a Leray-Ohya system [21,224] for which the results of Ref. [225] (see also Ref. [105]) apply. Thus, if the initial data are quasianalytic (see Ref. [75]), we obtain quasianalytic solutions.

Denote the initial dataset in the theorem by $\mathcal{D}$ and let $\mathcal{D}_\ell$ be a sequence of quasianalytic initial data converging to $\mathcal{D}$ in H^N (see Ref. [27] for the definition of H^N). Let $\mathbf{\Psi}_\ell$ be solutions corresponding to $\mathcal{D}_\ell$ (which exist by the foregoing). In order to finish the proof of the theorem, it suffices to show that $\mathbf{\Psi}_\ell$ has a limit in H^N. The limit will then be a solution with the desired properties because we can pass to the limit in the equations since $N \geq 5$.

According to the arguments given in Sec. 16.2 of Ref. [146] or in Refs. [107,108], the diagonalization obtained in Sec. V B implies that Ψ defined in Eq. (24) admits a uniform bound in H^{N-1}, and uniform difference bounds in H^{N-2} also holds. We apply these bounds to the vector Ψ_ℓ corresponding to $\mathbf{\Psi}_\ell$. We see at once that the uniform H^{N-1} bounds for Ψ_ℓ imply uniform H^N bounds for $\mathbf{\Psi}_\ell$, and the difference bounds imply that $\mathbf{\Psi}_\ell$ is a Cauchy sequence in H^{N-3}, thus converging in this space. But low-norm convergence combined with high-norm boundedness implies that the limit is in fact in H^N [226]. ∎

We observe that a similar local well-posedness result holds for the fluid equations in a fixed background.

We recall that a standard tensorial argument [22] guarantees that the solution established in Theorem II is intrinsically defined; i.e., given the data, which are defined independently of coordinates or gauge choices, there exists a spacetime where Einstein's equations are satisfied, and this spacetime is defined without any reference to coordinates or gauge choices—even if in the process of proving that this spacetime exists one has to work in a specific gauge and coordinate system. Therefore, even though we used the harmonic gauge in the proof, the existence of the solution is guaranteed for other choices as well. This logic is similar to showing that a map from a finite-dimensional vector space into itself is invertible: one can choose a basis, write the matrix of the linear transformation with respect to that basis, and compute its determinant. The map is invertible if and only if the determinant is nonzero, and this conclusion (the invertibility or not of the linear map) is independent of any basis choice—even if to show that the map is invertible we picked a basis and computed the determinant with respect to that basis.

We note, however, the following subtlety which is very relevant for numerical simulations. The fact that a unique solution is guaranteed to exist for given initial data, and that this solution is well defined regardless of gauge choices, does not imply that such a solution can always be *reconstructed* from an arbitrary gauge. In other words, suppose we write the equations in a different gauge. *If* we can numerically integrate them, we will obtain the solution found in Theorem II written on that gauge (modulo numerical accuracy). However, it is possible that the gauge we chose is not adequate to solve the equations numerically, so that our numerical simulation will not produce a solution. This does not mean, of course, that solutions do not exist; it simply means that the guaranteed-to-exist solution given by Theorem II cannot be accessed from that specific gauge. To use again our analogy with determinants, suppose we computed the determinant on a basis b_1 and found it to be nonzero, but now we are interested in computing the determinant numerically using another basis b_2. Depending on the basis b_2 and the numerical algorithm we use, this might not be possible, which, of course, does not mean that the determinant is zero or ill defined.

Thus, the practical matter of solving the equations numerically is not settled by an abstract existence and uniqueness result as Theorem II. Such theorems are naturally important as they provide the foundations on which numerical investigations can be built; i.e., it makes sense to look for solutions numerically because solutions do exist. But these theorems do not, in general, point to how to recover solutions numerically. That is why there is a great deal of work dedicated to writing Einstein's equations in different forms and special gauges, even if basic existence results for Einstein's equations coupled to most matter models are known, as reviewed in Refs. [2,29].

APPENDIX D: PROOF OF THEOREM III

From causality one obtains that $\det(n_\alpha \mathbb{A}^\alpha) \neq 0$ as far as n is timelike. Thus, we can rewrite Eq. (35) as

$$i\Omega\delta\Psi(K) = -i\kappa(-n_\alpha\mathbb{A}^\alpha)^{-1}\zeta_\beta\mathbb{A}^\beta\delta\Psi(K) - (-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}\delta\Psi(K). \tag{D1}$$

Since the eigenvalue problem (37) contains N linearly independent vectors $\mathfrak{r}_a$, one may write Eq. (37) as

$$(-n_\alpha\mathbb{A}^\alpha)^{-1}\zeta_\beta\mathbb{A}^\beta\mathfrak{r}_a = \Lambda_a\mathfrak{r}_a \tag{D2}$$

and define the $N\times N$ invertible matrix $R = [\mathfrak{r}_1 \cdots \mathfrak{r}_N]$ whose columns are the eigenvectors $\mathfrak{r}_1, \ldots, \mathfrak{r}_n$ and the $N\times N$ matrix

$$L \equiv R^{-1} = \begin{bmatrix} \mathfrak{l}_1 \\ \vdots \\ \mathfrak{l}_N \end{bmatrix},$$

where the rows $\mathfrak{l}_a$ are the left eigenvectors of $(-n_\alpha\mathbb{A}^\alpha)\zeta_\beta\mathbb{A}^\beta$ which, consequently, obey $\mathfrak{l}_a\mathfrak{r}_b = \delta_{ab}$ (because $RL = I_N$). Then, we can write

$$\delta\Psi(K) = RL\delta\Psi(K) = \sum_a c_a(K)\mathfrak{r}_a = R\mathfrak{c}, \quad (D3)$$

where $c_a(K) = \mathfrak{l}_a\delta\Psi(K)$ is a c-number and $\mathfrak{c}$ is the $N \times 1$ matrix

$$\mathfrak{c} = L\delta\Psi(K) = \begin{bmatrix} c_1(K) \\ \vdots \\ c_N(K) \end{bmatrix}.$$

Therefore, Eq. (D1) becomes

$$i\Omega R\mathfrak{c} = -i\kappa R\mathsf{D}\mathfrak{c} - (-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}, \quad (D4)$$

where D is the $N \times N$ real diagonal matrix $\mathsf{D} = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_N)$ and, thus, $(-n_\alpha\mathbb{A}^\alpha)^{-1}\zeta_\beta\mathbb{A}^\beta R = R\mathsf{D}$. By multiplying Eq. (D4) by $\mathfrak{c}^\dagger R^{-1}$ from the left one obtains that

$$i\Omega|\mathfrak{c}|^2 = -i\kappa\mathfrak{c}^\dagger\mathsf{D}\mathfrak{c} - \mathfrak{c}^\dagger R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}. \quad (D5)$$

Since D is real and diagonal (which gives $\mathfrak{c}^\dagger\mathsf{D}\mathfrak{c} \in \mathbb{R}$), $\Omega = \gamma_n(-i\Gamma + c^i k_i)$, and $\kappa = \gamma_\zeta(-i\hat{d}^j c_i\Gamma + \hat{d}^j k_j)$, then

$$\Gamma_R\mathfrak{c}^\dagger(\gamma_n I_N + \gamma_\zeta\hat{d}^j c_j\mathsf{D})\mathfrak{c} = -\mathrm{Re}[\mathfrak{c}^\dagger R^{-1}(n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}]. \quad (D6)$$

On the other hand, note that $\gamma_n I_N + \gamma_\zeta\hat{d}^j c_j\mathsf{D}$ is diagonal with elements

$$(\gamma_n I_N + \gamma_\zeta\hat{d}^j c_j\mathsf{D})_{aa} = \gamma_n + \gamma_\zeta\hat{d}^j c_j\Lambda_a > 0 \quad (D7)$$

because $|\hat{d}^j c_j| \leq |c^i| < 1$, $\Lambda \in [-1, 1]$, and $\gamma_n \geq \gamma_\zeta$ from Eq. (36). Hence, $\gamma_n I_N + \gamma_\zeta\hat{d}^j c_j\mathsf{D}$ is a positive Hermitian matrix and $\mathfrak{c}^\dagger(\gamma_n I_N + \gamma_\zeta\hat{d}^j c_j\mathsf{D})\mathfrak{c} > 0$. The consequence is that $\Gamma_R \leq 0$ if and only if

$$\mathrm{Re}[\mathfrak{c}^\dagger R^{-1}(n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}] \geq 0. \quad (D8)$$

Now, let $\mathcal{O}$ be the LRF and $\mathcal{O}'$ some other boosted frame. The connection between the two frames is given by the Lorentz transform $t' = \gamma(t - v^i x_i)$, $x'^i_\parallel = \gamma(x^i_\parallel - v^i t)$, and $x'^i_\perp = x^i_\perp$, where $\parallel$ and $\perp$ stand for the components parallel and perpendicular to v^i, respectively. This can be compactly written as $X'^\mu = \Lambda^\mu_\nu X^\nu$. Thus, one obtains that $K'^\mu = \Lambda^\mu_\nu K^\nu$ and $\delta\Psi'(K') = M\delta\Psi(K)$ from the structure of Eq. (33) (where M is an $N \times N$ invertible matrix), leading to $\mathbb{A}'^\mu = \Lambda^\mu_\nu M\mathbb{A}^\mu M^{-1}$ and $\mathbb{B}' = M\mathbb{B}M^{-1}$. In particular, $\zeta_\alpha\mathbb{A}^\alpha = M^{-1}\zeta'_\alpha\mathbb{A}'^\alpha M$ and $n_\alpha\mathbb{A}^\alpha = M^{-1}(n'_\alpha\mathbb{A}'^\alpha)M$. From Eq. (37), these relations give $R' = MR$, with the same eigenvalue Λ in both frames. Then, since $\delta\Psi(K) = R\mathfrak{c}$ and $\delta\Psi'(K') = R'\mathfrak{c}' = MR\mathfrak{c}$, one concludes that $\mathfrak{c} = \mathfrak{c}'$, i.e., $c'_a(K') = c_a(K)$. Therefore, one arrives at the following identity:

$$\mathfrak{c}'^\dagger R'^{-1}(-n'_\alpha\mathbb{A}'^\alpha)^{-1}\mathbb{B}'R'\mathfrak{c}' = \mathfrak{c}^\dagger R^{-1}(-n_\alpha\mathbb{A}^\alpha)^{-1}\mathbb{B}R\mathfrak{c}. \quad (D9)$$

However, if the system is stable in the LRF, then Eq. (D8) holds and, from Eq. (D9), one automatically obtains that $\Gamma'_R \leq 0$, proving that the system is also stable in any other frame $\mathcal{O}'$ obtained via a Lorentz transformation. ■

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[28] We say some time instead of all times because solutions can develop singularities and, thus, not be defined for all times. The fact that nonlinear equations of physical significance can indeed develop singularities, e.g., shock waves [2] in fluid dynamics or spacetime singularities in general relativity [20], is well known. The question of how to deal with such singularities, e.g., how to meaningfully extend the solutions past the singularity or replace the singular theory by a more complete one where singularities can be avoided, while of uttermost physical importance, is beyond the scope of this work.

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The Relativistic Euler Equations: Remarkable Null Structures and Regularity Properties

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Abstract. We derive a new formulation of the relativistic Euler equations that exhibits remarkable properties. This new formulation consists of a coupled system of geometric wave, transport, and transport-div-curl equations, sourced by nonlinearities that are null forms relative to the acoustical metric. Our new formulation is well-suited for various applications, in particular, for the study of stable shock formation, as it is surveyed in the paper. Moreover, using the new formulation presented here, we establish a local well-posedness result showing that the vorticity and the entropy of the fluid are one degree more differentiable compared to the regularity guaranteed by standard estimates (assuming that the initial data enjoy the extra differentiability). This gain in regularity is essential for the study of shock formation without symmetry assumptions. Our results hold for an arbitrary equation of state, not necessarily of barotropic type.

Mathematics Subject Classification. 35Q75; Secondary 35L10, 35Q35, 35L67.

Contents

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1. Introduction

The relativistic Euler equations are the most well-studied PDE system in relativistic fluid mechanics. In particular, they play a prominent role in cosmology, where they are often used to model the evolution of the average matter-energy content of the universe; see, for example, Weinberg's well-known monograph [40] for an account of the role that the relativistic Euler equations play in the standard model of cosmology. The equations are also widely used in astrophysics and high-energy nuclear physics, as is described, for example, in [28]. Our main result in this article is our derivation of a new formulation of the relativistic Euler equations that reveals remarkable new regularity and null structures that are not visible relative standard order formulations. The new formulation is available for an arbitrary equation of state, not necessarily of barotropic[1] type. Below we will describe potential applications that we anticipate will be the subject of future works. We mention already that our new formulation of the equations provides a viable framework for the rigorous mathematical study of stable shock formation without symmetry assumptions in solutions to the relativistic Euler equations; for reasons to be explained, standard first-order formulations are not adequate for tracking the behavior of solutions (without symmetry assumptions) all the way to the formation of a shock or for extending the solution (uniquely, in a weak sense tied to suitable selection criteria) past the first singularity.

We derive the new formulation by differentiating a standard first-order formulation with various geometric differential operators and observing remarkable cancellations.[2] The calculations are rather involved and make up the bulk of the article. We have carefully divided them into manageable pieces; see Sects. 4–8. Readers can jump ahead to Theorem 1.2 for a rough statement of the equations and Theorem 3.1 for the precise version.

As we alluded to above, the relativistic Euler equations are typically formulated as a first-order quasilinear hyperbolic PDE system. In our new formulation, the equations take the form of a system of covariant wave equations coupled to transport equations and to two transport-div-curl systems. The new formulation is well suited for various applications in ways that first-order formulations are not. In particular, the equations of Theorem 3.1 can be used to prove that *the vorticity and entropy are one degree more differentiable than one might naively expect* (assuming that the gain in differentiability is present in the initial data). This gain in differentiability is crucial for the rigorous mathematical study of some fundamental phenomena that occur in fluid dynamics. In particular, this gain, as well as other structural aspects of the new formulation, is essential for the study of shock waves (without symmetry assumptions) in relativistic fluid mechanics; see Sect. 1.2 for further discussion. Although the gain in differentiability for the vorticity had previously been observed relative to Lagrangian coordinates [13,15], Lagrangian coordinates are inadequate, for example, for the study of the formation of shock singularities because they are not adapted to the acoustic characteristics, whose intersection corresponds to a shock. Hence, it is of fundamental importance that our new formulation allows one to prove the gain in differentiability relative to *arbitrary vectorfield differential operators* (with suitably regular coefficients). In this vein, we also mention the works [9–11] on the non-relativistic compressible Euler equations, in which a gain in differentiability for the vorticity was shown relative to Lagrangian coordinates, and the first author's joint work [12], in which elliptic

[1] Barotropic equations of state are such that the pressure is a function of the proper energy density ρ alone.

[2] In observing many of the cancellations, the precise numerical coefficients in the equations are important; roughly, these cancellations lead to the presence of the null-form structures described below. However, for most applications, the overall coefficient of the null forms is not important; what matters is that the cancellations lead to null forms.

estimates were used to show that for the non-relativistic barotropic compressible Euler equations, it is possible to gain one derivative on the density relative to the velocity (again, assuming that the gain is present in the initial data).

We also highlight the following key advantage of our new formulation:

> It dramatically enlarges the set of energy estimate techniques that can be applied to the study of the relativistic Euler equations. More precisely, the new formulation partially decouples the "wave parts" and "transport parts" of the system and unlocks our ability to apply the full power of the commutator and multiplier vectorfield methods to the study of the wave part; see Sect. 9.6 for further discussion.

For applications to shock waves, it is fundamentally important that one is able to use the full scope of the vectorfield method on the wave part of the system; see the introduction of [23] for a discussion of this issue in the related context of the non-relativistic barotropic compressible Euler equations with vorticity. In particular, our new formulation of the equations allows one to derive a coercive energy estimate for the wave part of the system for any multiplier vectorfield that is causal relative to the acoustical metric g of Definition 2.6 and on any hypersurface that is null or spacelike relative to g; see Sect. 9.6.1 for further discussion. In contrast, for first-order hyperbolic systems (a special case of which is the relativistic Euler equations) without additional structure, there is, up to scalar function multiple, only one[3] available energy estimate on each causal or spacelike hypersurface.

Our second result in this article is that we provide a proof of local well-posedness for the relativistic Euler equations that relies on the new formulation; see Theorem 9.12. The new feature of Theorem 9.12 compared to standard proofs of local well-posedness for the relativistic Euler equations is that it provides the aforementioned gain in differentiability for the vorticity and entropy. Although many aspects of the proof of the theorem are standard, we also rely on some geometric and analytic insights that are tied to the special structure of our new formulation of the equations and thus are likely not known to the broader PDE research community; see the end of Sect. 1.2.3 for further discussion of this point.

[3]Here we further explain how standard first-order formulations of the relativistic Euler equations limit the available energy estimates. In deriving energy estimates for the relativistic Euler equations in their standard first-order form, one is effectively controlling the wave and transport parts of the system at the same time, and, up to a scalar function multiple, there is only one energy estimate available for transport equations. To see this limitation in a more concrete fashion, one can rewrite the relativistic Euler equations in first-order symmetric hyperbolic form as $A^{\alpha}(\mathbf{V})\partial_{\alpha}\mathbf{V} = 0$, where $\mathbf{V}$ is the array of solution variables and the A^{α} are symmetric matrices with A^0 positive definite; see, for example, [27] for a symmetric hyperbolic formulation of the general relativistic Euler equations in the barotropic case. The standard energy estimate for symmetric hyperbolic systems is obtained by taking the Euclidean dot product of both sides of the equation with $\mathbf{V}$ and then integrating by parts over an appropriate spacetime domain foliated by spacelike hypersurfaces. The key point is that for systems without additional structure, no other energy estimate is known, aside from rescaling the standard one by a scalar function.

For convenience, throughout the article, we restrict our attention to the special relativistic Euler equations, that is, the relativistic Euler equations on the Minkowski spacetime background $(\mathbb{R}^{1+3}, \eta)$, where η is the Minkowski metric. However, using arguments similar to the ones given in the present article, our results could be extended to apply to the relativistic Euler equations on a general Lorentzian manifold; such an extension could be useful, for example, in applications to fluid mechanics in the setting of general relativity. For use throughout the article, we fix a standard rectangular coordinate system $\{x^\alpha\}_{\alpha=0,1,2,3}$, relative to which $\eta_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$. See Sect. 2.1 for our index conventions. We clarify that in Sect. 9, we prove local well-posedness for the relativistic Euler equations (including the aforementioned gain in regularity for the vorticity and entropy) on the flat spacetime background $(\mathbb{R}\times\mathbb{T}^3, \eta)$, where the "spatial manifold" $\mathbb{T}^3$ is the three-dimensional torus and we recycle the notation in the sense that $\{x^\alpha\}_{\alpha=0,1,2,3}$ denotes standard coordinates on $\mathbb{R}\times\mathbb{T}^3$ (see Sect. 9.1.1 for further discussion) and η again denotes the Minkowski metric; the compactness of $\mathbb{T}^3$ allows for a simplified approach to some technical aspects of the argument while allowing us to illustrate the ideas needed to exhibit the gain in regularity for the vorticity and entropy.

Our work here can be viewed as extensions of the second author's previous joint work [22], in which the authors derived a similar formulation of the non-relativistic compressible Euler equations under an arbitrary barotropic equation of state, as well as the second author's work [33], which extended the results of [22] to a general equation of state. However, since the geo-analytic structures revealed by [22,33] are rather delicate (that is, quite unstable under perturbations of the equations), it is far from obvious that similar results hold in the relativistic case. We also stress that compared to the non-relativistic case, our work here is substantially more intricate in that it extensively relies on decompositions of various spacetime tensors into tensors that are parallel to the four-velocity u and tensors that are η-orthogonal to u. In particular, we heavily exploit that many of the tensorfields appearing in our analysis exhibit improved regularity under u-directional differentiation *or* contraction against u.

1.1. Rough Statement of the New Formulation

In this subsection, we provide a schematic version of our new formulation of the equations; in Sect. 1.2, we will refer to the schematic version when describing potential applications. In any formulation of the relativistic Euler equations, there is great freedom in choosing state-space variables (i.e., the fundamental unknowns in the system). In this article, as state-space variables, we use the logarithmic enthalpy h, the entropy s, and the four-velocity u, which is a future-directed timelike vectorfield normalized by $\eta_{\alpha\beta}u^\alpha u^\beta = -1$. Other fluid quantities such as the proper energy density ρ, the pressure p will also play a role in our discussion, but these quantities can be viewed as functions of the state-space variables; see Sect. 2 for detailed descriptions of all of these variables as well as the first-order formulation of the equations that forms the starting point for our ensuing analysis.

As we mentioned earlier, our new formulation comprises a system of covariant wave equations coupled to transport equations and to two transport-div-curl systems. Roughly, the wave equations correspond to the propagation of sound waves, while the transport equations correspond to the transporting of vorticity and entropy along the integral curves of u. The transport-div-curl systems are needed to control the top-order derivatives of the vorticity and the entropy and to exhibit the aforementioned gain in differentiability. In addition to the state-space variables h, s, and u, our formulation also involves a collection of auxiliary[4] fluid variables, including the entropy gradient one-form $S_\alpha := \partial_\alpha s$ and the vorticity ϖ^α, which is a vectorfield that is η-orthogonal to u (see Definition 2.2). Among these auxiliary variables, of crucial importance for our work is that we have identified new combinations of fluid variables that solve transport equations with unexpectedly good structure. These structures can be used to show that the combinations exhibit a gain in regularity compared to what can be inferred from a standard first-order formulation of the equations. We refer to these special combinations as "modified variables," and throughout, we denote them by $\mathcal{C}^\alpha$ and $\mathcal{D}$; see Definition 2.8.

The remaining discussion in this subsection relies on some schematic notation and refers to some geometric objects that are not precisely defined until later in the article:

- The notation "$\sim$" below means that we are only highlighting the maximum number of derivatives of the state-space variables that the auxiliary variables depend on. We note, however, that in practice, the precise structure of many of the terms that we encounter is important for observing the cancellations that lie behind our main results.
- "∂" schematically denotes the spacetime gradient with respect to the rectangular coordinates, and "∂^2" schematically denotes two differentiations with respect to the rectangular coordinates.
- $g = g(h, s, u)$ denotes the acoustical metric, which is Lorentzian (see Definition 2.6).
- $\varpi \sim \partial u + \partial h$ is the vorticity vectorfield (see Definition 2.2).
- $S_\alpha := \partial_\alpha s$ is the entropy gradient one-form.
- $\mathcal{C}^\alpha \sim \partial^2 u + \partial^2 h$ is a modified version of the vorticity of ϖ, that is, the vorticity of the vorticity (see Definition 2.8).
- $\mathcal{D} \sim \partial^2 s$ is a modified version of $\partial_\alpha S^\alpha$ (see Definition 2.8).
- $\mathfrak{Q}(\partial T_1, \ldots, \partial T_m)$ denotes *special* terms that are quadratic in the tensorfields $\partial T_1, \ldots, \partial T_m$. More precisely, the $\mathfrak{Q}(\partial T_1, \ldots, \partial T_m)$ are linear combinations of the standard **null forms relative to** g; see Definition 1.1 for the definitions of the standard null forms relative to g and Sect. 1.2.2 for a discussion of the significance that the special structure of these null forms plays in the context of the study of shock waves.
- $\mathfrak{L}(\partial T_1, \ldots, \partial T_m)$ denotes linear combinations of terms that are at most **linear** in $\partial T_1, \ldots, \partial T_m$; see Sect. 1.2.2 for a discussion of the significance of the linear dependence in the context of the study of shock waves.

[4]By "auxiliary," we mean that they are determined by h, s, and u.

Before schematically stating our main theorem, we first provide the definitions of the standard null forms relative to g.

Definition 1.1 (*Standard null forms relative to g*). We define the standard null forms relative to g (which we refer to as "standard g-null forms" for short) as follows, where ϕ and ψ are scalar functions and $0 \leq \mu < \nu \leq 3$:

$$\mathfrak{Q}^{(g)}(\partial\phi, \partial\psi) := (g^{-1})^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\psi),$$
$$\mathfrak{Q}_{\mu\nu}(\partial\phi, \partial\psi) := (\partial_\mu\phi)(\partial_\nu\psi) - (\partial_\nu\phi)(\partial_\mu\psi). \tag{1.1}$$

We now present the schematic version of our main theorem; see Theorem 3.1 for the precise statements.

Theorem 1.2 (New formulation of the relativistic Euler equation (schematic version)). *Assume that (h, s, u^α) is a C^3 solution to the (first-order) relativistic Euler equations (2.17)–(2.19) + (2.20). Then h, u^α, and s also verify the following covariant*[5] *wave equations, where the schematic notation "$\simeq$" below means that we have ignored the coefficients of the inhomogeneous terms and also harmless (from the point of view of applications to shock waves) lower-order terms, which are allowed to depend on h, s, u, S, and ϖ (but not their derivatives):*

$$\square_g h \simeq \mathcal{D} + \mathfrak{Q}(\partial h, \partial u) + \mathfrak{L}(\partial h), \tag{1.2a}$$
$$\square_g u^\alpha \simeq \mathcal{C}^\alpha + \mathfrak{Q}(\partial h, \partial u) + \mathfrak{L}(\partial h, \partial u), \tag{1.2b}$$
$$\square_g s \simeq \mathcal{D} + \mathfrak{L}(\partial h). \tag{1.2c}$$

In addition, s, S^α, and ϖ^α verify the following transport equations:

$$u^\kappa\partial_\kappa s = 0, \tag{1.3a}$$
$$u^\kappa\partial_\kappa S^\alpha \simeq \mathfrak{L}(\partial u), \tag{1.3b}$$
$$u^\kappa\partial_\kappa \varpi^\alpha \simeq \mathfrak{L}(\partial h, \partial u). \tag{1.3c}$$

Moreover, S^α verifies the following transport-div-curl system:

$$u^\kappa\partial_\kappa \mathcal{D} \simeq \mathcal{C} + \mathfrak{Q}(\partial S, \partial h, \partial u) + \mathfrak{L}(\partial h, \partial u), \tag{1.4a}$$
$$\mathrm{vort}^\alpha(S) = 0, \tag{1.4b}$$

where the vorticity operator vort *is defined in Definition 2.1.*

Finally, ϖ^α verifies the following transport-div-curl system:

$$\partial_\kappa \varpi^\kappa \simeq \mathfrak{L}(\partial h), \tag{1.5a}$$
$$u^\kappa\partial_\kappa \mathcal{C}^\alpha \simeq \mathcal{C} + \mathcal{D} + \mathfrak{Q}(\partial S, \partial\varpi, \partial h, \partial u) + \mathfrak{L}(\partial S, \partial\varpi, \partial h, \partial u). \tag{1.5b}$$

[5] Relative to arbitrary coordinates, for scalar functions f, we have

$$\square_g f = \frac{1}{\sqrt{|\mathrm{det} g|}}\partial_\alpha\left(\sqrt{|\mathrm{det} g|}(g^{-1})^{\alpha\beta}\partial_\beta f\right).$$

1.2. Connections to the Study of Shock Waves

As we have mentioned, the relativistic Euler equations are an example of a quasilinear hyperbolic PDE system. A central feature of the study of such systems is that initially smooth solutions can form shock singularities in finite time. By a "shock," we roughly mean that one of the solution's partial derivatives with respect to the standard coordinates blows up in finite time while the solution itself remains bounded. In the last decade, for interesting classes of quasilinear hyperbolic PDEs in multiple spatial dimensions, there has been dramatic progress [4,8,23–25,32,34,36,37] on our understanding of the formation of shocks as well as our understanding of the subsequent behavior of solutions past their singularities [5,7] (where the equations are verified in a weak sense past singularities).

The works cited above have roots in the work of John [16] on singularity formation for quasilinear wave equations in one spatial dimension as well as Alinhac's foundational works [2,3], which were the first to provide a constructive description of shock formation for quasilinear wave equations in more than one spatial dimension without symmetry assumptions. More precisely, Alinhac's approach allowed him to follow the solution precisely to the time of first blowup, but not further. His work yielded sharp information about the first singularity, but only for a subset of "non-degenerate" initial data such that the solution's first singularity is isolated in the constant-time hypersurface of first blowup; in particular, his proof did not apply to spherically symmetric initial data, where the "first" singularity typically corresponds to blowup on a sphere.

Subsequently, Christodoulou [4] proved a breakthrough result on the formation of shocks for solutions to the relativistic Euler equations in irrotational (that is, vorticity free) and isentropic regions of spacetime. More precisely, for the family of quasilinear wave equations that arise in the study of the irrotational and isentropic relativistic Euler equations,[6] Christodoulou gave a complete description of the maximal development of an open set (without symmetry assumptions) of initial data and showed in particular that an open subset of these data lead to shock-forming[7] solutions. Moreover, he gave a precise geometric description of the set of spacetime points where blowup occurs by showing that the singularity formation is exactly characterized by the intersection of the acoustic characteristics. In practice, he accomplished this by constructing an acoustical *eikonal function* U, whose level sets are acoustic

[6] For solutions with vanishing vorticity and constant entropy, one can introduce a potential function Φ and reformulate the relativistic Euler equations as a quasilinear wave equation in Φ.

[7] One of the key results of [4] is conditional: For small data, the only possible singularities that can form are shocks driven by the intersection of the acoustic characteristics. Here "small" means a small perturbation of the data of a non-vacuum constant fluid state, where the size of the perturbation is measured relative to a high-order Sobolev norm. Another result of [4] is that there is an open subset of small data, perhaps strictly contained in the aforementioned set of data, such that the acoustic characteristics do in fact intersect in finite time. The results of [4] leave open the possibility that there might exist some non-trivial small global solutions.

characteristics (see Sect. 1.2.1 for further discussion), and then constructing an initially positive geometric scalar function $\upmu \sim 1/\partial U$ known as the *inverse foliation density* of the characteristics, such that $\upmu \to 0$ corresponds to the intersection of the characteristics and the blowup of ∂U and of the fluid solution's derivatives too. Analytically, $\upmu$ plays the role of a weight that appears throughout the work [4], and the main theme of the proof is to control the solution all the way up to the region where $\upmu = 0$. We stress that [4] was the first work that provided sharp information about the boundary of the maximal development in more than one spatial dimension in the context of shock formation. Roughly, the maximal development is the largest possible classical solution that is uniquely determined by the initial data; see [29, 41] for further discussion.

To prove his results, Christodoulou relied on a novel formulation of the relativistic Euler equations. However, since he studied the shock formation only in irrotational and isentropic regions, he was able to introduce a potential function Φ, and his new formulation of the equations was drastically simpler than the equations of Theorem 1.2. In fact, the equations are exactly the covariant wave equation system $\square_{\widetilde{g}}\partial_\alpha\Phi = 0$ (with $\alpha = 0, 1, 2, 3$), where $\widetilde{g}$ is an appropriate scalar function multiple of the acoustical metric g and $\widetilde{g} = \widetilde{g}(\partial\Phi)$. In particular, Christodoulou was able to avoid deriving/relying on the transport-div-curl equations from Theorem 1.2, and he therefore did not need to derive elliptic estimates for the fluid variables. In total, the potential formulation leads to dramatic simplifications compared to the equations of Theorem 1.2, especially in the context of the study of shock waves; it seems quite miraculous that the equations of Theorem 1.2 have structures that are compatible with extending Christodoulou's results away from the irrotational and isentropic case (see below for further discussion).

Although the sharp information that Christodoulou derived about the maximal development is of interest in itself, it is also an essential ingredient for setting up the shock development problem. The shock development problem, which was recently partially[8] solved in the breakthrough work [5] (see also the precursor work [7] in spherical symmetry), is the problem of constructing the shock hypersurface of discontinuity (across which the solution jumps) as well as constructing a unique weak solution in a neighborhood of the shock hypersurface (uniqueness is enforced by selection criteria that are equivalent to the well-known Rankine–Hugoniot conditions). Christodoulou's description of the maximal development provided substantial new information that was not available under Alinhac's approach; as we mentioned above, due to some technical limitations tied to his reliance on Nash–Moser estimates, Alinhac was able to follow the solution only to the constant-time hypersurface of first blowup. In contrast, by exploiting some delicate tensorial regularity properties of eikonal functions for wave equations (see below for more details),

[8] In [5], Christodoulou solved the "restricted" shock development problem, in which he ignored the jump in entropy and vorticity across the shock hypersurface.

Christodoulou was able to avoid Nash–Moser estimates; this was a key ingredient in his following the solution to the boundary of the maximal development. Readers can consult [14] for a survey of some of these works, with a focus on the geometric and analytic techniques that lie behind the proofs.

We now aim to connect the works mentioned above to the new formulation of the relativistic Euler equations that we provide in this paper. To this end, for the equations in the works mentioned above, we first highlight the main structural features that allowed the proofs to go through. Specifically, the works [4,8,23–25,32,34,36,37] crucially relied on the following ingredients:

1. (**Nonlinear geometric optics**). The authors relied on geometric decompositions adapted to the characteristic hypersurfaces (also known as "characteristics" or "null hypersurfaces" in the context of wave equations) corresponding to the solution variable whose derivatives blow up. This was implemented with the help of an *eikonal function* U, whose level sets are characteristics. The eikonal function is a solution to the *eikonal equation*, which is a fully nonlinear transport equation that is coupled to the solution in the sense that the coefficients of the eikonal equation depend on the solution. Moreover, the authors showed that the intersection of the characteristics corresponds to the formation of a singularity in the derivatives of the eikonal function and in the derivatives of the solution.
2. (**Quasilinear null structure**). The authors found a formulation of the equations exhibiting remarkable null structures, where the notion of "null" is tied to the true characteristics, which are solution-dependent in view of the quasilinear nature of the equations. These structures allow one to derive sharp, fully nonlinear decompositions along characteristic hypersurfaces that reveal exactly which directional derivatives blow up and that precisely identify the terms driving the blowup (which are typically of Riccati-type, i.e., in analogy with the nonlinearities in the ODE $\dot{y} = y^2$).
3. (**Regularity properties and singular high-order energy estimates**). The authors' formulation allows one to derive sufficient L^2-type Sobolev regularity for all unknowns in the problem, including the eikonal function, *whose regularity properties are tied to the regularity of the solution through the dependence of the coefficients of the eikonal equation on the solution*. In particular, to close these estimates, *the authors had to show that various solution variables are one degree more differentiable compared to the degree of differentiability guaranteed by standard energy estimates.*
4. (**Structures amenable to commutations with geometric vectorfields**). The authors' formulation is such that one can commute *all* of the equations with geometric vectorfields constructed out of the eikonal function U, generating only controllable commutator error terms. By "controllable," we mean both from the point of view of regularity and from the point of view of the strength of their singular nature. In the works [23,34,36] that treat systems with multiple characteristic speeds, these are particularly delicate tasks that are quite sensitive to the structure of the equations;

one key reason behind their delicate nature is that the eikonal function (and thus the geometric vectorfields constructed from it) can be fully adapted only to "one speed," that is, to the characteristics whose intersection correspond to the singularity.

In the remainder of this subsection, we explain why our new formulation of the relativistic Euler equations has all four of the features listed above and is therefore well-suited for studying shocks without symmetry assumptions. Readers can consult the works [22,33,35] for related but extended discussion in the case of the non-relativistic compressible Euler equations.

1.2.1. Nonlinear Geometric Optics and Geometric Coordinates. First, to implement nonlinear geometric optics, one can construct an eikonal function. In the context of the relativistic Euler equations, one would construct an eikonal function U adapted to the acoustic characteristics, that is, a solution to the eikonal equation

$$(g^{-1})^{\alpha\beta}\partial_\alpha U \partial_\beta U = 0, \tag{1.6}$$

supplemented by appropriate initial conditions, where $g = g(h, s, u)$ is the acoustical metric (see Definition 2.6). Note that U is adapted to the "wave part" of the system and not the transport part. In the context of the relativistic Euler equations, this is reasonable in the sense that the transport part corresponds to the evolution of vorticity and entropy, and there are no known blowup results for these quantities, even in one spatial dimension.[9] Put differently, U is adapted to the "portion" of the relativistic Euler flow that is expected to develop singularities. More generally, eikonal functions are a natural tool for the study of wave-like systems, regardless of whether or not one is studying shocks. We also stress that introducing an eikonal function is essentially the same as relying on the method of characteristics. However, in more than one spatial dimension, the method of characteristics must be supplemented with an exceptionally technical ingredient that we further describe below: energy estimates that hold all the way up to the shock.

The first instance of an eikonal function being used to study the global properties of solutions to a quasilinear hyperbolic PDE occurred not in the context of singularity formation, but rather in a celebrated global existence result: the Christodoulou–Klainerman [6] proof of the stability of the Minkowski spacetime as a solution to the Einstein vacuum equations. Alinhac's aforementioned works [2,3] were the first instances in which an eikonal function was used to study a non-trivial set of solutions (without symmetry assumptions) to a quasilinear wave equation all the way up to the first singularity. Eikonal functions also played a fundamental role in all of the other shock formation results mentioned above. They have also played a role in other contexts, such as low-regularity local well-posedness for quasilinear wave equations [20,21,30,39]. In all of these works, the eikonal equation is a fully nonlinear hyperbolic PDE that is coupled to the PDE system of interest (here the relativistic Euler equations) through its coefficients [here through the acoustical metric, since $g = g(h, s, u)$].

[9] In one spatial dimension, the vorticity must vanish, but the entropy can be dynamic.

As we mentioned above, in the case of the relativistic Euler equations, the level sets of U are characteristics for the "wave part" of the system. Following Alinhac [2,3] and Christodoulou [4], in order to study the formation of shocks in relativistic Euler solutions, one completes U to a *geometric coordinate system*

$$(t, U, \vartheta^1, \vartheta^2) \tag{1.7}$$

on spacetime, where $t = x^0$ is the Minkowski time coordinate and the ϑ^A are solutions to the transport equation $(g^{-1})^{\alpha\beta}\partial_\alpha U \partial_\beta \vartheta^A = 0$ supplemented by appropriate initial conditions on the initial constant-time hypersurface Σ_0. Note that $(t, \vartheta^1, \vartheta^2)$ can be viewed as a coordinate system along each characteristic hypersurface $\{U = \mathrm{const}\}$.

1.2.2. Nonlinear Null Structure. We now aim to explain the role that the nonlinear null structure of the equations played in the works [4,8,23–25,32, 34,36,37] and to explain why the equations of Theorem 1.2 enjoy the same good structures. In total, one could say that the equations of Theorem 1.2 have been geometrically decomposed into terms that are capable of generating shocks and "harmless" terms, whose nonlinear structure is such that they do not interfere with the shock formation mechanisms. To flesh out these notions, we first provide some background material. In the works cited above, the main idea behind proving shock formation is to study the solution relative to the geometric coordinates (1.7) and to show that in fact, the solution remains rather smooth in these coordinates, all the way up to the shock. This approach allows one to transform the problem of shock formation into a more traditional one in which one tries to derive long-time estimates for the solution relative to the geometric coordinates. One then recovers the blowup of the solution's derivatives with respect to the *original* coordinates by showing that the geometric coordinates degenerate in a precise fashion relative to the standard rectangular coordinates as the shock forms; the degeneration is exactly tied to the vanishing of the inverse foliation density $\upmu$ that we mentioned earlier. Although the above description might seem compellingly simple, as we explain in Sect. 1.2.3, in implementing this approach, one encounters severe analytical difficulties.

We now highlight another key aspect of the proofs in the works cited above: showing that Euclidean-unit-length derivatives of the solution in directions *tangent* to the characteristics remain bounded all the way up to the shock. It turns out that in terms of the geometric coordinates (1.7), this is equivalent to showing that the $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \vartheta^A}$ derivatives of the solution remain bounded all the way up to the shock. Put differently, the following holds:

> The singularity occurs only for derivatives of the solution with respect to vectorfields that are transversal to the characteristics and non-degenerate[10] with respect to the rectangular coordinates.

[10] In all known shock formation results, at the location of shock singularities, the geometric partial derivative vectorfield $\frac{\partial}{\partial U}$ has vanishing Euclidean length (i.e., $\delta_{ab}\left(\frac{\partial}{\partial U}\right)^a\left(\frac{\partial}{\partial U}\right)^b = 0$, where $\left\{\left(\frac{\partial}{\partial U}\right)^a\right\}_{a=1,2,3}$ denotes the rectangular spatial components of $\frac{\partial}{\partial U}$ and δ_{ab} is the

In the works cited above, to prove all of these facts, the authors had to control various inhomogeneous error terms by showing that they enjoy a good nonlinear null structure *relative to the wave characteristics*. A key conclusion of the present article is that the derivative-quadratic inhomogeneous terms in the equations of Theorem 1.2 enjoy the same good structure (which we further describe just below). In fact, *all terms* on the RHSs of *all equations* of Theorem 1.2 are harmless in that they do not drive the Riccati-type blowup that lies behind shock formation. Consequently, the equations of Theorem 1.2 pinpoint the dangerous nonlinear terms in the relativistic Euler equations:

> The terms capable of driving shock formation are of Riccati-type and are hidden in the covariant wave operator terms on LHSs (1.2a)–(1.2b). These terms become visible only when the covariant wave operator terms are expanded relative to the standard coordinates.

In view of the above remarks, one might wonder why it is important to "hide" the dangerous terms in the covariant wave operator. The answer is that there is an advanced framework for constructing geometric vectorfields adapted to wave equations, and *the framework is tailored to covariant wave operators.*[11] As we explain later in this subsection, this geometric framework seems to be essential in more than one spatial dimension,[12] when one is forced to commute the wave equations with suitable vectorfields and to derive energy estimates.

We now further describe the good structure found in the terms on the RHSs of the equations of Theorem 1.2. The good nonlinear "null structure" is found precisely in the (quadratic) null-form terms $\mathfrak{Q}$ appearing on the RHSs of the equations of Theorem 1.2. More precisely, these $\mathfrak{Q}$ are *null forms relative to the acoustical metric* g, which means that they are linear combinations (with coefficients that are allowed to depend on the solution variables—but not their derivatives) of the standard null forms relative to g (see Definition 1.1). The key property of null forms relative to g is that given *any* hypersurface $\mathscr{H}$ that is characteristic relative to g [e.g., any level set of any eikonal function U that solves Eq. (1.6)], we have the following well-known schematic decomposition:

$$\mathfrak{Q}(\partial\phi, \partial\psi) = \mathscr{T}\phi \cdot \partial\psi + \mathscr{T}\psi \cdot \partial\phi, \tag{1.8}$$

where $\mathscr{T}$ denotes a differentiation in a direction *tangent* to $\mathscr{H}$ and ∂ denotes a generic directional derivative; see, for example, [22] for a standard proof of (1.8). Equation (1.8) implies that even though $\mathfrak{Q}$ is quadratic, it *never involves two differentiations in directions transversal to any characteristic.* Since,

Footnote 10 continued
Kronecker delta). That is, at the shock singularities, $\frac{\partial}{\partial U}$ degenerates with respect to the rectangular coordinates. Due to this degeneracy, the solution's $\frac{\partial}{\partial U}$ derivatives can remain bounded all the way up to the shock, even though $\frac{\partial}{\partial U}$ is transversal to the characteristics.

[11] Roughly, these covariant wave operators are equivalent to divergence-form wave operators. In this way, one could say that a better theory is available for divergence-form wave operators than for non-divergence-form wave operators. This reminds one of the situation in elliptic PDE theory, where better results are known for elliptic PDEs in divergence form compared to ones in non-divergence form.

[12] In one spatial dimension, one can rely exclusively on the method of characteristics and thus avoid energy estimates.

in all known proofs, it is precisely the transversal derivatives that blow up when a shock forms (since the Riccati-type terms that drive the blowup are precisely quadratic in the transversal derivatives), we see that g-null forms are linear in the tensorial component of the solution that blows up. This can be viewed as *the absence of the worst possible combinations of terms in* $\mathfrak{Q}$. In terms of the geometric coordinates (1.7), null forms do not contain any "dangerous" terms proportional to $\frac{\partial}{\partial U}\phi \cdot \frac{\partial}{\partial U}\psi$. We also note that, obviously, the terms $\mathfrak{L}$ from Theorem 1.2 cannot contain any dangerous quadratic terms since they are linear in the solution's derivatives. In contrast, upon expanding the covariant wave operator terms on LHSs (1.2a)–(1.2b) relative to the standard coordinates, one typically encounters terms that are quadratic in derivatives of h and u that are transversal to the characteristics; as we highlighted above, it is precisely such "Riccati-type" terms that can drive the formation of a shock. We stress that near a shock, such transversal-derivative-quadratic terms are much larger than the null form terms. We also stress that for the relativistic Euler equations, one encounters such transversal-derivative-quadratic terms on LHSs (1.2a)–(1.2b) under *any* equation of state aside from a single exceptional one. In the irrotational and isentropic case (in which case the relativistic Euler equations reduce to a quasilinear wave equation satisfied by a potential function), this exceptional equation of state was identified in [4]; it corresponds to the quasilinear wave equation satisfied by a timelike minimal surface graph in an ambient Minkowski spacetime, which can be expressed as follows: $\partial_\alpha \left\{ \frac{(\eta^{-1})^{\alpha\beta}\partial_\beta\Phi}{\sqrt{1+(\eta^{-1})^{\kappa\lambda}(\partial_\kappa\Phi)(\partial_\lambda\Phi)}} \right\} = 0$.

In view of the previous paragraph, we would like to highlight the following point:

> Proofs of shock formation are unstable under typical perturbations of the equations by nonlinear terms that are of quadratic or higher order in derivatives. However, proofs of shock formation for wave equations typically *are* stable under perturbations of the equations by null forms that are adapted to the metric of the shock-forming wave. By "stable," we mean in the following sense: as the shock forms, null form terms become "asymptotically negligible" compared to the shock-driving terms (for the reasons described above).

The reason that the precise structure of the nonlinearities is so important for the proofs is that the known framework is designed precisely to handle specific kinds of singularity-driving derivative-quadratic terms: the kind that are hidden in the covariant wave operator terms on LHSs (1.2a)–(1.2b). In the context of the relativistic Euler equations, this means that if *any* of the equations of Theorem 1.2 had contained, on the right-hand side, an inhomogeneous non-g-null-form quadratic term of type $(\partial h)^2$, $\partial u \cdot \partial h$, $(\partial u)^2$, etc., or a term of type $(\partial h)^3$, $(\partial h)^4$, etc., then the *only known framework for proving shock formation would not work.* The difficulty is that adding such terms to the equation could in principle radically alter the expected blowup rate or even altogether prevent the formation of a singularity; either way, this would invalidate[13] the known approach for proving shock formation. One might draw an analogy with the Riccati ODE $\dot{y} = y^2$, which we suggest as a caricature model for the formation of shocks (in the case of the relativistic Euler equations, y should be identified with ∂h and/or ∂u). Note that for all data $y(0) = y_0$ with $y_0 > 0$, the solution to the Riccati ODE blows up in finite time. Now if one perturbs the Riccati ODE to obtain the perturbed equation $\dot{y} = y^2 \pm \epsilon y^3$, with ϵ a small positive number, then depending on the sign of $\pm$, the perturbed solutions with $y_0 > 0$ will either exist for all time or will blow up at a quite different rate compared to the blowup rate for the unperturbed equation.

1.2.3. Regularity Properties and Singular High-Order Energy Estimates. In the rigorous mathematical study of quasilinear hyperbolic PDEs in more than one spatial dimension, one is forced to derive energy estimates for the solution's higher derivatives by commuting the equations with appropriate differential operators. Indeed, all known approaches to studying even the basic local well-posedness theory for such equations rely on deriving estimates in L^2-based Sobolev spaces. In the works [4,8,23–25,32,34,36,37] on shock formation in multiple spatial dimensions, the authors controlled the solutions' higher geometric derivatives by differentiating the equations with geometric "commutator vectorfields" Z that are adapted to the characteristics, more precisely to the characteristics corresponding to the variables that form a shock singularity. As we mentioned earlier, the Z are designed to avoid generating uncontrollable commutator error terms. It turns out that all Z that have been successfully used to study shock formation have the schematic structure $Z^\alpha \sim \partial U$, where Z^α denotes a rectangular component of Z and U is the eikonal function.

Although the geometric vectorfields Z exhibit good commutation properties with the differential operators corresponding to the characteristics to which they are adapted, the regularity theory of the vectorfields themselves is very delicate and is intimately tied to that of the solution. We now further explain this fact in the context of wave equations whose principal operator is $(g^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta$. The corresponding eikonal equation is the nonlinear transport equation $(g^{-1})^{\alpha\beta}\partial_\alpha U \partial_\beta U = 0$. The key point is that the standard regularity theory of transport equations yields only that U is as regular as its coefficients, that is, as regular as $g_{\alpha\beta}$. In the context of the relativistic Euler equations (where the formation of a shock corresponds to the intersection of the wave characteristics and $g = g(h, s, u)$), this suggests that one might expect U to be only as regular as h, s, and u. Since, as we mentioned in the previous paragraph, we have $Z^\alpha \sim \partial U$, this leads to the following severe difficulty: In commuting equation the wave equation (1.2a) with Z, one obtains the wave equation

[13] As is explained in [22], in the known framework for proving shock formation, one crucially relies on the fact that the derivatives of the solution blow up at a linear rate, that is like $\frac{C}{T_{(\mathrm{Lifespan})} - t}$, where C is a constant and $T_{(\mathrm{Lifespan})} > 0$ is the (future) classical lifespan of the solution; if one perturbs the equation by adding terms that are expected to alter this blowup rate, then one should expect that the known approach for proving shock formation will not work (at least in its current form).

$\square_g(Zh) = \square_g Z^\alpha \cdot \partial_\alpha h + \cdots \sim \partial^3 U \cdot \partial h + \cdots$ (one would obtain similar wave equations for Zs and Zu^α upon commuting equations (1.2b) and (1.2c) with Z). The difficulty is that the above discussion suggests that the factor $\partial^3 U$ can be controlled only in terms of *three* derivatives of h, s, and u, while standard energy estimates for the wave equations $\square_g(Zh) = \cdots$, $\square_g(Zs) = \cdots$, and $\square_g(Zu^\alpha) = \cdots$ yield control of only *two* derivatives of h, s, and u. This suggests that there is a loss of regularity and in fact, this is the reason that Alinhac used Nash–Moser estimates in his works [2,3]. However, for wave equations, one can in fact overcome this loss of regularity by exploiting some delicate tensorial properties of the eikonal equation $(g^{-1})^{\alpha\beta}\partial_\alpha U \partial_\beta U = 0$ and of the wave equation itself relative to geometric coordinates, which together can be used to show that in directions *tangent to the characteristics*, some geometric tensors constructed out of the derivatives of U are *one degree more differentiable than one might naively expect*. In particular, the factor $\partial^3 U$ in the aforementioned product $\partial^3 U \cdot \partial h$ has special structure and enjoys this gain in regularity. These crucial structures were first observed by Christodoulou–Klainerman in their proof [6] of the stability of Minkowski spacetime as a solution to Einstein's equations, and later by Klainerman–Rodnianski in their proof of improved regularity local well-posedness [20] for a general class of scalar quasilinear wave equations. In total, using this gain in regularity along the characteristics and carefully accounting for the precise tensorial structure of the product $\partial^3 U \cdot \partial h$ highlighted above, one can avoid the loss of derivatives tied to the product $\partial^3 U \cdot \partial h$.

Despite the fact that the procedure described above allows one to avoid losing derivatives, at least in the context of wave equations,[14] one pays a steep price: It turns out that upon implementing this procedure, one introduces a dangerous factor into the wave equation energy identities, one that in fact blows up as the shock forms. More precisely, the singular factor is $1/\mu$, where μ is the inverse foliation density mentioned earlier, with $\mu \to 0$ signifying the formation of a shock. This leads to singular top-order a priori energy estimates for the wave equation solutions relative to the geometric coordinates. At first glance, these singular geometric energy estimates might seem to obstruct the philosophy of obtaining regular estimates relative to the geometric coordinates. However, below the top derivative level, one can allow the loss of a derivative, and it turns out that this allows one to derive improved (i.e., less singular) energy estimates below the top derivative level. In fact, by an induction-from-the-top-down argument, one can show that the mid-derivative-level and below geometric energies remain bounded up to the shock. This allows one to show that indeed, the solution remains rather smooth relative to the geometric coordinates, which in practice is a crucial ingredient that is needed to close the proof. It also turns out that many steps are needed to descend to the level of a non-singular energy, which in practice means that one must assume that the

[14]Actually, it is not known whether or not the derivative-loss-avoiding procedure can be implemented for general systems of wave equations featuring more than one distinct wave operator. From this perspective, we find it fortunate that the equations of Theorem 1.2 feature only one wave operator.

data have a lot of Sobolev regularity to close the proof; see [14] for an in-depth overview of these issues in the context of quasilinear wave equations.

The structures described above, which allow one to avoid the loss of derivatives in eikonal functions for quasilinear wave equations, are rather delicate. Thus, it is not a priori clear that one can also avoid the loss of derivatives in eikonal functions for the relativistic Euler equations. A key advantage of our new formulation of the relativistic Euler equations is that it can be used to prove that *one can still avoid the loss of derivatives, even though there is deep coupling between the wave and transport equations in the new formulation.* That is, one can show that the acoustic eikonal function U [see (1.6), where $g = g(h, s, u)$ is the acoustical metric from Definition 2.6] for the relativistic Euler equations has enough regularity to be used in the study of shock formation; see three paragraphs below for further discussion. However, this requires one to first prove that the fluid variables have a consistent amount of regularity among themselves. At first thought, the desired consistency of regularity might seem to follow from standard local well-posedness. However, all standard local well-posedness results for the relativistic Euler equations are based on first-order formulations, which are not known to be sufficient for avoiding a loss of derivatives in the eikonal function U; the above outline for how to avoid derivative loss in U implicitly relied on the assumption that h, s, and u^α solve wave equations whose source terms have an allowable amount of regularity, which, as we will explain, for the relativistic Euler equations is a true—but deep—fact. Moreover, the first-order formulations do not seem to be sufficient for studying solutions all the way up to a shock; as we have mentioned, the known framework for studying shocks crucially relies on the special null structures exhibited by the equations of Theorem 1.2.

In view of the regularity concerns raised in the previous paragraph, one must carefully check that (under suitable assumptions on the initial data), all terms in the equations of Theorem 1.2 have a *consistent amount of regularity.* We stress that this is not obvious, as we now illustrate by counting derivatives. For example, to control ∂u^α in L^2 using standard energy estimates for the wave equation (1.2b), one must control, also in L^2, the source term $\mathcal{C}^\alpha$ on RHS (1.2b). Note that from the point of view of regularity, we have the schematic relationship [see (2.16a) for the definition of $\mathcal{C}^\alpha$] $\mathcal{C}^\alpha \sim \mathrm{vort}^\alpha(\varpi) \sim \partial\varpi$. Moreover, since ϖ solves the transport equation (1.3c), whose source term depends on ∂u and ∂h, this suggests that $\partial\varpi$ should be no more regular[15] than $(\partial^2 u, \partial^2 h)$ and thus $\mathcal{C}^\alpha$ should be no more regular than $(\partial^2 u, \partial^2 h)$. In total, this discussion suggests that the wave equation for u has the following schematic structure from the point of view of regularity: $\square_g u^\alpha = \partial^2 u + \cdots$. That is, this discussion *suggests* that in order to control ∂u in L^2 using standard energy estimates for wave equations, we must control $\partial^2 u$ in L^2. This approach therefore *seems* to lead to a loss in derivatives, which is a serious obstacle to using the equations of Theorem 1.2 to prove any rigorous

[15] In the absence of special structures, solutions to transport equations are not more regular than their source terms.

result. Similar difficulties arise in the study of h and s, due to the source term $\mathcal{D}$ in the wave equations (1.2a) and (1.2c).

A crucial feature of the equations of Theorem 1.2 is that *one can in fact overcome the loss of derivative difficulty for the fluid variables described in the previous paragraph.* To this end, one must rely on the transport-div-curl equations for ϖ and S; see Sect. 9.5 and the proofs of Proposition 9.22 and Theorem 9.12 for the details on how one can use these equations and elliptic estimates to avoid the loss of derivatives. Equally important for applications to shock waves is the fact that the elliptic div-curl estimates, which occur across space, are compatible with the proof of the formation of a spatially localized shock singularity and with the singular high-order geometric energy estimates described earlier in this subsubsection. These are delicate issues, especially since the elliptic estimates involve derivatives in directions transversal to the characteristics, i.e., in the singular directions; see [22] for an overview of how to derive the relevant elliptic estimates in the context of shock-forming solutions to the non-relativistic compressible Euler equations.

We now return to the issue of the regularity of the acoustic eikonal function U for the relativistic Euler equations [see (1.6), where $g = g(h, s, u)$ is the acoustical metric from Definition 2.6]. As we explained above, in order to avoid a loss of regularity in U, one needs to show that its regularity theory is compatible with the regularity of the fluid variables. It turns out that this requires proving, in particular, that $\square_g h$, $\square_g s$, and $\square_g u^\alpha$ have the same regularity as ∂h, ∂s, and ∂u^α. The connection between $\square_g h$, $\square_g s$, and $\square_g u^\alpha$ and the regularity theory of U is through the null mean curvature of the level sets of U, a critically important geometric quantity whose evolution equation[16] depends on a certain component of the Ricci curvature tensor of the Lorentzian metric $g(h, s, u)$, whose rectangular components can be shown to depend on $\square_g h$, $\square_g s$, and $\square_g u^\alpha$. We will not further discuss this crucial technical issue here; we instead refer readers to [14, Section 3.4] for further discussion of the regularity theory of eikonal functions in the context of shock formation for quasilinear wave equations. In view of the wave equations (1.2a)–(1.2c), we see that obtaining the desired regularity for $\square_g h$, $\square_g s$, and $\square_g u^\alpha$ requires, in particular, establishing that the source terms $\mathcal{C}^\alpha$ and $\mathcal{D}$ on RHSs (1.2a)–(1.2c) have the same regularity as ∂h, ∂s, and ∂u^α. This is again tantamount to showing that the vorticity and entropy are one degree more differentiable compared to the regularity guaranteed by deriving standard energy estimates for first-order formulations of the equations; to obtain the desired extra regularity for $\mathcal{C}^\alpha$ and $\mathcal{D}$, one can again rely on the transport-div-curl equations mentioned in the previous paragraph. We prove a rigorous version of this gain in regularity in Theorem 9.12, in which we use the new formulation of the relativistic Euler equations to prove a local well-posedness result that, in particular, yields the desired extra differentiability (assuming that it is present in the initial data).

[16]The evolution equation is in fact the famous *Raychaudhuri equation*, which plays an important role in general relativity.

Although one might view the results of Theorem 9.12 as expected consequences of our new formulation of the relativistic Euler equations, we highlight that its proof relies on a few ingredients that are not entirely straightforward:

(i) *Time-continuity* for the L^2 norms of the vorticity and entropy at top-order, i.e., including the extra differentiability of these variables, is non-standard in view of the necessity of invoking elliptic–hyperbolic estimates.
(ii) The transport-div-curl systems featured in the new formulation of the equations involve *spacetime* divergence and curl operators, but we need to extract L^2 regularity along the constant-time hypersurfaces. This requires connecting the spacetime divergence and curl to spatial elliptic estimates, which in turn requires some geometric and technical insights.
(iii) For the wave equation energy estimates, one cannot use the multiplier[17] ∂_t when the three-velocity is large, since the corresponding energy will not necessarily be coercive[18] in this case. Consequently, one has to use the four-velocity as a multiplier.[19]

1.2.4. Structures Amenable to Commutations with Geometric Vectorfields. A key point is that the geometric vectorfields Z described in Sect. 1.2.3 are adapted only to the principal part of the shock-forming solution variables, e.g., the operator $\square_g$ in the case that a wave equation solution is the shock-forming variable. However, to close the proof of shock formation for a system in which wave equations of the type $\square_g \cdot = \cdots$ are *coupled* to other equations, one must commute that Z through *all* of the equations in the system. One then has to handle the commutator terms generated by commuting the Z through the other equations. It turns out, perhaps not surprisingly, that commuting Z through a generic second-order differential operator ∂^2 leads to uncontrollable error terms, from the point of view of regularity and from the point of view of the singular nature of the commutator error terms; see the work [22] on the non-relativistic compressible barotropic Euler equations for further discussion on this point. However, as was first shown in [22], it is possible to commute the Z through an arbitrary first-order differential operator ∂ by first weighting it by μ (where μ is the inverse foliation density mentioned above); it can be shown

[17] See Sect. 9.6.1 for additional details regarding the multiplier method in the context of wave equations.

[18] Equations (2.11), (2.20), and (2.13a) collectively imply that when $\sum_{a=1}^{3} |u^a|$ is large, $g(\partial_t, \partial_t) = g_{00} = -1 + (c^{-2} - 1)u_a u^a$ can be positive, i.e., ∂_t can be spacelike with respect to the acoustical metric g; it is well known that this can lead to indefinite energies if the standard partial time derivative vectorfield ∂_t is used as a multiplier in the wave equation energy estimates.

[19] The use of u as a multiplier is likely familiar to researchers who have previously studied the relativistic Euler equations, but it might be unknown to the broader PDE community. We also remark that in searching the literature, we were unable to find results that, given our new formulation of the relativistic Euler equations, could be directly applied to establish points (i) and (ii) above. Moreover, we were not able to locate a local well-posedness result for elliptic–hyperbolic systems that can be directly applied to our new formulation of the equations. In particular, we could not locate a result that would directly imply continuous dependence of solutions on the initial data up to top order, i.e., a result that applies in the case when the vorticity and entropy enjoy the aforementioned extra regularity.

that this leads to commutator error terms that are controllable under the scope of the approach. **It is for this reason that we have formulated Theorem 1.2 in such a way that all of the equations are of the type $\square_g \cdot = \cdots$ or are first-order**; i.e., the equations of Theorem 1.2 are such that the approach described in [22] can be applied. Put differently, the geometric vectorfields Z that are of essential importance for commuting the wave equations of Theorem 1.2 can also be commuted through all of the remaining equations, generating only controllable error terms.

2. A First-Order Formulation of the Relativistic Euler Equations, Geometric Tensorfields, and the Modified Fluid Variables

In this section, we introduce some notation, define the fluid variables that play a role in the subsequent discussion, introduce some geometric tensorfields associated to the flow, and provide the standard first-order formulation of the relativistic Euler equations that will serve as a starting point for our main results. Most of the discussion here is standard and therefore, we are somewhat terse; we refer readers to [4, Chapter 1] for a detailed introduction to the relativistic Euler equations. Section 2.2.5, however, is not standard. In that subsubsection, we define *modified fluid variables*, which are special combinations of the derivatives of the vorticity and entropy. The structures revealed by Theorem 3.1 imply (see the proof of Theorem 9.12 for additional details) that these special combinations enjoy a gain of one derivative compared to the regularity afforded by standard estimates. As we mentioned in the introduction, this gain is crucial for applications to shock waves.

2.1. Notation and Conventions

We somewhat follow the setup of [4], but there are some differences, including sign differences and notational differences.

Greek "spacetime" indices $\alpha, \beta, \ldots$ take on the values $0, 1, 2, 3$, while Latin "spatial" indices $a, b, \ldots$ take on the values $1, 2, 3$. Repeated indices are summed over (from 0 to 3 if they are Greek, and from 1 to 3 if they are Latin). Greek and Latin indices are lowered and raised with the Minkowski metric η and its inverse η^{-1}, and **not with the acoustical metric g of Definition 2.6**. Moreover, $\epsilon_{\alpha\beta\gamma\delta}$ denotes the fully antisymmetric symbol normalized by $\epsilon_{0123} = 1$. Note that $\epsilon^{0123} = -1$.

If X^α is a vectorfield and $\xi^{\alpha_1 \cdots \alpha_l}_{\beta_1 \cdots \beta_m}$ is a type $\binom{l}{m}$ tensorfield, then

$$(\mathcal{L}_X \xi)^{\alpha_1 \cdots \alpha_l}_{\beta_1 \cdots \beta_m} = X^\kappa \partial_\kappa \xi^{\alpha_1 \cdots \alpha_l}_{\beta_1 \cdots \beta_m} - \sum_{a=1}^{l} (\partial_\kappa X^{\alpha_a}) \xi^{\alpha_1 \cdots \alpha_{a-1} \kappa \alpha_{a+1} \cdots \alpha_l}_{\beta_1 \cdots \beta_m} + \sum_{b=1}^{m} (\partial_{\beta_b} X^\kappa) \xi^{\alpha_1 \cdots \alpha_l}_{\beta_1 \cdots \beta_{b-1} \kappa \beta_{b+1} \cdots \beta_m} \tag{2.1}$$

denotes the Lie derivative of ξ with respect to X.

We derive all of our results relative to a Minkowski-rectangular coordinate system $\{x^\alpha\}_{\alpha=0,1,2,3}$, that is, a coordinate system on $\mathbb{R}^{1+3}$ in which the Minkowski metric η takes the form $\eta_{\alpha\beta} := \mathrm{diag}(-1,1,1,1)$. $\{\partial_\alpha\}_{\alpha=0,1,2,3}$ denotes the corresponding rectangular coordinate partial derivative vectorfields. We sometimes use the alternate notation $x^0 := t$ and $\partial_t := \partial_0$.

Throughout, d denotes the exterior derivative operator. In particular, if f is a scalar function, then $(df)_\alpha := \partial_\alpha f$, and if V is a one-form, then $(dV)_{\alpha\beta} := \partial_\alpha V_\beta - \partial_\beta V_\alpha$. We use the notation $V_\flat$ to denote the one-form that is η-dual to the vectorfield V, i.e., $(V_\flat)_\alpha := \eta_{\alpha\kappa} V^\kappa$.

2.2. Definitions of the Fluid Variables and Related Geometric Quantities

In this subsection, we define the fluid variables and geometric quantities that play a role in the subsequent discussion.

2.2.1. The Basic Fluid Variables. The fluid four velocity u^α is future-directed and normalized by $u_\alpha u^\alpha = -1$. p denotes the pressure, ρ denotes the proper energy density, n denotes the proper number density, s denotes the entropy per particle, θ denotes the temperature, and

$$H = (\rho + p)/n \tag{2.2}$$

is the enthalpy per particle. Thermodynamics supplies the following laws:

$$H = \frac{\partial \rho}{\partial n}\Big|_s, \qquad \theta = \frac{1}{n}\frac{\partial \rho}{\partial s}\Big|_n, \qquad dH = \frac{dp}{n} + \theta ds, \tag{2.3}$$

where $\frac{\partial}{\partial n}|_s$ denotes partial differentiation with respect to n at fixed s and $\frac{\partial}{\partial s}|_n$ denotes partial differentiation with respect to s at fixed n. Below we employ similar partial differentiation notation, and in Definition 2.7, we introduce alternate partial differentiation notation, which we use throughout the remainder of the article.

2.2.2. The u-orthogonal Vorticity of a One-Form and Auxiliary Fluid Variables. In this subsubsection, we define some auxiliary fluid variables that will play a role throughout the paper. By "auxiliary," we mean that they are determined by the variables introduced in Sect. 2.2.1.

We start by defining the u-orthogonal vorticity of a one form.

Definition 2.1 (*The u-orthogonal vorticity of a one form*). Given a one-form V, we define the corresponding u-orthogonal vorticity vectorfield as follows:

$$\mathrm{vort}^\alpha(V) := -\epsilon^{\alpha\beta\gamma\delta} u_\beta \partial_\gamma V_\delta. \tag{2.4}$$

Definition 2.2 (*Vorticity vectorfield*). We define the vorticity vectorfield ϖ^α as follows:

$$\varpi^\alpha := \mathrm{vort}^\alpha(Hu) = -\epsilon^{\alpha\beta\gamma\delta} u_\beta \partial_\gamma (H u_\delta). \tag{2.5}$$

We find it convenient to work with the natural log of the enthalpy.

Definition 2.3 (*Logarithmic enthalpy*). Let $\overline{H} > 0$ be a fixed constant value of the enthalpy. We define the (dimensionless) logarithmic enthalpy h as follows:

$$h := \ln\left(H/\overline{H}\right). \tag{2.6}$$

Definition 2.4 (*The quantity q*). We define the quantity q as follows:

$$q := \frac{\theta}{H}. \tag{2.7}$$

Definition 2.5 (*Entropy gradient one-form*). We define the entropy gradient one-form S_α as follows:

$$S_\alpha := \partial_\alpha s. \tag{2.8}$$

2.2.3. Equation of State and Speed of Sound. To obtain a closed system of equations, we assume an *equation of state* of the form $p = p(\rho, s)$. The *speed of sound* is defined by

$$c := \sqrt{\frac{\partial p}{\partial \rho} |_s}. \tag{2.9}$$

For reasons that will become clear in Sect. 2.3, in the rest of the article, we view the speed of sound to be a function of h and s:

$$c = c(h, s). \tag{2.10}$$

In this article, we will confine our study to equations of state and solutions that verify

$$0 < c \leq 1. \tag{2.11}$$

The upper bound in (2.11) signifies that the speed of sound is no bigger than the speed of light. In this article, we exploit both inequalities in (2.11). We use the bound $c \leq 1$ to ensure that we can always solve for time derivatives of the solution in terms of spatial derivatives; see the discussion surrounding Eq. (2.28). The bound $c > 0$ is important because some of the equations featured in Theorem 3.1 contain factors of c^{-1}.

2.2.4. Projection Onto the Minkowski-Orthogonal Complement of the Four-Velocity and the Acoustical Metric. We start by introducing the tensorfield $\Pi^{\alpha\beta}$, defined by

$$\Pi^{\alpha\beta} := (\eta^{-1})^{\alpha\beta} + u^\alpha u^\beta. \tag{2.12}$$

It is straightforward to see Π is the projection onto the η-orthogonal complement of u. In particular, $\Pi^{\alpha\kappa} u_\kappa = 0$.

We now introduce the acoustical metric g. It is a Lorentzian[20] metric that drives the propagation of sound waves.

Definition 2.6 (*Acoustical metric and its inverse*). We define the acoustical metric $g_{\alpha\beta}$ and its inverse[21] $(g^{-1})^{\alpha\beta}$ as follows:

$$g_{\alpha\beta} := c^{-2}\eta_{\alpha\beta} + (c^{-2} - 1)u_\alpha u_\beta, \tag{2.13a}$$

$$(g^{-1})^{\alpha\beta} := c^2\Pi^{\alpha\beta} - u^\alpha u^\beta = c^2(\eta^{-1})^{\alpha\beta} + (c^2 - 1)u^\alpha u^\beta. \tag{2.13b}$$

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[20] That is, the signature of the 4×4 matrix $g_{\alpha\beta}$, viewed as a quadratic form, is $(-, +, +, +)$.

[21] It is straightforward to check that $(g^{-1})^{\alpha\kappa} g_{\kappa\beta} = \delta^\alpha_\beta$, where δ^α_β is the Kronecker delta. That is, g^{-1} is indeed the inverse of g.

It is straightforward to compute that relative to the rectangular coordinates, we have

$$\det g = -c^{-6}, \tag{2.14a}$$

$$|\det g|^{1/2}(g^{-1})^{\alpha\beta} = c^{-1}(\eta^{-1})^{\alpha\beta} + (c^{-1} - c^{-3})u^{\alpha}u^{\beta}. \tag{2.14b}$$

The notation featured in the next definition will allow for a simplified presentation of various equations.

Definition 2.7 (*Partial derivatives with respect to h and s*). If Q is a quantity that can be expressed as a function of (h, s), then

$$Q_{;h} = Q_{;h}(h,s) := \frac{\partial Q}{\partial h}|_s\ , \tag{2.15a}$$

$$Q_{;s} = Q_{;s}(h,s) := \frac{\partial Q}{\partial s}|_h\ , \tag{2.15b}$$

where $\frac{\partial}{\partial h}|_s$ denotes partial differentiation with respect to h at fixed s and $\frac{\partial}{\partial s}|_h$ denotes partial differentiation with respect to s at fixed h.

2.2.5. Modified Fluid Variables. In our analysis, we will have to control the vorticity of the vorticity, that is, $\mathrm{vort}^{\alpha}(\varpi)$. The following modified version of $\mathrm{vort}^{\alpha}(\varpi)$, denoted by $\mathcal{C}^{\alpha}$ obeys a transport equation [see (3.11b)] with a better structure (from the point of view of the regularity of the RHS and also the null structure of the RHS) than the one satisfied by $\mathrm{vort}^{\alpha}(\varpi)$. Similar remarks apply to the modified version of the divergence of entropy gradient, which we denote by $\mathcal{D}$ [see Eq. (3.9a) for the transport equation verified by $\mathcal{D}$].

Definition 2.8 (*Modified fluid variables*).

$$\begin{aligned}\mathcal{C}^{\alpha} := {}& \mathrm{vort}^{\alpha}(\varpi) + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)\varpi_{\delta} \\ & + (\theta - \theta_{;h})S^{\alpha}(\partial_{\kappa}u^{\kappa}) + (\theta - \theta_{;h})u^{\alpha}(S^{\kappa}\partial_{\kappa}h) \\ & + (\theta_{;h} - \theta)S^{\kappa}((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}u_{\kappa}),\end{aligned} \tag{2.16a}$$

$$\mathcal{D} := \frac{1}{n}(\partial_{\kappa}S^{\kappa}) + \frac{1}{n}(S^{\kappa}\partial_{\kappa}h) - \frac{1}{n}c^{-2}(S^{\kappa}\partial_{\kappa}h). \tag{2.16b}$$

2.3. A Standard First-Order Formulation of the Relativistic Euler Equations

In formulating the relativistic Euler equations as a first-order hyperbolic system, we will consider h, s, and $\{u^{\alpha}\}_{\alpha=0,1,2,3}$ to be the fundamental unknowns.[22] In terms of these variables and the quantities defined in (2.9), (2.12), and (2.7), the relativistic Euler equations are

$$u^{\kappa}\partial_{\kappa}h + c^2\partial_{\kappa}u^{\kappa} = 0, \tag{2.17}$$

$$u^{\kappa}\partial_{\kappa}u^{\alpha} + \Pi^{\alpha\kappa}\partial_{\kappa}h - q(\eta^{-1})^{\alpha\kappa}\partial_{\kappa}s = 0, \tag{2.18}$$

$$u^{\kappa}\partial_{\kappa}s = 0. \tag{2.19}$$

[22] On might argue that it is more accurate to think of u^0 as being "redundant" in the sense that it is algebraically determined in terms of $\{u^a\}_{a=1,2,3}$ via the condition $u^0 > 0$ and the normalization condition (2.20). In fact, in most of Sect. 9, we adopt this point of view. However, prior to Sect. 9, we do not adopt this point of view.

It is straightforward to see that the following constraint is preserved by the flow of Eqs. (2.18)–(2.19).

$$u_\kappa u^\kappa = -1. \tag{2.20}$$

Remark 2.9 (*More common first-order formulations*). Many authors define the relativistic Euler equations to be the system comprising (2.20), (2.25), and the four equations $\partial_\kappa T^{\alpha\kappa} = 0$, where $T^{\alpha\beta} := (\rho + p)u^\alpha u^\beta + p(\eta^{-1})^{\alpha\beta}$ is the fluid's energy–momentum tensor. These equations are in fact equivalent (at least in the case of C^1 solutions with $\rho > 0$) to Eqs. (2.17)–(2.20). We refer readers to [4, Chapter 1] for background material that is sufficient for understanding the equivalence.

Note that (2.19) is equivalent to

$$u^\kappa S_\kappa = 0. \tag{2.21}$$

Equation (2.18) can be written more explicitly as

$$u^\kappa \partial_\kappa u_\alpha + \partial_\alpha h + u_\alpha u^\kappa \partial_\kappa h - qS_\alpha = 0. \tag{2.22}$$

Also, from (2.22), we easily derive

$$u^\kappa \partial_\kappa (H u_\alpha) + \partial_\alpha H - \theta S_\alpha = 0. \tag{2.23}$$

Moreover, differentiating (2.19) with a rectangular coordinate partial derivative, we deduce

$$u^\kappa \partial_\kappa S_\alpha = -S_\kappa(\partial_\alpha u^\kappa). \tag{2.24}$$

In our analysis, we will also use the following evolution equation for n:

$$u^\kappa \partial_\kappa n + n \partial_\kappa u^\kappa = 0. \tag{2.25}$$

To obtain (2.25), we first use Eqs. (2.17) and (2.19), the thermodynamic relation $dH = dp/n + \theta ds$, and the relation $H = (\rho + p)/n$ to deduce $u^\kappa \partial_\kappa p + c^2(\rho + p)\partial_\kappa u^\kappa = 0$. We then use this equation, (2.9), and (2.19) to deduce $u^\kappa \partial_\kappa \rho + (\rho + p)\partial_\kappa u^\kappa = 0$. Next, using this equation and Eq. (2.19), we deduce $\frac{\partial \rho(n,s)}{\partial n}|_s \, u^\kappa \partial_\kappa n + (\rho + p)\partial_\kappa u^\kappa = 0$. Finally, from this equation and the thermodynamic relation $\rho + p = n \frac{\partial \rho(n,s)}{\partial n}|_s$, we conclude (2.25).

For future use, we also note that Eqs. (2.17)–(2.19) can be written [using (2.20)] in the form

$$A^\alpha \partial_\alpha \begin{pmatrix} h \\ u^0 \\ u^1 \\ u^2 \\ u^3 \\ s \end{pmatrix} = 0, \tag{2.26}$$

where for $\alpha = 0, 1, 2, 3$, A^α is a 6×6 matrix that is a smooth function of the solution array $(h, u^0, u^1, u^2, u^3, s)$. In particular, we compute that

$$A^0 = \begin{pmatrix} u^0 & c^2 & 0 & 0 & 0 & 0 \\ u_a u^a & u^0 & 0 & 0 & 0 & q \\ u^0 u^1 & 0 & u^0 & 0 & 0 & 0 \\ u^0 u^2 & 0 & 0 & u^0 & 0 & 0 \\ u^0 u^3 & 0 & 0 & 0 & u^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u^0 \end{pmatrix}, \tag{2.27}$$

and we compute that

$$\det A^0 = (u^0)^6 - c^2 (u^0)^4 u_a u^a = (1 + u_a u^a)^4 \left\{ 1 + (1 - c^2) u_b u^b \right\}. \tag{2.28}$$

In particular, in view of (2.11), we deduce from (2.28) that A^0 is invertible.

3. The New Formulation of the Relativistic Euler Equations

In the next theorem, we provide the main result of the article: the new formulation of the relativistic Euler equations.

Theorem 3.1 (New formulation of the relativistic Euler equations). *For C^3 solutions (h, s, u^α) to the relativistic Euler equations (2.17)–(2.19) + (2.20), the following equations hold, where the phrase "g-**null form**" refers to a linear combination of the standard g-null forms of Definition 1.1 with coefficients that are allowed to depend on the quantities $(h, s, u^\alpha, S^\alpha, \varpi^\alpha)$ (but **not** their derivatives).*

Wave equations. *The logarithmic enthalpy h verifies the following covariant wave equation (see Footnote 5 on pg. 6 for a formula for the covariant wave operator):*

$$\square_g h = n c^2 q \mathcal{D} + \mathfrak{Q}_{(h)} + \mathfrak{L}_{(h)}, \tag{3.1}$$

*where $\mathfrak{Q}_{(h)}$ is the g-**null form** defined by*

$$\begin{aligned} \mathfrak{Q}_{(h)} := & - c^{-1} c_{;h} (g^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h) \\ & + c^2 \left\{ (\partial_\kappa u^\kappa)(\partial_\lambda u^\lambda) - (\partial_\lambda u^\kappa)(\partial_\kappa u^\lambda) \right\}, \end{aligned} \tag{3.2a}$$

and $\mathfrak{L}_{(h)}$, which is at most linear in the derivatives of $(h, s, u^\alpha, S^\alpha, \varpi^\alpha)$, is defined by

$$\mathfrak{L}_{(h)} := \left\{ (1 - c^2) q + c^2 q_{;h} - c c_{;s} \right\} (S^\kappa \partial_\kappa h) + c^2 q_{;s} S_\kappa S^\kappa. \tag{3.2b}$$

Moreover, the rectangular four-velocity components[23] *u^α verify the following covariant wave equations:*

$$\square_g u^\alpha = - \frac{c^2}{H} \mathcal{C}^\alpha + \mathfrak{Q}_{(u^\alpha)} + \mathfrak{L}_{(u^\alpha)}, \tag{3.3}$$

[23] We stress that on LHS (3.3), the components u^α are treated as scalar functions under the action of the covariant wave operator $\square_g$.

where $\mathfrak{Q}_{(u^\alpha)}$ *is the* g-***null form*** *defined by*

$$\begin{aligned}\mathfrak{Q}_{(u^\alpha)} := (\eta^{-1})^{\alpha\lambda}\left\{(\partial_\kappa u^\kappa)(\partial_\lambda h) - (\partial_\lambda u^\kappa)(\partial_\kappa h)\right\} \\ + c^2 u^\alpha\left\{(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\lambda u^\lambda)(\partial_\kappa u^\kappa)\right\} \\ - \left\{1 + c^{-1}c_{;h}\right\}(g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u^\alpha), \end{aligned} \tag{3.4a}$$

and $\mathfrak{L}_{(u^\alpha)}$, *which is at most linear in the derivatives of* $(h, s, u^\alpha, S^\alpha, \varpi^\alpha)$, *is defined by*

$$\begin{aligned}\mathfrak{L}_{(u^\alpha)} := &-\frac{c^2}{H}\epsilon^{\alpha\beta\gamma\delta}(\partial_\beta u_\gamma)\varpi_\delta + \frac{(1-c^2)}{H}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi_\delta \\ &+ \frac{(1-c^2)q}{H}\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma \varpi_\delta \\ &+ \{q - cc_{;s}\}(S^\kappa\partial_\kappa u^\alpha) + q(c^2-1)u^\alpha S^\kappa(u^\lambda\partial_\lambda u_\kappa) \\ &+ S^\kappa\left\{c^2 q + \frac{(\theta - \theta_{;h})c^2}{H}\right\}((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\kappa) \\ &+ \left\{2c^{-1}c_{;h}qS^\alpha + 2c^{-1}c_{;s}S^\alpha - q_{;h}S^\alpha\right\}(u^\kappa\partial_\kappa h) \\ &+ S^\alpha\left\{\frac{(\theta-\theta_{;h})c^2}{H} - q\right\}(\partial_\kappa u^\kappa) + \frac{(\theta-\theta_{;h})c^2}{H}u^\alpha(S^\kappa\partial_\kappa h). \end{aligned} \tag{3.4b}$$

Auxiliary wave equation for s. *The entropy s verifies the following covariant wave equation*[24]*:*

$$\square_g s = c^2 n\mathcal{D} + \mathfrak{L}_{(s)}, \tag{3.5}$$

where $\mathfrak{L}_{(s)}$, *which is at most linear in the derivatives of* $(h, s, u^\alpha, S^\alpha, \varpi^\alpha)$, *is defined by*

$$\mathfrak{L}_{(s)} := \left\{1 - c^2 - cc_{;h}\right\}(S^\kappa\partial_\kappa h) - cc_{;s}S_\kappa S^\kappa. \tag{3.6}$$

Transport equations. *The rectangular components of the entropy gradient vectorfield* S^α, *whose* η-*dual is defined in* (2.8), *verify the following transport equations:*

$$u^\kappa\partial_\kappa S^\alpha = -S_\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda u^\kappa). \tag{3.7}$$

Moreover, the rectangular components of the vorticity vectorfield ϖ^α, *which is defined in* (2.5), *verify the following transport equations:*

$$\begin{aligned}u^\kappa\partial_\kappa\varpi^\alpha = -u^\alpha(\varpi^\kappa\partial_\kappa h) + \varpi^\kappa\partial_\kappa u^\alpha - \varpi^\alpha(\partial_\kappa u^\kappa) \\ + (\theta - \theta_{;h})\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)S_\delta + qu^\alpha\varpi^\kappa S_\kappa. \end{aligned} \tag{3.8}$$

Transport-div-curl systems. *The modified divergence of the entropy gradient* $\mathcal{D}$ *[which is defined in* (2.16b)*] and the rectangular components* $\text{vort}^\alpha(S)$

[24]The wave equation (3.5) is auxiliary in the sense that we do not use it in our proof of Theorem 9.12. However, in applications (for example, in the study of shock formation), one has to compute $\square_g$ applied to the scalar component functions $g_{\alpha\beta}$, and, by virtue of the chain rule, the quantity $\square_g s$ arises in such computations. It is for this reason that we have included Eq. (3.5) in this paper.

of the u-orthogonal vorticity of the entropy gradient vectorfield [see definition (2.4)*] verify the following transport-div-curl system:*

$$u^\kappa \partial_\kappa \mathcal{D} = \frac{2}{n}\left\{(\partial_\kappa S^\kappa)(\partial_\lambda u^\lambda) - (\partial_\lambda S^\kappa)(\partial_\kappa u^\lambda)\right\} + \frac{1}{n}c^{-2}u^\kappa\left\{(\partial_\kappa h)(\partial_\lambda S^\lambda) - (\partial_\lambda h)(\partial_\kappa S^\lambda)\right\} + \frac{S_\kappa \mathcal{C}^\kappa}{nH} + \mathfrak{Q}_{(\mathcal{D})} + \mathfrak{L}_{(\mathcal{D})}, \tag{3.9a}$$

$$\mathrm{vort}^\alpha(S) = 0, \tag{3.9b}$$

where $\mathfrak{Q}_{(\mathcal{D})}$ *is the g-**null form** defined by*

$$\mathfrak{Q}_{(\mathcal{D})} := \frac{1}{n}c^{-2}S^\kappa\left\{(\partial_\kappa u^\lambda)(\partial_\lambda h) - (\partial_\lambda u^\lambda)(\partial_\kappa h)\right\}, \tag{3.10a}$$

and $\mathfrak{L}_{(\mathcal{D})}$*, which is linear in the derivatives of* $(h, s, u^\alpha, S^\alpha, \varpi^\alpha)$*, is defined by*

$$\mathfrak{L}_{(\mathcal{D})} := \frac{(1-c^{-2})}{nH}\epsilon^{\alpha\beta\gamma\delta}S_\alpha u_\beta(\partial_\gamma h)\varpi_\delta + \frac{1}{nH}\epsilon^{\alpha\beta\gamma\delta}S_\alpha(\partial_\beta u_\gamma)\varpi_\delta + \frac{S^\kappa S^\lambda}{n}\left\{\frac{(\theta - \theta_{;h})}{H} - 2q\right\}(\partial_\kappa u_\lambda) + \frac{S_\kappa S^\kappa}{n}\left\{\frac{(\theta_{;h} - \theta)}{H} + 2c^{-1}c_{;s} - c^2 q_{;h} + q\right\}(\partial_\lambda u^\lambda). \tag{3.10b}$$

Finally, the divergence of the vorticity vectorfield ϖ^α *(which is defined in* (2.5)*) and the rectangular components* $\mathcal{C}^\alpha$ *of the modified vorticity of the vorticity (which is defined in* (2.16a)*) verify the following equations:*

$$\partial_\alpha \varpi^\alpha = -\varpi^\kappa \partial_\kappa h + 2q\varpi^\kappa S_\kappa, \tag{3.11a}$$

$$\begin{aligned} u^\kappa \partial_\kappa \mathcal{C}^\alpha = {} & \mathcal{C}^\kappa \partial_\kappa u^\alpha - 2\mathcal{C}^\alpha(\partial_\kappa u^\kappa) + u^\alpha \mathcal{C}^\kappa(u^\lambda \partial_\lambda u_\kappa) \\ & - 2\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma \varpi^\kappa)(\partial_\delta u_\kappa) \\ & + (\theta_{;h} - \theta)\left\{(\eta^{-1})^{\alpha\kappa} + 2u^\alpha u^\kappa\right\}\left\{(\partial_\kappa h)(\partial_\lambda S^\lambda) - (\partial_\lambda h)(\partial_\kappa S^\lambda)\right\} \\ & + (\theta - \theta_{;h})nu^\alpha(u^\kappa \partial_\kappa h)\mathcal{D} \\ & + (\theta - \theta_{;h})qS^\alpha(\partial_\kappa S^\kappa) + (\theta_{;h} - \theta)qS_\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda S^\kappa) \\ & + \mathfrak{Q}_{(\mathcal{C}^\alpha)} + \mathfrak{L}_{(\mathcal{C}^\alpha)}, \end{aligned} \tag{3.11b}$$

where $\mathfrak{Q}_{(\mathcal{C}^\alpha)}$ *is the g-**null form** defined by*

$$\begin{aligned} \mathfrak{Q}_{(\mathcal{C}^\alpha)} := {} & -c^{-2}\epsilon^{\kappa\beta\gamma\delta}(\partial_\kappa u^\alpha)u_\beta(\partial_\gamma h)\varpi_\delta \\ & + (c^{-2} + 2)\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa) \\ & + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta\varpi_\delta\left\{(\partial_\kappa u^\kappa)(\partial_\gamma h) - (\partial_\gamma u^\kappa)(\partial_\kappa h)\right\} \\ & + \left\{(\theta_{;h;h} - \theta_h) + c^{-2}(\theta - \theta_{;h})\right\}u^\kappa(\eta^{-1})^{\alpha\lambda}S^\beta \times \\ & \quad \left\{(\partial_\kappa h)(\partial_\lambda u_\beta) - (\partial_\lambda h)(\partial_\kappa u_\beta)\right\} \\ & + (\theta_{;h} - \theta)S^\kappa u^\lambda\left\{(\partial_\kappa u^\alpha)(\partial_\lambda h) - (\partial_\lambda u^\alpha)(\partial_\kappa h)\right\} \\ & + (\theta_{;h} - \theta)\left\{(\eta^{-1})^{\alpha\kappa} + u^\alpha u^\kappa\right\}S^\beta\left\{(\partial_\kappa u_\beta)(\partial_\lambda u^\lambda) - (\partial_\lambda u_\beta)(\partial_\kappa u^\lambda)\right\} \end{aligned}$$

$$
\begin{aligned}
&+ (\theta_{;h} - \theta) S^\alpha \left\{ (\partial_\kappa u^\lambda) \partial_\lambda u^\kappa - (\partial_\lambda u^\lambda)(\partial_\kappa u^\kappa) \right\} \\
&+ (\theta_{;h} - \theta) S^\kappa \left\{ (\partial_\kappa u^\alpha)(\partial_\lambda u^\lambda) - (\partial_\lambda u^\alpha)(\partial_\kappa u^\lambda) \right\} \\
&+ S^\alpha \left\{ c^{-2}(\theta_h - \theta_{;h;h}) + c^{-4}(\theta_{;h} - \theta) \right\} (g^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h), \quad (3.12a)
\end{aligned}
$$

and $\mathfrak{L}_{(\mathcal{C}^\alpha)}$*, which is linear in the derivatives of* $(h, s, u^\alpha, S^\alpha, \varpi^\alpha)$*, is defined by*

$$
\begin{aligned}
\mathfrak{L}_{(\mathcal{C}^\alpha)} := &\frac{2q}{H}(\varpi^\kappa S_\kappa \varpi^\alpha) - \frac{2}{H}\varpi^\alpha(\varpi^\kappa \partial_\kappa h) \\
&+ 2c^{-3} c_{;s} \epsilon^{\alpha\beta\gamma\delta} u_\beta S_\gamma \varpi_\delta (u^\kappa \partial_\kappa h) \\
&- 2q \epsilon^{\alpha\beta\gamma\delta} u_\beta S_\gamma \varpi^\kappa (\partial_\delta u_\kappa) - q \epsilon^{\alpha\beta\gamma\delta} S_\beta u_\gamma \varpi_\delta (\partial_\kappa u^\kappa) \\
&+ \frac{1}{H}(\theta - \theta_{;h}) \epsilon^{\kappa\beta\gamma\delta} (\partial_\kappa u^\alpha) S_\beta u_\gamma \varpi_\delta + c^{-2} q \epsilon^{\alpha\beta\gamma\delta} S_\beta (\partial_\gamma h) \varpi_\delta \\
&- c^{-2} q u^\alpha \epsilon^{\kappa\beta\gamma\delta} S_\kappa u_\beta (\partial_\gamma h) \varpi_\delta \\
&+ (\theta_{;h} - \theta) q S_\kappa S^\kappa (u^\lambda \partial_\lambda u^\alpha) \\
&+ u^\alpha S_\kappa S^\kappa \left\{ (\theta_{;h} - \theta) q + (\theta_{;h;s} - \theta_{;s}) \right\} (u^\lambda \partial_\lambda h) \\
&+ S^\alpha \left\{ (\theta_{;s} - \theta_{;h;s}) + (\theta - \theta_{;h}) q_{;h} \right\} (S^\kappa \partial_\kappa h) \\
&+ S_\kappa S^\kappa \{ (\theta_{;h;h} - \theta_h) q + (\theta_{;h;s} - \theta_{;s}) + (\theta - \theta_{;h}) q c^{-2} + (\theta_{;h} - \theta) q_{;h} \} \times \\
&\qquad ((\eta^{-1})^{\alpha\lambda} \partial_\lambda h). \quad (3.12b)
\end{aligned}
$$

Remark 3.2 (*Special structure of the inhomogeneous terms*). We emphasize the following two points, which are of crucial importance for applications to shock waves (see Sect. 1.2.2 for further discussion): (i) all inhomogeneous terms on the RHSs of the equations of the theorem are at most quadratic in the derivatives of $(h, s, u^\alpha, S^\alpha, \varpi^\alpha)$ and (ii) **all derivative-quadratic terms on the RHSs of the equations of the theorem are linear combinations of standard g-null forms**. In particular, the following are linear combinations of standard g-null forms, even though we did not explicitly state so in the theorem: the terms on the first and second lines of RHS (3.9a) and the terms on the second and third lines of RHS (3.11b). We have separated these null forms, which involve the derivatives of ϖ and S, because they need to be handled with elliptic estimates, at least at the top derivative level (see the proof of Theorem 9.12). This is different compared to the terms $\mathfrak{Q}_{(h)}$, $\mathfrak{Q}_{(u^\alpha)}$, $\mathfrak{Q}_{(\mathcal{D})}$, and $\mathfrak{Q}_{(\mathcal{C}^\alpha)}$, which can be handled with standard energy estimates at all derivative levels.

Proof. Theorem 3.1 follows from a lengthy series of calculations, most of which we derive later in the paper, except that we have somewhat reorganized (using only simple algebra) the terms on the right-hand sides of the equations of the theorem. More precisely, we prove (3.1)–(3.2b) in Proposition 5.2.

We prove (3.3)–(3.4b) in Proposition 5.3.

We prove (3.5)–(3.6) in Proposition 5.4.

Equation (3.7) follows from raising the indices of (2.24) with the inverse Minkowski metric.

We prove (3.8) in Proposition 7.1.

Except for (3.9b), (3.9a)–(3.10b) follow from Proposition 6.2. Equation (3.9b) is a simple consequence of definition (2.4) and the symmetry property $\partial_\alpha S_\beta = \partial_\beta S_\alpha$ [see (4.1)].

Finally, we prove (3.11a)–(3.12b) in Proposition 8.2. □

4. Preliminary Identities

In the next lemma, we derive some preliminary identities that we will later use when deriving the equations stated in Theorem 3.1.

Lemma 4.1 (Some useful identities). *Assume that (h, s, u^α) is a C^2 solution to (2.17)–(2.19) + (2.20), and let V_α be any C^1 one-form. Then the following identities hold:*

$$\partial_\alpha S_\beta = \partial_\beta S_\alpha, \tag{4.1}$$

$$\varpi^\kappa u_\kappa = 0, \tag{4.2}$$

$$\kappa \partial_\alpha u_\kappa = 0, \tag{4.3}$$

$$u^\kappa \partial_\alpha S_\kappa = -S^\kappa \partial_\alpha u_\kappa, \tag{4.4}$$

$$u^\kappa \partial_\alpha \varpi_\kappa = -\varpi^\kappa \partial_\alpha u_\kappa, \tag{4.5}$$

$$\partial_\alpha = -u_\alpha u^\kappa \partial_\kappa + \Pi_\alpha^{\ \kappa} \partial_\kappa, \tag{4.6}$$

$$\partial_\kappa V^\kappa = -u_\kappa u^\lambda \partial_\lambda V^\kappa + \Pi^{\kappa\lambda} \partial_\kappa V_\lambda, \tag{4.7}$$

$$\begin{aligned} \partial_\alpha V_\beta - \partial_\beta V_\alpha = {} & \epsilon_{\alpha\beta\gamma\delta} u^\gamma \text{vort}^\delta(V) + u_\alpha u^\kappa \partial_\beta V_\kappa - u_\beta u^\kappa \partial_\alpha V_\kappa \\ & + u_\beta u^\kappa \partial_\kappa V_\alpha - u_\alpha u^\kappa \partial_\kappa V_\beta, \end{aligned} \tag{4.8}$$

$$\Pi^{\alpha\beta}\Pi^{\gamma\delta}(\partial_\alpha V_\gamma - \partial_\gamma V_\alpha)(\partial_\beta V_\delta - \partial_\delta V_\beta) = 2\Pi^{\alpha\beta}\text{vort}_\alpha(V)\text{vort}_\beta(V). \tag{4.9}$$

Moreover, if $u^\kappa V_\kappa = 0$, then

$$\begin{aligned} \partial_\alpha V_\beta - \partial_\beta V_\alpha = {} & \epsilon_{\alpha\beta\gamma\delta} u^\gamma \text{vort}^\delta(V) - u_\alpha V_\kappa \partial_\beta u^\kappa + u_\beta V_\kappa \partial_\alpha u^\kappa \\ & + u_\beta u^\kappa \partial_\kappa V_\alpha - u_\alpha u^\kappa \partial_\kappa V_\beta. \end{aligned} \tag{4.10}$$

In addition, the following identity holds, where the indices for ϵ on LHS (4.11) are raised before Lie differentiation:

$$\mathcal{L}_u(\epsilon^{\alpha\beta\gamma\delta}) = (-\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}. \tag{4.11}$$

Furthermore, the following identities hold:

$$\mathcal{L}_u(u_\flat)_\alpha = u^\kappa \partial_\kappa u_\alpha = -\partial_\alpha h - u_\alpha u^\kappa \partial_\kappa h + q S_\alpha, \tag{4.12}$$

$$\mathcal{L}_u d(Hu_\flat) = d\mathcal{L}_u(Hu_\flat), \tag{4.13}$$

$$[\mathcal{L}_u d(Hu_\flat)]_{\alpha\beta} = \theta_{;h}(\partial_\alpha h)\partial_\beta s - \theta_{;h}(\partial_\alpha s)\partial_\beta h, \tag{4.14}$$

$$\partial_\alpha(Hu_\beta) - \partial_\beta(Hu_\alpha) = \epsilon_{\alpha\beta\gamma\delta} u^\gamma \varpi^\delta + \theta\left\{S_\alpha u_\beta - S_\beta u_\alpha\right\}, \tag{4.15}$$

$$\epsilon^{\alpha\beta\gamma\delta}\partial_\gamma(Hu_\delta) = \varpi^\alpha u^\beta - u^\alpha \varpi^\beta + \theta \epsilon^{\alpha\beta\gamma\delta} S_\gamma u_\delta, \tag{4.16}$$

$$\begin{aligned} \partial_\alpha u_\beta - \partial_\beta u_\alpha = {} & \frac{1}{H}\epsilon_{\alpha\beta\gamma\delta} u^\gamma \varpi^\delta - (\partial_\alpha h)u_\beta + (\partial_\beta h)u_\alpha \\ & + q\left\{S_\alpha u_\beta - S_\beta u_\alpha\right\}, \end{aligned} \tag{4.17}$$

$$(u^\kappa \partial_\kappa u_\lambda) S^\lambda = -S^\kappa \partial_\kappa h + q S_\kappa S^\kappa, \tag{4.18}$$

$$(u^\kappa \partial_\kappa S_\lambda) u^\lambda = S^\kappa \partial_\kappa h - q S_\kappa S^\kappa, \tag{4.19}$$

$$\begin{aligned} S^\kappa \partial_\alpha u_\kappa &= S^\kappa \partial_\kappa u_\alpha + (S^\kappa \partial_\kappa h) u_\alpha - q S^\kappa S_\kappa u_\alpha + \frac{1}{H} \epsilon_{\alpha\kappa\gamma\delta} S^\kappa u^\gamma \varpi^\delta \\ &= S^\kappa \partial_\kappa u_\alpha - (S^\kappa u^\lambda \partial_\lambda u_\kappa) u_\alpha + \frac{1}{H} \epsilon_{\alpha\beta\gamma\delta} S^\beta u^\gamma \varpi^\delta, \end{aligned} \tag{4.20}$$

$$\varpi^\kappa \partial_\kappa u_\alpha = \varpi^\kappa \partial_\alpha u_\kappa - (\varpi^\kappa \partial_\kappa h) u_\alpha + q \varpi^\kappa S_\kappa u_\alpha, \tag{4.21}$$

$$\epsilon^{\alpha\beta\gamma\delta} \partial_\gamma u_\delta = \frac{1}{H} \varpi^\alpha u^\beta - \frac{1}{H} u^\alpha \varpi^\beta - \epsilon^{\alpha\beta\gamma\delta} (\partial_\gamma h) u_\delta + q \epsilon^{\alpha\beta\gamma\delta} S_\gamma u_\delta, \tag{4.22}$$

$$\epsilon^{\alpha\beta\gamma\delta} u_\beta \partial_\gamma u_\delta = -\frac{1}{H} \varpi^\alpha, \tag{4.23}$$

$$\begin{aligned} \partial_\gamma \varpi_\delta - \partial_\delta \varpi_\gamma &= \epsilon_{\gamma\delta\kappa\lambda} u^\kappa \mathrm{vort}^\lambda(\varpi) - (u^\kappa \partial_\kappa \varpi_\delta) u_\gamma + u^\kappa (\partial_\delta \varpi_\kappa) u_\gamma \\ &\quad + (u^\kappa \partial_\kappa \varpi_\gamma) u_\delta - u^\kappa (\partial_\gamma \varpi_\kappa) u_\delta, \end{aligned} \tag{4.24}$$

$$\begin{aligned} \epsilon^{\alpha\beta\gamma\delta} \partial_\gamma \varpi_\delta &= \mathrm{vort}^\alpha(\varpi) u^\beta - u^\alpha \mathrm{vort}^\beta(\varpi) + \epsilon^{\alpha\beta\gamma\delta} (u^\kappa \partial_\kappa \varpi_\gamma) u_\delta \\ &\quad - \epsilon^{\alpha\beta\gamma\delta} u^\kappa (\partial_\gamma \varpi_\kappa) u_\delta. \end{aligned} \tag{4.25}$$

Proof. (4.1) follows from definition (2.8) and the symmetry property $\partial_\alpha \partial_\beta s = \partial_\beta \partial_\alpha s$. Equation (4.2) is a simple consequence of definition (2.2). Equation (4.3) follows from differentiating (2.20) with ∂_α. Equation (4.4) follows from differentiating (2.21) with ∂_α. Equation (4.5) follows from differentiating (4.2) with ∂_α. Equation (4.6) follows directly from definition (2.12). Equation (4.7) then follows from (4.6).

To prove (4.8), we first use definition (2.4) to express the first product on RHS (4.8) as follows:

$$\epsilon_{\alpha\beta\gamma\delta} u^\gamma \mathrm{vort}^\delta(V) = -\epsilon_{\alpha\beta\gamma\delta} \epsilon^{\delta\theta\kappa\lambda} u^\gamma u_\theta \partial_\kappa V_\lambda. \tag{4.26}$$

Next, we observe the following identity for the first two factors on RHS (4.26):

$$\begin{aligned} -\epsilon_{\alpha\beta\gamma\delta} \epsilon^{\delta\theta\kappa\lambda} &= \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\theta\kappa\lambda\delta} \\ &= \delta_\alpha^\theta \delta_\gamma^\kappa \delta_\beta^\lambda - \delta_\alpha^\theta \delta_\gamma^\lambda \delta_\beta^\kappa + \delta_\alpha^\lambda \delta_\gamma^\theta \delta_\beta^\kappa - \delta_\alpha^\lambda \delta_\gamma^\kappa \delta_\beta^\theta + \delta_\alpha^\kappa \delta_\gamma^\lambda \delta_\beta^\theta - \delta_\alpha^\kappa \delta_\gamma^\theta \delta_\beta^\lambda. \end{aligned} \tag{4.27}$$

Using (4.27) to substitute on RHS (4.26), we deduce, in view of (2.20), the following identity:

$$\begin{aligned} -\epsilon_{\alpha\beta\gamma\delta} \epsilon^{\delta\theta\kappa\lambda} u_\theta \partial_\kappa V_\lambda u^\gamma &= u_\alpha u^\kappa \partial_\kappa V_\beta - u_\alpha u^\kappa \partial_\beta V_\kappa - \partial_\beta V_\alpha \\ &\quad - u_\beta u^\kappa \partial_\kappa V_\alpha + u_\beta u^\kappa \partial_\alpha V_\kappa + \partial_\alpha V_\beta. \end{aligned} \tag{4.28}$$

Combining (4.26) and (4.28) and rearranging the terms, we arrive at the desired identity (4.8). Equation (4.10) then follows from (4.8) and the relation $u^\kappa \partial_\alpha V_\kappa = -V_\kappa \partial_\alpha u^\kappa$, which follows from differentiating the assumed identity $u^\kappa V_\kappa = 0$ with ∂_α.

To prove (4.9), we first use (4.8) to deduce

$$\Pi^{\alpha\beta}\Pi^{\gamma\delta}(\partial_\alpha V_\gamma - \partial_\gamma V_\alpha)(\partial_\beta V_\delta - \partial_\delta V_\beta)$$
$$= \Pi^{\alpha\beta}\Pi^{\gamma\delta}\epsilon_{\alpha\gamma\kappa\lambda}\epsilon_{\beta\delta\mu\nu}u^\kappa \mathrm{vort}^\lambda(V)u^\mu \mathrm{vort}^\nu(V). \tag{4.29}$$

Next, we note the following identity, which follows easily from definition (2.12):

$$\Pi^{\alpha\beta}\Pi^{\gamma\delta}\epsilon_{\alpha\gamma\kappa\lambda}\epsilon_{\beta\delta\mu\nu}u^\kappa \mathrm{vort}^\lambda(V)u^\mu \mathrm{vort}^\nu(V)$$
$$= (\eta^{-1})^{\alpha\beta}(\eta^{-1})^{\gamma\delta}\epsilon_{\alpha\gamma\kappa\lambda}\epsilon_{\beta\delta\mu\nu}u^\kappa \mathrm{vort}^\lambda(V)u^\mu \mathrm{vort}^\nu(V)$$
$$= \epsilon^{\alpha\beta\kappa\lambda}\epsilon_{\alpha\beta\mu\nu}u_\kappa \mathrm{vort}_\lambda(V)u^\mu \mathrm{vort}^\nu(V). \tag{4.30}$$

From (4.30), the identity $\epsilon^{\alpha\beta\kappa\lambda}\epsilon_{\alpha\beta\mu\nu} = 2\delta_\mu^\lambda\delta_\nu^\kappa - 2\delta_\mu^\kappa\delta_\nu^\lambda$, (2.20), and the simple identity $u_\alpha \mathrm{vort}^\alpha(V) = 0$ [which follows easily from definition (2.4)], we find that RHS (4.30) $= 2\mathrm{vort}_\alpha(V)\mathrm{vort}^\alpha(V)$. Again using that $u_\alpha \mathrm{vort}^\alpha(V) = 0$, we conclude, in view of definition (2.12), the desired identity (4.9). Equation (4.11) is a standard geometric identity, as is (4.13).

To prove (4.12), we first note the Lie differentiation identity $\mathcal{L}_u(u_\flat)_\alpha = u^\kappa\partial_\kappa u_\alpha + u_\kappa\partial_\alpha u^\kappa$, which follows from (2.1). Equation (4.12) follows from this identity, (2.22), and (4.3).

To prove (4.14), we first use (2.23) and the Lie derivative formula (2.1) to deduce that $\mathcal{L}_u(Hu_\flat)_\alpha = u^\kappa\partial_\kappa(Hu_\alpha) + Hu_\kappa\partial_\alpha u^\kappa = -\partial_\alpha H + \theta\partial_\alpha s + Hu_\kappa\partial_\alpha u^\kappa$. From (4.3), we see that the last product on the RHS of this identity vanishes. Hence, taking the exterior derivative of the identity, we obtain $[d\mathcal{L}_u(Hu_\flat)]_{\alpha\beta} = \theta_{;h}(\partial_\alpha h)\partial_\beta s - \theta_{;h}(\partial_\alpha s)\partial_\beta h$. The desired identity (4.14) now follows from this identity and (4.13).

To prove (4.15), we first use definition (2.5) to compute that

$$\epsilon_{\alpha\beta\gamma\delta}u^\gamma\varpi^\delta = -\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\delta\kappa\theta\lambda}u^\gamma u_\kappa\partial_\theta(Hu_\lambda).$$

Using the identity $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\delta\kappa\theta\lambda} = -\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\lambda\kappa\theta\delta} = \delta_\alpha^\lambda\delta_\beta^\kappa\delta_\gamma^\theta - \delta_\alpha^\lambda\delta_\beta^\theta\delta_\gamma^\kappa + \delta_\alpha^\kappa\delta_\beta^\theta\delta_\gamma^\lambda - \delta_\alpha^\kappa\delta_\beta^\lambda\delta_\gamma^\theta + \delta_\alpha^\theta\delta_\beta^\lambda\delta_\gamma^\kappa - \delta_\alpha^\theta\delta_\beta^\kappa\delta_\gamma^\lambda$, we deduce, in view of (2.20), that

$$-\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\delta\kappa\theta\lambda}u^\gamma u_\kappa\partial_\theta(Hu_\lambda) = \partial_\alpha(Hu_\beta) - \partial_\beta(Hu_\alpha)$$
$$- u_\beta u^\kappa\partial_\kappa(Hu_\alpha) - u_\alpha u^\kappa\partial_\beta(Hu_\kappa)$$
$$+ u_\alpha u^\kappa\partial_\kappa(Hu_\beta) + u_\beta u^\kappa\partial_\alpha(Hu_\kappa). \tag{4.31}$$

Using (2.20), (2.23), and (4.3), we compute that the last four products on RHS (4.31) sum to $\theta(u_\alpha S_\beta - u_\beta S_\alpha)$, which yields the desired identity (4.15).

To prove (4.16), we first contract $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}$ against (4.15) to obtain the identity

$$\epsilon^{\alpha\beta\gamma\delta}\partial_\gamma(Hu_\delta) = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda}u^\kappa\varpi^\lambda + \theta\epsilon^{\alpha\beta\gamma\delta}S_\gamma u_\delta. \tag{4.32}$$

(4.16) now follows from using the identity $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda} = \delta_\kappa^\beta\delta_\lambda^\alpha - \delta_\kappa^\alpha\delta_\lambda^\beta$ to substitute for the factor $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda}$ on RHS (4.32). Equation (4.17) follows from (4.15) and simple computations.

To prove (4.18), we contract S^α against Eq. (2.22) and use Eq. (2.21). Equation (4.19) then follows from (4.4) and (4.18).

To prove the first equality in (4.20), we contract S^β against (4.17) and use Eq. (2.21). To obtain the second equality in (4.20), we use the first equality and the identity (4.18). Equation (4.21) follows from contracting (4.17) against ϖ^α and using (4.2).

To prove (4.22), we first use (4.17) to deduce that

$$\epsilon^{\alpha\beta\gamma\delta}\partial_\gamma u_\delta = \frac{1}{2}\frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda}u^\kappa\varpi^\lambda - \epsilon^{\alpha\beta\gamma\delta}(\partial_\gamma h)u_\delta + q\epsilon^{\alpha\beta\gamma\delta}S_\gamma u_\delta. \quad (4.33)$$

(4.22) now follows from using the identity $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda} = \delta_\kappa^\beta\delta_\lambda^\alpha - \delta_\kappa^\alpha\delta_\lambda^\beta$ to substitute for the product $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda}$ on RHS (4.33).

To prove (4.23), we contract (4.22) against u_β and use (2.20) and (4.2).

To prove (4.24), we first use definition (2.4) to express the first product on RHS (4.24) as follows:

$$\epsilon_{\gamma\delta\kappa\lambda}u^\kappa \mathrm{vort}^\lambda(\varpi) = -\epsilon_{\gamma\delta\kappa\lambda}\epsilon^{\lambda\theta\alpha\beta}u^\kappa u_\theta\partial_\alpha\varpi_\beta. \quad (4.34)$$

Next, we use the identity $-\epsilon_{\gamma\delta\kappa\lambda}\epsilon^{\lambda\theta\alpha\beta} = \epsilon_{\gamma\delta\kappa\lambda}\epsilon^{\theta\alpha\beta\lambda} = \delta_\gamma^\theta\delta_\delta^\beta\delta_\kappa^\alpha - \delta_\gamma^\theta\delta_\delta^\alpha\delta_\kappa^\beta + \delta_\gamma^\alpha\delta_\delta^\theta\delta_\kappa^\beta - \delta_\gamma^\alpha\delta_\delta^\beta\delta_\kappa^\theta + \delta_\gamma^\beta\delta_\delta^\alpha\delta_\kappa^\theta - \delta_\gamma^\beta\delta_\delta^\theta\delta_\kappa^\alpha$ to substitute on RHS (4.34), thereby obtaining, in view of (2.20), the following identity:

$$\begin{aligned}\epsilon_{\gamma\delta\kappa\lambda}u^\kappa \mathrm{vort}^\lambda(\varpi) &= u_\gamma u^\kappa\partial_\kappa\varpi_\delta - u_\gamma u^\kappa\partial_\delta\varpi_\kappa + u_\delta u^\kappa\partial_\gamma\varpi_\kappa + \partial_\gamma\varpi_\delta \\ &\quad - \partial_\delta\varpi_\gamma - u_\delta u^\kappa\partial_\kappa\varpi_\gamma. \quad (4.35)\end{aligned}$$

Finally, we note that it is straightforward to see that (4.35) is equivalent to the desired identity (4.24).

To prove (4.25), we first contract (4.24) against $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}$ to deduce

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}\partial_\gamma\varpi_\delta &= \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda}u^\kappa \mathrm{vort}^\lambda(\varpi) + \epsilon^{\alpha\beta\gamma\delta}(u^\kappa\partial_\kappa\varpi_\gamma)u_\delta \\ &\quad - \epsilon^{\alpha\beta\gamma\delta}u^\kappa(\partial_\gamma\varpi_\kappa)u_\delta. \quad (4.36)\end{aligned}$$

Using the identity $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta\kappa\lambda} = \frac{1}{2}\epsilon^{\gamma\delta\alpha\beta}\epsilon_{\gamma\delta\kappa\lambda} = \delta_\kappa^\beta\delta_\lambda^\alpha - \delta_\kappa^\alpha\delta_\lambda^\beta$ to substitute in the first product on RHS (4.36), we arrive at the desired identity (4.25). □

5. Wave Equations

In this section, with the help of the preliminary identities of Lemma 4.1, we derive the covariant wave equations (3.1), (3.3), and (5.18).

5.1. Covariant Wave Operator

We start by establishing a formula for the covariant wave operator of the acoustical metric acting on a scalar function.

Lemma 5.1 (Covariant wave operator of g). *Assume that (h, s, u^α) is a C^2 solution to (2.17)–(2.19) + (2.20). Then the covariant wave operator of the acoustical metric $g = g(h, s, u)$ (see Definition 2.6) acts on scalar functions ϕ as follows, where RHS (5.1) is expressed relative to the rectangular coordinates:*

$$\begin{aligned}\square_g \phi &= (c^2-1)u^\kappa \partial_\kappa (u^\lambda \partial_\lambda \phi) + c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa \partial_\lambda \phi) \\ &\quad + (c^2-1)(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda \phi) + 2c^{-1}c_{;h}(u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda \phi) \\ &\quad - c^{-1}c_{;h}(g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda \phi) \\ &\quad - cc_{;s}(S^\kappa \partial_\kappa \phi). \end{aligned} \tag{5.1}$$

Proof. It is a standard fact that relative to arbitrary coordinates (and in particular relative to the rectangular coordinates), we have

$$\square_g \phi = \frac{1}{\sqrt{|\mathrm{det} g|}}\partial_\kappa \left(\sqrt{|\mathrm{det} g|}(g^{-1})^{\kappa\lambda}\partial_\lambda \phi \right).$$

Using this formula and (2.14a)–(2.14b), we compute that

$$\begin{aligned}\square_g \phi &= c^3 \partial_\kappa \left\{ -(c^{-3}-c^{-1})u^\kappa (u^\lambda \partial_\lambda \phi) + c^{-1}((\eta^{-1})^{\kappa\lambda}\partial_\lambda \phi)\right\} \\ &= -(1-c^2)u^\kappa \partial_\kappa (u^\lambda \partial_\lambda \phi) - (1-c^2)(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda \phi) \\ &\quad + (3c^{-1}-c)(u^\kappa \partial_\kappa c)(u^\lambda \partial_\lambda \phi) - c(\eta^{-1})^{\kappa\lambda}(\partial_\kappa c)(\partial_\lambda \phi) \\ &\quad + c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa \partial_\lambda \phi). \end{aligned} \tag{5.2}$$

The desired identity (5.1) now follows from (5.2), (2.13b), the evolution equation (2.19), and straightforward computations. □

5.2. Covariant Wave Equation for the Logarithmic Enthalpy

We now derive the covariant wave equation (3.1).

Proposition 5.2 (Covariant wave equation for the logarithmic enthalpy). *Assume that* (h, s, u^α) *is a* C^2 *solution to* (2.17)–(2.19) + (2.20)*. Then the logarithmic enthalpy* h *verifies the following covariant wave equation:*

$$\begin{aligned}\square_g h &= nc^2 q\mathcal{D} - c^{-1}c_{;h}(g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h) \\ &\quad + c^2\left\{ (\partial_\kappa u^\kappa)(\partial_\lambda u^\lambda) - (\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa)\right\} \\ &\quad + (1-c^2)q(S^\kappa \partial_\kappa h) - cc_{;s}(S^\kappa \partial_\kappa h) + c^2 q_{;h}(S^\kappa \partial_\kappa h) + c^2 q_{;s}S_\kappa S^\kappa. \end{aligned} \tag{5.3}$$

Proof. From (5.1) with $\phi := h$, we deduce

$$\begin{aligned}\square_g h &= (c^2-1)u^\kappa \partial_\kappa (u^\lambda \partial_\lambda h) + c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa \partial_\lambda h) \\ &\quad + (c^2-1)(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) + 2c^{-1}c_{;h}(u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) \\ &\quad - c^{-1}c_{;h}(g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h) \\ &\quad - cc_{;s}(S^\kappa \partial_\kappa h). \end{aligned} \tag{5.4}$$

Next, we differentiate Eq. (2.22) with ∂_β, contract against $(\eta^{-1})^{\alpha\beta}$, and multiply by c^2 to obtain the identity

$$\begin{aligned}c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa \partial_\lambda h) &= -c^2(u^\kappa \partial_\kappa \partial_\lambda u^\lambda) - c^2(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) \\ &\quad - c^2 u^\kappa \partial_\kappa (u^\lambda \partial_\lambda h) - c^2(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) \\ &\quad + c^2 q(\partial_\kappa S^\kappa) + c^2 q_{;h}(S^\kappa \partial_\kappa h) + c^2 q_{;s}S^\kappa S_\kappa. \end{aligned} \tag{5.5}$$

Next, we use (2.17) and the evolution equation (2.19) to rewrite the first product on RHS (5.5) as follows:

$$-c^2(u^\kappa \partial_\kappa \partial_\lambda u^\lambda) = c^2 u^\kappa \partial_\kappa (c^{-2} u^\lambda \partial_\lambda h)$$
$$= u^\kappa \partial_\kappa (u^\lambda \partial_\lambda h) - 2c^{-1} c_{;h} (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h). \tag{5.6}$$

Using (5.6) to substitute for the first product on RHS (5.5) and then using the resulting identity to substitute for the product $c^2(\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h$ on RHS (5.4), we deduce

$$\begin{aligned}\square_g h = & -c^2(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) \\ & - c^{-1} c_{;h} (g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h) \\ & - c c_{;s}(S^\kappa \partial_\kappa h) + c^2 q(\partial_\kappa S^\kappa) + c^2 q_{;h}(S^\kappa \partial_\kappa h) + c^2 q_{;s} S^\kappa S_\kappa. \end{aligned} \tag{5.7}$$

Finally, we use Eq. (2.17) to substitute for the factor $u^\lambda \partial_\lambda h$ in the second product on RHS (5.7), and we use definition (2.16b) to express the product $c^2 q(\partial_\kappa S^\kappa)$ on RHS (5.7) as $nc^2 q\mathcal{D} + (1-c^2)q(S^\kappa \partial_\kappa h)$, which in total yields the desired Eq. (5.3). $\square$

5.3. Covariant Wave Equation for the Rectangular Components of the Four-Velocity

We now derive the covariant wave equation (3.3).

Proposition 5.3 (Covariant wave equation for the rectangular four-velocity components). *Assume that* (h, s, u^α) *is a* C^2 *solution to* (2.17)–(2.19) + (2.20). *Then the rectangular velocity components* u^α *verify the following covariant wave equations:*

$$\begin{aligned}\square_g u^\alpha = & -\frac{c^2}{H}\mathcal{C}^\alpha \\ & - \frac{c^2}{H}\epsilon^{\alpha\beta\gamma\delta}(\partial_\beta u_\gamma)\varpi_\delta + \frac{(1-c^2)}{H}\epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma h)\varpi_\delta \\ & + \frac{(1-c^2)q}{H}\epsilon^{\alpha\beta\gamma\delta} S_\beta u_\gamma \varpi_\delta \\ & - (g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u^\alpha) - c^{-1}c_{;h}(g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u^\alpha) \\ & + (\eta^{-1})^{\alpha\lambda}\left\{(\partial_\kappa u^\kappa)(\partial_\lambda h) - (\partial_\lambda u^\kappa)(\partial_\kappa h)\right\} \\ & + c^2 u^\alpha \left\{(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\lambda u^\lambda)(\partial_\kappa u^\kappa)\right\} \\ & - c c_{;s}(S^\kappa \partial_\kappa u^\alpha) + q(S^\kappa \partial_\kappa u^\alpha) \\ & + (c^2-1)q u^\alpha (S^\kappa u^\lambda \partial_\lambda u_\kappa) + c^2 q S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\kappa) \\ & + 2c^{-1}c_{;s}S^\alpha(u^\kappa \partial_\kappa h) + 2c^{-1}c_{;h} q S^\alpha (u^\kappa \partial_\kappa h) \\ & - q_{;h} S^\alpha (u^\kappa \partial_\kappa h) - q S^\alpha (\partial_\kappa u^\kappa) \\ & + (\theta - \theta_{;h})\frac{c^2}{H} S^\alpha (\partial_\kappa u^\kappa) + (\theta - \theta_{;h})\frac{c^2}{H} u^\alpha (S^\kappa \partial_\kappa h) \\ & + (\theta_{;h} - \theta)\frac{c^2}{H} S^\kappa ((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\kappa). \end{aligned} \tag{5.8}$$

Proof. From (5.1) with $\phi := u_\alpha$, we deduce

$$\begin{aligned}\square_g u_\alpha &= (c^2-1)u^\kappa \partial_\kappa (u^\lambda \partial_\lambda u_\alpha) + c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda u_\alpha) \\ &\quad + (c^2-1)(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda u_\alpha) + 2c^{-1}c_{;h}(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda u_\alpha) \\ &\quad - c^{-1}c_{;h}(g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u_\alpha) - cc_{;s}(S^\kappa \partial_\kappa u_\alpha). \end{aligned} \tag{5.9}$$

Next, we use Eqs. (2.19), (2.22), and the second line of (6.1) [where below, we derive (6.1) using an independent argument] to rewrite the first product on RHS (5.9) as follows:

$$\begin{aligned}(c^2-1)u^\kappa\partial_\kappa(u^\lambda\partial_\lambda u_\alpha) &= (1-c^2)(u^\kappa\partial_\kappa\partial_\alpha h) + (1-c^2)\left\{u^\kappa\partial_\kappa(u^\lambda\partial_\lambda h)\right\}u_\alpha \\ &\quad + (1-c^2)(u^\kappa\partial_\kappa u_\alpha)(u^\lambda\partial_\lambda h) + (c^2-1)u^\kappa\partial_\kappa(qS_\alpha) \\ &= (1-c^2)(u^\kappa\partial_\kappa\partial_\alpha h) + (1-c^2)\left\{u^\kappa\partial_\kappa(u^\lambda\partial_\lambda h)\right\}u_\alpha \\ &\quad + (1-c^2)(u^\kappa\partial_\kappa u_\alpha)(u^\lambda\partial_\lambda h) + (c^2-1)q_{;h}(u^\kappa\partial_\kappa h)S_\alpha \\ &\quad + (1-c^2)q(S^\kappa\partial_\kappa u_\alpha) + \frac{1}{H}(1-c^2)q\epsilon_{\alpha\beta\gamma\delta}S^\beta u^\gamma\varpi^\delta \\ &\quad + (c^2-1)qS^\kappa(u^\lambda\partial_\lambda u_\kappa)u_\alpha. \end{aligned} \tag{5.10}$$

Next, we use definition (2.16b), the identity (4.17), and the evolution equations (2.17), (2.19), and (2.24) to rewrite the second product on RHS (5.9) as follows:

$$\begin{aligned} c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda u_\alpha) &= c^2(\partial_\alpha\partial_\kappa u^\kappa) \\ &\quad + c^2(\eta^{-1})^{\kappa\lambda}\partial_\kappa\Big\{\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}u^\gamma\varpi^\delta - (\partial_\lambda h)u_\alpha + (\partial_\alpha h)u_\lambda + \\ &\qquad qS_\lambda u_\alpha - qS_\alpha u_\lambda\Big\} \\ &= (c^2-1)(u^\kappa\partial_\kappa\partial_\alpha h) - (\partial_\alpha u^\kappa)(\partial_\kappa h) \\ &\quad + 2c^{-1}c_{;h}(\partial_\alpha h)(u^\kappa\partial_\kappa h) + 2c^{-1}c_{;s}S_\alpha(u^\kappa\partial_\kappa h) \\ &\quad - c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa h)u^\gamma\varpi^\delta \\ &\quad + c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa u^\gamma)\varpi^\delta \\ &\quad + c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}u^\gamma((\eta^{-1})^{\kappa\lambda}\partial_\kappa\varpi^\delta) \\ &\quad - c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h)u_\alpha - c^2(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u_\alpha) \\ &\quad + c^2(\partial_\alpha h)(\partial_\kappa u^\kappa) \\ &\quad + c^2 q_{;h}(S^\kappa\partial_\kappa h)u_\alpha + c^2 q_{;s}S_\kappa S^\kappa u_\alpha \\ &\quad + c^2 q(\partial_\kappa S^\kappa)u_\alpha + c^2 q(S^\kappa\partial_\kappa u_\alpha) \\ &\quad - c^2 q_{;h}(u^\kappa\partial_\kappa h)S_\alpha - c^2 q(u^\kappa\partial_\kappa S_\alpha) - c^2 q(\partial_\kappa u^\kappa)S_\alpha \\ &= nc^2 q\mathcal{D}u_\alpha \\ &\quad + (c^2-1)(u^\kappa\partial_\kappa\partial_\alpha h) - (\partial_\alpha u^\kappa)(\partial_\kappa h) \end{aligned}$$

$$
\begin{aligned}
&+ 2c^{-1}c_{;h}(\partial_\alpha h)(u^\kappa\partial_\kappa h)\\
&- c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa h)u^\gamma\varpi^\delta\\
&+ c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa u^\gamma)\varpi^\delta\\
&+ c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}u^\gamma((\eta^{-1})^{\kappa\lambda}\partial_\kappa\varpi^\delta)\\
&- c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h)u_\alpha - c^2(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u_\alpha)\\
&+ c^2(\partial_\alpha h)(\partial_\kappa u^\kappa)\\
&+ c^2 q_{;h}(S^\kappa\partial_\kappa h)u_\alpha + c^2 q_{;s}S_\kappa S^\kappa u_\alpha + c^2 q(S^\kappa\partial_\kappa u_\alpha)\\
&- c^2 q_{;h}(u^\kappa\partial_\kappa h)S_\alpha + c^2 q(\partial_\alpha u^\kappa)S_\kappa - c^2 q(\partial_\kappa u^\kappa)S_\alpha\\
&+ (1-c^2)q(S^\kappa\partial_\kappa h)u_\alpha + 2c^{-1}c_{;s}(u^\kappa\partial_\kappa h)S_\alpha. \qquad (5.11)
\end{aligned}
$$

Next, we use the identity (5.1) with $\phi := h$ to substitute for the term $\square_g h$ on LHS (5.3), which yields the identity

$$
\begin{aligned}
c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h) &= c^2\left\{(\partial_\kappa u^\kappa)(\partial_\lambda u^\lambda) - (\partial_\lambda u^\kappa)(\partial_\kappa u^\lambda)\right\}\\
&\quad + (1-c^2)u^\kappa\partial_\kappa(u^\lambda\partial_\lambda h) + (1-c^2)(\partial_\kappa u^\kappa)(u^\lambda\partial_\lambda h)\\
&\quad - 2c^{-1}c_{;h}(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&\quad + nc^2 q\mathcal{D}\\
&\quad + (1-c^2)q(S^\kappa\partial_\kappa h) + c^2 q_{;h}(S^\kappa\partial_\kappa h) + c^2 q_{;s}S_\kappa S^\kappa. \qquad (5.12)
\end{aligned}
$$

From (5.12), it follows that the product $-c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h)u_\alpha$ on RHS (5.11) can be expressed as

$$
\begin{aligned}
-c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h)u_\alpha &= c^2\left\{(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\lambda u^\lambda)(\partial_\kappa u^\kappa)\right\}u_\alpha\\
&\quad + (c^2-1)\left\{u^\kappa\partial_\kappa(u^\lambda\partial_\lambda h)\right\}u_\alpha\\
&\quad + (c^2-1)(\partial_\kappa u^\kappa)(u^\lambda\partial_\lambda h)u_\alpha\\
&\quad + 2c^{-1}c_{;h}(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)u_\alpha\\
&\quad - nc^2 q\mathcal{D}u_\alpha\\
&\quad + (c^2-1)q(S^\kappa\partial_\kappa h)u_\alpha - c^2 q_{;h}(S^\kappa\partial_\kappa h)u_\alpha\\
&\quad - c^2 q_{;s}S_\kappa S^\kappa u_\alpha. \qquad (5.13)
\end{aligned}
$$

Using (5.13) to substitute for the term $-c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h)u_\alpha$ on RHS (5.11), we obtain the identity

$$
\begin{aligned}
c^2((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda u_\alpha) &= (c^2-1)(u^\kappa\partial_\kappa\partial_\alpha h) - (\partial_\alpha u^\kappa)(\partial_\kappa h)\\
&\quad + 2c^{-1}c_{;h}(\partial_\alpha h)(u^\kappa\partial_\kappa h)\\
&\quad - c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa h)u^\gamma\varpi^\delta\\
&\quad + c^2\frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa u^\gamma)\varpi^\delta
\end{aligned}
$$

$$
\begin{aligned}
&+ c^2 \frac{1}{H} \epsilon_{\lambda\alpha\gamma\delta} u^\gamma ((\eta^{-1})^{\kappa\lambda} \partial_\kappa \varpi^\delta) \\
&+ c^2 \left\{ (\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\lambda u^\lambda)(\partial_\kappa u^\kappa) \right\} u_\alpha \\
&+ (c^2 - 1) \left\{ u^\kappa \partial_\kappa (u^\lambda \partial_\lambda h) \right\} u_\alpha \\
&+ (c^2 - 1)(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) u_\alpha \\
&+ 2c^{-1} c_{;h} (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) u_\alpha \\
&- c^2 (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda u_\alpha) + c^2 (\partial_\alpha h)(\partial_\kappa u^\kappa) \\
&+ c^2 q (S^\kappa \partial_\kappa u_\alpha) \\
&- c^2 q_{;h} (u^\kappa \partial_\kappa h) S_\alpha + c^2 q (\partial_\alpha u^\kappa) S_\kappa - c^2 q (\partial_\kappa u^\kappa) S_\alpha \\
&+ 2c^{-1} c_{;s} (u^\kappa \partial_\kappa h) S_\alpha. \qquad (5.14)
\end{aligned}
$$

Using (5.10) and (5.14) to substitute for the first and second products on RHS (5.9), and reorganizing the terms, we deduce [where we have added and subtracted $(\partial_\kappa u^\kappa)(\partial_\alpha h)$ on the third and fourth lines of RHS (5.15)]

$$
\begin{aligned}
\square_g u_\alpha = {} & c^2 \frac{1}{H} \epsilon_{\lambda\alpha\gamma\delta} u^\gamma ((\eta^{-1})^{\kappa\lambda} \partial_\kappa \varpi^\delta) \\
&+ (1 - c^2)(u^\kappa \partial_\kappa u_\alpha)(u^\lambda \partial_\lambda h) - c^2 (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda u_\alpha) \\
&+ \{ (\partial_\kappa u^\kappa)(\partial_\alpha h) - (\partial_\alpha u^\kappa)(\partial_\kappa h) \} \\
&+ (c^2 - 1)(\partial_\kappa u^\kappa)(\partial_\alpha h) + (c^2 - 1)(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) u_\alpha \\
&+ (c^2 - 1)(\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda u_\alpha) \\
&+ 2c^{-1} c_{;h} (\partial_\alpha h)(u^\kappa \partial_\kappa h) + 2c^{-1} c_{;h} (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) u_\alpha \\
&+ 2c^{-1} c_{;h} (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda u_\alpha) \\
&- c^2 \frac{1}{H} \epsilon_{\lambda\alpha\gamma\delta} ((\eta^{-1})^{\kappa\lambda} \partial_\kappa h) u^\gamma \varpi^\delta + c^2 \frac{1}{H} \epsilon_{\lambda\alpha\gamma\delta} ((\eta^{-1})^{\kappa\lambda} \partial_\kappa u^\gamma) \varpi^\delta \\
&+ c^2 \left\{ (\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\lambda u^\lambda)(\partial_\kappa u^\kappa) \right\} u_\alpha \\
&- c^{-1} c_{;h} (g^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda u_\alpha) - c c_{;s} (S^\kappa \partial_\kappa u_\alpha) \\
&+ 2c^{-1} c_{;s} (u^\kappa \partial_\kappa h) S_\alpha - q_{;h} (u^\kappa \partial_\kappa h) S_\alpha + c^2 q (\partial_\alpha u^\kappa) S_\kappa \\
&- c^2 q (\partial_\kappa u^\kappa) S_\alpha + q (S^\kappa \partial_\kappa u_\alpha) \\
&+ \frac{1}{H} (1 - c^2) q \epsilon_{\alpha\beta\gamma\delta} S^\beta u^\gamma \varpi^\delta + (c^2 - 1) q (S^\kappa u^\lambda \partial_\lambda u_\kappa) u_\alpha. \qquad (5.15)
\end{aligned}
$$

Next, using (2.13b), we observe the following identity for the two terms on the second line of RHS (5.15):

$$
\begin{aligned}
&(1 - c^2)(u^\kappa \partial_\kappa u_\alpha)(u^\lambda \partial_\lambda h) - c^2 (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda u_\alpha) \\
&\quad = -(g^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda u_\alpha). \qquad (5.16)
\end{aligned}
$$

Moreover, using Eq. (2.22), we see that the terms on the fourth through seventh lines of RHS (5.15) sum to $(c^2 - 1) q (\partial_\kappa u^\kappa) S_\alpha + 2c^{-1} c_{;h} q (u^\kappa \partial_\kappa h) S_\alpha$. In addition, appealing to definition (2.4) with $V_\alpha := \varpi_\alpha$, we obtain the following identity for the first product on RHS (5.15): $c^2 \frac{1}{H} \epsilon_{\lambda\alpha\gamma\delta} u^\gamma ((\eta^{-1})^{\kappa\lambda} \partial_\kappa \varpi^\delta) =$

$-c^2 \frac{1}{H}\mathrm{vort}_\alpha(\varpi)$. From these facts, (5.15), and (5.16), we obtain the following equation:

$$\begin{aligned}
\square_g u_\alpha &= -c^2 \frac{1}{H}\mathrm{vort}_\alpha(\varpi) \\
&\quad - c^2 \frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa h)u^\gamma \varpi^\delta + c^2 \frac{1}{H}\epsilon_{\lambda\alpha\gamma\delta}((\eta^{-1})^{\kappa\lambda}\partial_\kappa u^\gamma)\varpi^\delta \\
&\quad + \{(\partial_\kappa u^\kappa)(\partial_\alpha h) - (\partial_\alpha u^\kappa)(\partial_\kappa h)\} - (g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u_\alpha) \\
&\quad + c^2 \left\{(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\lambda u^\lambda)(\partial_\kappa u^\kappa)\right\} u_\alpha \\
&\quad - c^{-1} c_{;h}(g^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda u_\alpha) - c c_{;s}(S^\kappa \partial_\kappa u_\alpha) \\
&\quad + 2c^{-1} c_{;s}(u^\kappa \partial_\kappa h) S_\alpha - q_{;h}(u^\kappa \partial_\kappa h) S_\alpha + c^2 q(\partial_\alpha u^\kappa) S_\kappa \\
&\quad + q(S^\kappa \partial_\kappa u_\alpha) + \frac{1}{H}(1-c^2) q \epsilon_{\alpha\beta\gamma\delta} S^\beta u^\gamma \varpi^\delta \\
&\quad + (c^2-1) q (S^\kappa u^\lambda \partial_\lambda u_\kappa) u_\alpha \\
&\quad - q(\partial_\kappa u^\kappa) S_\alpha + 2c^{-1} c_{;h} q (u^\kappa \partial_\kappa h) S_\alpha. \qquad (5.17)
\end{aligned}$$

Using definition (2.16a) to express the product $-c^2 \frac{1}{H}\mathrm{vort}_\alpha(\varpi)$ on RHS (5.17) as $-c^2 \frac{1}{H}\mathcal{C}_\alpha + \cdots$, reorganizing the terms on the RHS of the resulting identity, and raising the α index with η^{-1}, we arrive at the desired identity (5.8). □

5.4. Covariant Wave Equation for the Entropy

In this subsection, we derive the covariant wave equation (3.5).

Proposition 5.4 (Covariant wave equation for s). *Assume that (h, s, u^α) is a C^2 solution to (2.17)–(2.19) + (2.20). Then the entropy s verifies the following covariant wave equation:*

$$\square_g s = c^2 n\mathcal{D} + S^\kappa \partial_\kappa h - c^2 S^\kappa \partial_\kappa h - c c_{;h} S^\kappa \partial_\kappa h - c c_{;s} S_\kappa S^\kappa. \qquad (5.18)$$

Proof. Applying (5.1) with $\phi := s$, using (2.13b) to algebraically substitute for the factor of $(g^{-1})^{\kappa\lambda}$ on RHS (5.1), and using the evolution equation (2.19) [which implies that many factors on RHS (5.1) vanish], we deduce, in view of definition (2.8), that

$$\square_g s = c^2 \partial_\kappa S^\kappa - c c_{;h} S^\kappa \partial_\kappa h - c c_{;s} S_\kappa S^\kappa. \qquad (5.19)$$

We then solve for $\partial_\kappa S^\kappa$ in terms of the remaining terms in definition (2.16b) and then use the resulting identity to algebraically substitute for the factor $\partial_\kappa S^\kappa$ in the first product on RHS (5.19), which in total yields the desired Eq. (5.18). □

6. Transport Equations for the Entropy Gradient and the Modified Divergence of the Entropy

In this section, with the help of the preliminary identities of Lemma 4.1, we derive Eqs. (3.7) and (3.9a). We start by deriving (3.7) (more precisely, its η-dual).

Proposition 6.1 (Transport equation for the entropy gradient). *Assume that* (h, s, u^α) *is a* C^2 *solution to* (2.17)–(2.19) + (2.20). *Then the rectangular components the* S_α *of the entropy gradient vectorfield (see Definition* 2.5*) verify the following transport equations:*

$$\begin{aligned} u^\kappa \partial_\kappa S_\alpha &= -S^\kappa \partial_\kappa u_\alpha - \frac{1}{H}\epsilon_{\alpha\beta\gamma\delta} S^\beta u^\gamma \varpi^\delta - (S^\kappa \partial_\kappa h) u_\alpha + q S_\kappa S^\kappa u_\alpha \\ &= -S^\kappa \partial_\kappa u_\alpha - \frac{1}{H}\epsilon_{\alpha\beta\gamma\delta} S^\beta u^\gamma \varpi^\delta + S^\kappa (u^\lambda \partial_\lambda u_\kappa) u_\alpha. \end{aligned} \tag{6.1}$$

Proof. From Eq. (2.24), the identity (4.17), (2.20), and (2.21), we deduce

$$\begin{aligned} u^\kappa \partial_\kappa S_\alpha &= -S^\kappa \partial_\kappa u_\alpha - \frac{1}{H}\epsilon_{\alpha\beta\gamma\delta} S^\beta u^\gamma \varpi^\delta + (\partial_\alpha h) S^\kappa u_\kappa - (S^\kappa \partial_\kappa h) u_\alpha \\ &\quad - q\left\{S_\alpha S^\kappa u_\kappa - S^\kappa S_\kappa u_\alpha\right\} \\ &= -S^\kappa \partial_\kappa u_\alpha - \frac{1}{H}\epsilon_{\alpha\beta\gamma\delta} S^\beta u^\gamma \varpi^\delta - (S^\kappa \partial_\kappa h) u_\alpha + q S^\kappa S_\kappa u_\alpha, \end{aligned} \tag{6.2}$$

which yields the first line of (6.1). To obtain the second line of (6.1) from the first, we use the identity (4.18). □

We now derive Eq. (3.9a).

Proposition 6.2 (Transport equation for the modified divergence of the entropy). *Assume that* (h, s, u^α) *is a* C^3 *solution to* (2.17)–(2.19) + (2.20). *Then the modified divergence of the entropy gradient* $\mathcal{D}$*, which is defined in* (2.16b)*, verifies the following transport equation:*

$$\begin{aligned} u^\kappa \partial_\kappa \mathcal{D} &= \frac{2}{n}\left\{(\partial_\kappa S^\kappa)(\partial_\lambda u^\lambda) - (\partial_\lambda S^\kappa)(\partial_\kappa u^\lambda)\right\} \\ &\quad + \frac{1}{n} c^{-2} u^\kappa \left\{(\partial_\kappa h)(\partial_\lambda S^\lambda) - (\partial_\lambda h)(\partial_\kappa S^\lambda)\right\} \\ &\quad + \frac{1}{n} c^{-2} S^\kappa \left\{(\partial_\kappa u^\lambda)(\partial_\lambda h) - (\partial_\lambda u^\lambda)(\partial_\kappa h)\right\} \\ &\quad + \frac{S_\kappa \mathcal{C}^\kappa}{nH} \\ &\quad + \frac{(1-c^{-2})}{nH}\epsilon^{\alpha\beta\gamma\delta} S_\alpha u_\beta (\partial_\gamma h)\varpi_\delta + \frac{1}{nH}\epsilon^{\alpha\beta\gamma\delta} S_\alpha (\partial_\beta u_\gamma)\varpi_\delta \\ &\quad + \frac{(\theta - \theta_{;h})}{nH} S^\kappa (S^\lambda \partial_\lambda u_\kappa) - \frac{2q}{n} S^\kappa (S^\lambda \partial_\lambda u_\kappa) \\ &\quad + \frac{(\theta_{;h} - \theta)}{nH} S_\kappa S^\kappa (\partial_\lambda u^\lambda) + \frac{2c^{-1}c_{;s}}{n} S_\kappa S^\kappa (\partial_\lambda u^\lambda) \\ &\quad - \frac{c^2 q_{;h}}{n} S_\kappa S^\kappa (\partial_\lambda u^\lambda) + \frac{q}{n} S_\kappa S^\kappa (\partial_\lambda u^\lambda). \end{aligned} \tag{6.3}$$

Proof. We apply $(\eta^{-1})^{\alpha\lambda}\partial_\lambda$ to Eq. (6.1) [where we use the first equality in (6.1)] and use the evolution equation (2.19) and the identity (4.18) to deduce

$$\begin{aligned} u^\kappa \partial_\kappa \partial_\lambda S^\lambda &= -u^\kappa \partial_\kappa (S^\lambda \partial_\lambda h) - 2(\partial_\lambda S^\kappa)(\partial_\kappa u^\lambda) \\ &\quad - S^\kappa \partial_\kappa \partial_\lambda u^\lambda - (S^\kappa \partial_\kappa h)(\partial_\lambda u^\lambda) \end{aligned}$$

$$+\frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha h)S_\beta u_\gamma\varpi_\delta - \frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma(\partial_\alpha\varpi_\delta)$$
$$-\frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}S_\beta(\partial_\alpha u_\gamma)\varpi_\delta$$
$$+q_{;h}S_\kappa S^\kappa(u^\lambda\partial_\lambda h) + 2qS^\kappa(u^\lambda\partial_\lambda S_\kappa) + qS_\kappa S^\kappa(\partial_\lambda u^\lambda). \tag{6.4}$$

Next, we use the evolution equations (2.17) and (2.19) to rewrite the third product on RHS (6.4) as follows:

$$\begin{aligned}
-S^\kappa\partial_\kappa\partial_\lambda u^\lambda &= S^\kappa\partial_\kappa(c^{-2}u^\lambda\partial_\lambda h)\\
&= c^{-2}(S^\kappa u^\lambda\partial_\lambda\partial_\kappa h) - 2c^{-3}c_{;h}(S^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&\quad - 2c^{-3}c_{;s}S_\kappa S^\kappa(u^\lambda\partial_\lambda h) + c^{-2}(S^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h)\\
&= u^\kappa\partial_\kappa(c^{-2}S^\lambda\partial_\lambda h) + c^{-2}(S^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h) - c^{-2}(u^\kappa\partial_\kappa S^\lambda)(\partial_\lambda h)\\
&\quad - 2c^{-3}c_{;s}S_\kappa S^\kappa(u^\lambda\partial_\lambda h).
\end{aligned} \tag{6.5}$$

Next, with the help of the evolution equation (2.17), we decompose the second and third products on RHS (6.5) as follows:

$$\begin{aligned}
c^{-2}(S^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h) &= c^{-2}(S^\kappa\partial_\kappa h)(\partial_\lambda u^\lambda)\\
&\quad + c^{-2}\left\{(S^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h) - (\partial_\lambda u^\lambda)(S^\kappa\partial_\kappa h)\right\},
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
-c^{-2}(u^\kappa\partial_\kappa S^\lambda)(\partial_\lambda h) &= -c^{-2}(u^\kappa\partial_\kappa h)(\partial_\lambda S^\lambda)\\
&\quad + c^{-2}\left\{(u^\kappa\partial_\kappa h)(\partial_\lambda S^\lambda) - (u^\kappa\partial_\kappa S^\lambda)(\partial_\lambda h)\right\}\\
&= (\partial_\kappa u^\kappa)(\partial_\lambda S^\lambda)\\
&\quad + c^{-2}\left\{(u^\kappa\partial_\kappa h)(\partial_\lambda S^\lambda) - (u^\kappa\partial_\kappa S^\lambda)(\partial_\lambda h)\right\}.
\end{aligned} \tag{6.7}$$

Using (6.6)–(6.7) to substitute for the second and third products on RHS (6.5) and then using the resulting identity to substitute for the third product on RHS (6.4), we obtain the following equation:

$$\begin{aligned}
&u^\kappa\partial_\kappa\left\{\partial_\lambda S^\lambda + S^\lambda\partial_\lambda h - c^{-2}(S^\lambda\partial_\lambda h)\right\}\\
&\quad = (\partial_\kappa S^\kappa)(\partial_\lambda u^\lambda) - 2(\partial_\lambda S^\kappa)(\partial_\kappa u^\lambda)\\
&\qquad - (S^\kappa\partial_\kappa h)(\partial_\lambda u^\lambda) + c^{-2}(S^\kappa\partial_\kappa h)(\partial_\lambda u^\lambda)\\
&\qquad + c^{-2}\left\{(S^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h) - (\partial_\lambda u^\lambda)(S^\kappa\partial_\kappa h)\right\}\\
&\qquad + c^{-2}\left\{(u^\kappa\partial_\kappa h)(\partial_\lambda S^\lambda) - (u^\kappa\partial_\kappa S^\lambda)(\partial_\lambda h)\right\}\\
&\qquad + \frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha h)S_\beta u_\gamma\varpi_\delta - \frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma(\partial_\alpha\varpi_\delta) - \frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}S_\beta(\partial_\alpha u_\gamma)\varpi_\delta\\
&\qquad + q_{;h}S_\kappa S^\kappa(u^\lambda\partial_\lambda h) + 2qS^\kappa(u^\lambda\partial_\lambda S_\kappa) + qS_\kappa S^\kappa(\partial_\lambda u^\lambda)\\
&\qquad - 2c^{-3}c_{;s}S_\kappa S^\kappa(u^\lambda\partial_\lambda h).
\end{aligned} \tag{6.8}$$

We now multiply both sides of (6.8) by $1/n$, commute the factor of $1/n$ under the operator $u^\kappa\partial_\kappa$ on LHS (6.8), use Eq. (2.25) (which in particular implies that $u^\kappa\partial_\kappa(1/n) = (1/n)\partial_\kappa u^\kappa$), and use Eq. (2.17) to replace the two factors of

$u^\lambda\partial_\lambda h$ on the last and next-to-last lines of RHS (6.8) with $-c^2\partial_\lambda u^\lambda$, thereby obtaining the following equation:

$$\begin{aligned} u^\kappa\partial_\kappa &\left\{\frac{1}{n}(\partial_\lambda S^\lambda) + \frac{1}{n}(S^\lambda\partial_\lambda h) - \frac{1}{n}c^{-2}(S^\lambda\partial_\lambda h)\right\} \\ &= \frac{2}{n}\left\{(\partial_\kappa S^\kappa)(\partial_\lambda u^\lambda) - (\partial_\lambda S^\kappa)(\partial_\kappa u^\lambda)\right\} \\ &\quad + \frac{1}{n}c^{-2}\left\{(S^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h) - (\partial_\lambda u^\lambda)(S^\kappa\partial_\kappa h)\right\} \\ &\quad + \frac{1}{n}c^{-2}\left\{(u^\kappa\partial_\kappa h)(\partial_\lambda S^\lambda) - (\partial_\lambda h)(u^\kappa\partial_\kappa S^\lambda)\right\} \\ &\quad + \frac{1}{nH}\epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha h)S_\beta u_\gamma\varpi_\delta - \frac{1}{nH}\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma(\partial_\alpha\varpi_\delta) \\ &\quad - \frac{1}{nH}\epsilon^{\alpha\beta\gamma\delta}S_\beta(\partial_\alpha u_\gamma)\varpi_\delta \\ &\quad + \frac{2q}{n}S^\kappa(u^\lambda\partial_\lambda S_\kappa) + \frac{2c^{-1}c_{;s}}{n}S_\kappa S^\kappa(\partial_\lambda u^\lambda) - \frac{c^2 q_{;h}}{n}S_\kappa S^\kappa(\partial_\lambda u^\lambda) \\ &\quad + \frac{q}{n}S_\kappa S^\kappa(\partial_\lambda u^\lambda). \end{aligned} \tag{6.9}$$

Next, we use definitions (2.4) and (2.16a) and the identity (2.21) to obtain the following identity for the second product on the fourth line of RHS (6.9):

$$\begin{aligned} -\frac{1}{nH}\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma(\partial_\alpha\varpi_\delta) &= -\frac{1}{nH}\epsilon^{\alpha\beta\gamma\delta}S_\alpha u_\beta(\partial_\gamma\varpi_\delta) \\ &= \frac{\mathcal{C}^\kappa S_\kappa}{nH} - \frac{1}{nH}c^{-2}\epsilon^{\alpha\beta\gamma\delta}S_\alpha u_\beta(\partial_\gamma h)\varpi_\delta \\ &\quad + \frac{(\theta_{;h}-\theta)}{nH}S_\kappa S^\kappa(\partial_\lambda u^\lambda) + \frac{(\theta-\theta_{;h})}{nH}S^\kappa(S^\lambda\partial_\lambda u_\kappa). \end{aligned} \tag{6.10}$$

Using (6.10) to substitute for the second product on the fourth line of RHS (6.9), using (2.24) to express the first product on the next-to-last line of RHS (6.9) as $\frac{2q}{n}S^\kappa(u^\lambda\partial_\lambda S_\kappa) = -\frac{2q}{n}S^\kappa(S^\lambda\partial_\lambda u_\kappa)$, and noting that the terms in parentheses on LHS (6.9) are equal to $\mathcal{D}$ [see (2.16b)], we arrive at the desired evolution equation (6.3). □

7. Transport Equation for the Vorticity

In this section, with the help of the preliminary identities of Lemma 4.1, we derive Eq. (3.8). We also derive some preliminary identities that, in the next section, we will use when deriving Eq. (3.11b). We collect all of these results in the following proposition.

Proposition 7.1 (Transport equation for the vorticity). *Assume that* (h, s, u^α) *is a* C^3 *solution to* (2.17)–(2.19) + (2.20). *Then the rectangular components* ϖ^α *of the vorticity vectorfield defined in* (2.5) *verify the following transport equations:*

$$u^\kappa \partial_\kappa \varpi^\alpha = \varpi^\kappa \partial_\kappa u^\alpha - (\partial_\kappa u^\kappa)\varpi^\alpha - (\varpi^\kappa \partial_\kappa h)u^\alpha + (\theta - \theta_{;h})\epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma h) S_\delta + q\varpi^\kappa S_\kappa u^\alpha. \tag{7.1}$$

Moreover, the following identity holds:

$$(\mathcal{L}_u \varpi_\flat)_\alpha = \varpi^\kappa \partial_\kappa u_\alpha + \varpi^\kappa (\partial_\alpha u_\kappa) - (\partial_\kappa u^\kappa)\varpi_\alpha + (u^\kappa \partial_\kappa u_\lambda) u_\alpha \varpi^\lambda + (\theta - \theta_{;h})\epsilon_\alpha{}^{\beta\gamma\delta} u_\beta (\partial_\gamma h) S_\delta. \tag{7.2}$$

In addition, the following identity holds:

$$\begin{aligned}
(d\mathcal{L}_u \varpi_\flat)_{\alpha\beta}
&= (\partial_\alpha \varpi^\kappa)(\partial_\kappa u_\beta) - (\partial_\beta \varpi^\kappa)(\partial_\kappa u_\alpha) \\
&\quad + \varpi^\kappa \partial_\kappa \partial_\alpha u_\beta - \varpi^\kappa \partial_\kappa \partial_\beta u_\alpha \\
&\quad + (\partial_\alpha \varpi^\kappa)(\partial_\beta u_\kappa) - (\partial_\beta \varpi^\kappa)(\partial_\alpha u_\kappa) \\
&\quad - (\partial_\alpha \partial_\kappa u^\kappa)\varpi_\beta + (\partial_\beta \partial_\kappa u^\kappa)\varpi_\alpha \\
&\quad - (\partial_\kappa u^\kappa)(\partial_\alpha \varpi_\beta) + (\partial_\kappa u^\kappa)(\partial_\beta \varpi_\alpha) \\
&\quad + (\partial_\alpha u_\beta)\varpi^\lambda (u^\kappa \partial_\kappa u_\lambda) - (\partial_\beta u_\alpha)\varpi^\lambda (u^\kappa \partial_\kappa u_\lambda) \\
&\quad + u_\beta (\partial_\alpha \varpi^\lambda)(u^\kappa \partial_\kappa u_\lambda) - u_\alpha (\partial_\beta \varpi^\lambda)(u^\kappa \partial_\kappa u_\lambda) \\
&\quad + u_\beta \varpi^\lambda (\partial_\alpha u^\kappa)(\partial_\kappa u_\lambda) - u_\alpha \varpi^\lambda (\partial_\beta u^\kappa)(\partial_\kappa u_\lambda) \\
&\quad + u_\beta \varpi^\lambda (u^\kappa \partial_\kappa \partial_\alpha u_\lambda) - u_\alpha \varpi^\lambda (u^\kappa \partial_\kappa \partial_\beta u_\lambda) \\
&\quad + (\theta_h - \theta_{;h;h})\epsilon_{\beta\kappa}{}^{\gamma\delta} u^\kappa (\partial_\alpha h)(\partial_\gamma h) S_\delta \\
&\quad + (\theta_{;h;h} - \theta_h)\epsilon_{\alpha\kappa}{}^{\gamma\delta} u^\kappa (\partial_\beta h)(\partial_\gamma h) S_\delta \\
&\quad + (\theta_{;s} - \theta_{;h;s})\epsilon_{\beta\kappa}{}^{\gamma\delta} u^\kappa S_\alpha (\partial_\gamma h) S_\delta + (\theta_{;h;s} - \theta_{;s})\epsilon_{\alpha\kappa}{}^{\gamma\delta} u^\kappa S_\beta (\partial_\gamma h) S_\delta \\
&\quad + (\theta - \theta_{;h})\epsilon_{\beta\kappa}{}^{\gamma\delta} (\partial_\alpha u^\kappa)(\partial_\gamma h) S_\delta + (\theta_{;h} - \theta)\epsilon_{\alpha\kappa}{}^{\gamma\delta} (\partial_\beta u^\kappa)(\partial_\gamma h) S_\delta \\
&\quad + (\theta - \theta_{;h})\epsilon_{\beta\kappa}{}^{\gamma\delta} u^\kappa (\partial_\alpha \partial_\gamma h) S_\delta + (\theta_{;h} - \theta)\epsilon_{\alpha\kappa}{}^{\gamma\delta} u^\kappa (\partial_\beta \partial_\gamma h) S_\delta \\
&\quad + (\theta - \theta_{;h})\epsilon_{\beta\kappa}{}^{\gamma\delta} u^\kappa (\partial_\gamma h)(\partial_\alpha S_\delta) + (\theta_{;h} - \theta)\epsilon_{\alpha\kappa}{}^{\gamma\delta} u^\kappa (\partial_\gamma h)(\partial_\beta S_\delta).
\end{aligned} \tag{7.3}$$

Finally, the rectangular components $\mathrm{vort}^\alpha(\varpi)$ *of the vorticity of the vorticity, which is defined by* (2.4) *and* (2.5)*, verify the following transport equations:*

$$\begin{aligned}
u^\kappa \partial_\kappa \mathrm{vort}^\alpha(\varpi) &= \mathrm{vort}^\kappa(\varpi)\partial_\kappa u^\alpha - (\partial_\kappa u^\kappa)\mathrm{vort}^\alpha(\varpi) \\
&\quad + u^\alpha (u^\kappa \partial_\kappa u_\beta)\mathrm{vort}^\beta(\varpi) \\
&\quad + \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma \partial_\kappa u^\kappa)\varpi_\delta - \epsilon^{\alpha\beta\gamma\delta} u_\beta (\varpi^\kappa \partial_\kappa \partial_\gamma u_\delta) \\
&\quad + \epsilon^{\alpha\beta\gamma\delta} (u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\lambda \varpi_\delta) u_\gamma + \epsilon^{\alpha\beta\gamma\delta} (u^\kappa \partial_\kappa u_\beta)\varpi^\lambda (\partial_\delta u_\lambda) u_\gamma \\
&\quad - \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma \varpi^\kappa)(\partial_\delta u_\kappa) - \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma \varpi^\kappa)(\partial_\kappa u_\delta) \\
&\quad + \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\kappa u^\kappa)(\partial_\gamma \varpi_\delta) \\
&\quad - \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma u_\delta)\varpi^\lambda (u^\kappa \partial_\kappa u_\lambda) \\
&\quad + (\theta_h - \theta_{;h;h}) S^\alpha (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h)
\end{aligned}$$

$$
\begin{aligned}
&+ (\theta_h - \theta_{;h;h}) S^\alpha (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) \\
&+ (\theta_{;h;h} - \theta_h) u^\alpha (S^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) \\
&+ (\theta_{;h;h} - \theta_h)((\eta^{-1})^{\alpha\kappa} \partial_\kappa h)(S^\lambda \partial_\lambda h) \\
&+ (\theta_{;s} - \theta_{;h;s}) S^\alpha (S^\kappa \partial_\kappa h) + (\theta_{;h;s} - \theta_{;s}) u^\alpha S_\kappa S^\kappa (u^\lambda \partial_\lambda h) \\
&+ (\theta_{;h;s} - \theta_{;s}) S_\kappa S^\kappa ((\eta^{-1})^{\alpha\lambda} \partial_\lambda h) \\
&+ (\theta - \theta_{;h}) S^\alpha (\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) + (\theta_{;h} - \theta)(S^\kappa \partial_\kappa u^\alpha)(u^\lambda \partial_\lambda h) \\
&+ (\theta - \theta_{;h}) S^\alpha ((\eta^{-1})^{\kappa\lambda} \partial_\kappa \partial_\lambda h) + (\theta - \theta_{;h}) S^\alpha (u^\kappa u^\lambda \partial_\kappa \partial_\lambda h) \\
&+ (\theta_{;h} - \theta) u^\alpha (S^\kappa u^\lambda \partial_\kappa \partial_\lambda h) + (\theta_{;h} - \theta)(\eta^{-1})^{\alpha\lambda} (S^\kappa \partial_\kappa \partial_\lambda h) \\
&+ (\theta - \theta_{;h})(\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda S^\alpha) + (\theta - \theta_{;h})(u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda S^\alpha) \\
&+ (\theta_{;h} - \theta) u^\alpha (u^\kappa \partial_\kappa h)(\partial_\lambda S^\lambda) + (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa} \partial_\kappa h)(\partial_\lambda S^\lambda) \\
&+ (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa} \partial_\kappa h) u_\beta (u^\lambda \partial_\lambda S^\beta) \\
&+ (\theta - \theta_{;h}) u^\alpha (\eta^{-1})^{\kappa\lambda} \partial_\kappa h) u_\beta (\partial_\lambda S^\beta). \qquad (7.4)
\end{aligned}
$$

Remark 7.2 Note that RHS (7.4) features some terms that explicitly depend on two derivatives of u, falsely suggesting that there is derivative loss, that is, that $\mathrm{vort}^\alpha(\varpi)$ cannot be more regular than $\partial^2 u$. For this reason, Eq. (7.4) is not suitable for obtaining top-order energy estimates for $\mathrm{vort}^\alpha(\varpi)$. To overcome this difficulty, we will derive a transport-div-curl system for ϖ that does not lose derivatives; see Proposition 8.2.

Proof of Proposition 7.1. We first prove (7.1). From definition (2.5) and the Lie differentiation formula (2.1), we deduce that

$$
u^\kappa \partial_\kappa \varpi^\alpha - \varpi^\kappa \partial_\kappa u^\alpha = \mathcal{L}_u \varpi^\alpha = -\frac{1}{2} \mathcal{L}_u \left\{ \epsilon^{\alpha\beta\gamma\delta} u_\beta (d(H u_\flat))_{\gamma\delta} \right\}. \qquad (7.5)
$$

Using (7.5), the Leibniz rule for Lie derivatives, definition (2.5), (4.11), the first identity in (4.12), (4.14), and (4.16), we compute that

$$
\begin{aligned}
u^\kappa \partial_\kappa \varpi^\alpha &= \varpi^\kappa \partial_\kappa u^\alpha - (\partial_\kappa u^\kappa) \varpi^\alpha + (u^\kappa \partial_\kappa u_\lambda) u^\alpha \varpi^\lambda - (u^\kappa \partial_\kappa u_\lambda) u^\lambda \varpi^\alpha \\
&\quad - \theta \epsilon^{\alpha\beta\gamma\delta} (u^\kappa \partial_\kappa u_\beta) S_\gamma u_\delta - \theta_{;h} \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma h) S_\delta. \qquad (7.6)
\end{aligned}
$$

Using (4.3), we see that the fourth product on RHS (7.6) vanishes. Next, we use (2.22) and (4.2) to obtain the following identity for the third product on RHS (7.6): $(u^\kappa \partial_\kappa u_\lambda) u^\alpha \varpi^\lambda = -(\varpi^\kappa \partial_\kappa h) u^\alpha + q \varpi^\kappa S_\kappa u^\alpha$. Next, we use (2.22) to obtain the following identity for the fifth product on RHS (7.6): $-\theta \epsilon^{\alpha\beta\gamma\delta} (u^\kappa \partial_\kappa u_\beta) S_\gamma u_\delta = \theta \epsilon^{\alpha\beta\gamma\delta} (\partial_\beta h) S_\gamma u_\delta = \theta \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma h) S_\delta$. Substituting these two identities for the third and fifth products on RHS (7.6), we arrive at the desired identity (7.1).

Equation (7.2) follows from the Lie derivative identity

$$
(\mathcal{L}_u \varpi_\flat)_\alpha = u^\kappa \partial_\kappa \varpi_\alpha + \varpi^\kappa \partial_\alpha u_\kappa
$$

[see (2.1)], from using (4.3) to observe the vanishing of the fourth product on RHS (7.6), and from using the identity for the fifth product on RHS (7.6) proved in the previous paragraph. Equation (7.3) then follows from taking the exterior derivative of Eq. (7.2) and carrying out straightforward computations.

To derive (7.4), we first use definition (2.4) to deduce

$$\mathcal{L}_u \mathrm{vort}^\alpha(\varpi) = -\frac{1}{2}\mathcal{L}_u(\epsilon^{\alpha\beta\gamma\delta} u_\beta (d\varpi_\flat)_{\gamma\delta}). \tag{7.7}$$

Next, we use (7.7), the Leibniz rule for Lie derivatives, (4.11), the first equality in (4.12), and the standard commutation identity $\mathcal{L}_u d\varpi_\flat = d\mathcal{L}_u \varpi_\flat$ to deduce

$$\begin{aligned}\mathcal{L}_u \mathrm{vort}^\alpha(\varpi) = &-(\partial_\kappa u^\kappa)\mathrm{vort}^\alpha(\varpi) - \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(d\varpi_\flat)_{\gamma\delta} \\ &- \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} u_\beta (d\mathcal{L}_u \varpi_\flat)_{\gamma\delta}.\end{aligned} \tag{7.8}$$

Next, using (4.25), we express the second product on RHS (7.8) as follows:

$$\begin{aligned}-\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(d\varpi_\flat)_{\gamma\delta} = &-\mathrm{vort}^\alpha(\varpi)(u^\kappa \partial_\kappa u_\beta)u^\beta \\ &+ u^\alpha(u^\kappa \partial_\kappa u_\beta)\mathrm{vort}^\beta(\varpi) \\ &- \epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\lambda \varpi_\gamma)u_\delta \\ &+ \epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\gamma \varpi_\lambda)u_\delta.\end{aligned} \tag{7.9}$$

Next, using (4.3), we observe that the first product on RHS (7.9) vanishes. From this fact, the Lie derivative identity $\mathcal{L}_u \mathrm{vort}^\alpha(\varpi) = u^\kappa \partial_\kappa \mathrm{vort}^\alpha(\varpi) - \mathrm{vort}^\kappa(\varpi)\partial_\kappa u^\alpha$ [see (2.1)], (7.8), and (7.9), we deduce

$$\begin{aligned}u^\kappa \partial_\kappa \mathrm{vort}^\alpha(\varpi) = &\ \mathrm{vort}^\kappa(\varpi)\partial_\kappa u^\alpha - (\partial_\kappa u^\kappa)\mathrm{vort}^\alpha(\varpi) \\ &+ u^\alpha(u^\kappa \partial_\kappa u_\beta)\mathrm{vort}^\beta(\varpi) \\ &+ \epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\gamma \varpi_\lambda)u_\delta \\ &- \epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\lambda \varpi_\gamma)u_\delta \\ &- \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} u_\beta (d\mathcal{L}_u \varpi_\flat)_{\gamma\delta}.\end{aligned} \tag{7.10}$$

Next, we use (4.5) and the antisymmetry of $\epsilon^{\cdots}$ to express the product on the third line of RHS (7.10) as

$$\epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\gamma \varpi_\lambda)u_\delta = \epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)\varpi^\lambda(\partial_\delta u_\lambda)u_\gamma,$$

use the antisymmetry of $\epsilon^{\cdots}$ to express the product on the fourth line of RHS (7.10)

$$-\epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\lambda \varpi_\gamma)u_\delta = \epsilon^{\alpha\beta\gamma\delta}(u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\lambda \varpi_\delta)u_\gamma,$$

use (7.3) to substitute for the factor $(d\mathcal{L}_u \varpi_\flat)_{\gamma\delta}$ in the last product on RHS (7.10), and carry out straightforward computations, thereby deducing that

$$\begin{aligned}u^\kappa \partial_\kappa \mathrm{vort}^\alpha(\varpi) = &\ \mathrm{vort}^\kappa(\varpi)\partial_\kappa u^\alpha - (\partial_\kappa u^\kappa)\mathrm{vort}^\alpha(\varpi) \\ &+ u^\alpha(u^\kappa \partial_\kappa u_\beta)\mathrm{vort}^\beta(\varpi) \\ &+ \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma \partial_\kappa u^\kappa)\varpi_\delta - \epsilon^{\alpha\beta\gamma\delta} u_\beta (\varpi^\kappa \partial_\kappa \partial_\gamma u_\delta)\end{aligned}$$

$$
\begin{aligned}
&+ \epsilon^{\alpha\beta\gamma\delta}(u^\kappa\partial_\kappa u_\beta)(u^\lambda\partial_\lambda\varpi_\delta)u_\gamma \\
&+ \epsilon^{\alpha\beta\gamma\delta}(u^\kappa\partial_\kappa u_\beta)\varpi^\lambda(\partial_\delta u_\lambda)u_\gamma \\
&- \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa) - \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\kappa u_\delta) \\
&+ \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\kappa u^\kappa)(\partial_\gamma\varpi_\delta) \\
&- \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma u_\delta)\varpi^\lambda(u^\kappa\partial_\kappa u_\lambda) \\
&+ (\theta_{;h;h} - \theta_h)\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\kappa}{}^{\mu\nu}u_\beta u^\kappa(\partial_\gamma h)(\partial_\mu h)S_\nu \\
&+ (\theta_{;h;s} - \theta_{;s})\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\kappa}{}^{\mu\nu}u_\beta u^\kappa S_\gamma(\partial_\mu h)S_\nu \\
&+ (\theta_{;h} - \theta)\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\kappa}{}^{\mu\nu}u_\beta(\partial_\gamma u^\kappa)(\partial_\mu h)S_\nu \\
&+ (\theta_{;h} - \theta)\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\kappa}{}^{\mu\nu}u_\beta u^\kappa(\partial_\gamma\partial_\mu h)S_\nu \\
&+ (\theta_{;h} - \theta)\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\kappa}{}^{\mu\nu}u_\beta u^\kappa(\partial_\mu h)(\partial_\gamma S_\nu).
\end{aligned}
\tag{7.11}
$$

Finally, we use the identity

$$
\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\kappa}{}^{\mu\nu} = {} & (\eta^{-1})^{\nu\alpha}\delta_\kappa^\beta(\eta^{-1})^{\mu\gamma} - (\eta^{-1})^{\nu\alpha}\delta_\kappa^\gamma(\eta^{-1})^{\mu\beta} \\
& + (\eta^{-1})^{\nu\gamma}\delta_\kappa^\alpha(\eta^{-1})^{\mu\beta} - (\eta^{-1})^{\nu\gamma}\delta_\kappa^\beta(\eta^{-1})^{\mu\alpha} \\
& + (\eta^{-1})^{\nu\beta}\delta_\kappa^\gamma(\eta^{-1})^{\mu\alpha} - (\eta^{-1})^{\nu\beta}\delta_\kappa^\alpha(\eta^{-1})^{\mu\gamma}
\end{aligned}
$$

to substitute for the five products $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\kappa}{}^{\mu\nu}$ on RHS (7.11). Also using (2.20), (2.21), and (4.3), we arrive at the desired identity (7.4). □

8. The Transport-div-curl System for the Vorticity

Our main goal in this section is to derive Eqs. (3.11a) and (3.11b). We accomplish this in Proposition 8.2. Before proving the proposition, we will first establish some preliminary identities.

8.1. Preliminary Identities

In the next lemma, we derive a large collection of identities that we will use in deriving the transport equation verified by the vectorfield $\mathcal{C}^\alpha$ defined in (2.16a).

Lemma 8.1 (Identification of the null structure of some terms tied to the transport-div-curl system for the vorticity). *Assume that (h, s, u^α) is a C^2 solution to (2.17)–(2.19) + (2.20). Then the following identities hold for some of the terms on the third through seventh lines of RHS (7.4):*

$$
\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\partial_\kappa u^\kappa)\varpi_\delta = {} & -u^\kappa\partial_\kappa\left\{c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi_\delta\right\} \\
& - 2(\partial_\kappa u^\kappa)c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi_\delta \\
& + c^{-2}\epsilon^{\alpha\beta\gamma\delta}(u^\kappa\partial_\kappa u_\beta)(\partial_\gamma h)\varpi_\delta \\
& + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa) \\
& + c^{-2}(\theta - \theta_{;h})(S^\kappa\partial_\kappa h)((\eta^{-1})^{\alpha\lambda}\partial_\lambda h) \\
& + c^{-2}(\theta - \theta_{;h})u^\alpha(S^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h) \\
& + c^{-2}(\theta_{;h} - \theta)S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)
\end{aligned}
$$

$$
\begin{aligned}
&+ c^{-2}(\theta_{;h} - \theta) S^{\alpha} (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h) \\
&+ c^{-2} \epsilon^{\alpha\beta\gamma\delta} u_\beta \left\{ (\partial_\kappa u^\kappa)(\partial_\gamma h) - (\partial_\gamma u^\kappa)(\partial_\kappa h) \right\} \varpi_\delta \\
&+ 2c^{-3} c_{;s} (u^\kappa \partial_\kappa h) \epsilon^{\alpha\beta\gamma\delta} u_\beta S_\gamma \varpi_\delta, \qquad (8.1a)
\end{aligned}
$$

$$
\begin{aligned}
-\epsilon^{\alpha\beta\gamma\delta} u_\beta (\varpi^\kappa \partial_\kappa \partial_\gamma u_\delta) &= \frac{1}{H}(\varpi^\kappa \partial_\kappa \varpi^\alpha) - \frac{1}{H} \varpi^\alpha (\varpi^\kappa \partial_\kappa h) \\
&- \frac{1}{H} u^\alpha \varpi^\lambda (\varpi^\kappa \partial_\kappa u_\lambda) + \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma h) \varpi^\kappa (\partial_\delta u_\kappa) \\
&- q \epsilon^{\alpha\beta\gamma\delta} u_\beta S_\gamma \varpi^\kappa (\partial_\delta u_\kappa), \qquad (8.1b)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta} (u^\kappa \partial_\kappa u_\beta)(u^\lambda \partial_\lambda \varpi_\delta) u_\gamma &= -\epsilon^{\alpha\beta\gamma\delta} (\partial_\beta h) u_\gamma \varpi^\kappa (\partial_\delta u_\kappa) \\
&+ (\partial_\kappa u^\kappa) \epsilon^{\alpha\beta\gamma\delta} (\partial_\beta h) u_\gamma \varpi_\delta \\
&+ (\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa} \partial_\kappa h)(S^\lambda \partial_\lambda h) \\
&+ (\theta - \theta_{;h}) u^\alpha (u^\kappa \partial_\kappa h)(S^\lambda \partial_\lambda h) \\
&+ (\theta_{;h} - \theta) S^\alpha (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) \\
&+ (\theta_{;h} - \theta) S^\alpha (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h) \\
&+ q \epsilon^{\alpha\beta\gamma\delta} S_\beta u_\gamma \varpi^\kappa (\partial_\delta u_\kappa) \\
&- q (\partial_\kappa u^\kappa) \epsilon^{\alpha\beta\gamma\delta} S_\beta u_\gamma \varpi_\delta \\
&+ q (\theta_{;h} - \theta)((\eta^{-1})^{\kappa\alpha} \partial_\kappa h) S^\lambda S_\lambda \\
&+ q (\theta_{;h} - \theta) u^\alpha (u^\kappa \partial_\kappa h) S^\lambda S_\lambda \\
&+ q (\theta - \theta_{;h}) S^\alpha (S^\kappa \partial_\kappa h), \qquad (8.1c)
\end{aligned}
$$

$$
\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta} (u^\kappa \partial_\kappa u_\beta) u_\gamma \varpi^\lambda (\partial_\delta u_\lambda) &= -\epsilon^{\alpha\beta\gamma\delta} (\partial_\beta h) u_\gamma \varpi^\lambda (\partial_\delta u_\lambda) \\
&+ q \epsilon^{\alpha\beta\gamma\delta} S_\beta u_\gamma \varpi^\lambda (\partial_\delta u_\lambda), \qquad (8.1d)
\end{aligned}
$$

$$
\begin{aligned}
-\epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma \varpi^\kappa)(\partial_\kappa u_\delta) &= -\epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma \varpi^\kappa)(\partial_\delta u_\kappa) \\
&- \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma h) \varpi^\kappa (\partial_\delta u_\kappa) \\
&- \frac{1}{H}(\varpi^\kappa \partial_\kappa \varpi^\alpha) + \frac{1}{H} \varpi^\alpha (\partial_\kappa \varpi^\kappa) \\
&- \frac{1}{H} \varpi^\alpha \varpi^\lambda (u^\kappa \partial_\kappa u_\lambda) + \frac{1}{H} u^\alpha \varpi^\lambda (\varpi^\kappa \partial_\kappa u_\lambda) \\
&- q \epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma u^\kappa) \varpi_\kappa S_\delta, \qquad (8.1e)
\end{aligned}
$$

$$
\epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\kappa u^\kappa)(\partial_\gamma \varpi_\delta) = -(\partial_\kappa u^\kappa) \mathrm{vort}^\alpha(\varpi), \qquad (8.1f)
$$

$$
-\epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma u_\delta) \varpi^\lambda (u^\kappa \partial_\kappa u_\lambda) = \frac{1}{H} \varpi^\alpha \varpi^\lambda (u^\kappa \partial_\kappa u_\lambda). \qquad (8.1g)
$$

Moreover, we have

$$
\begin{aligned}
&(\theta - \theta_{;h}) S^\alpha ((\eta^{-1})^{\kappa\lambda} \partial_\kappa \partial_\lambda h) + (\theta - \theta_{;h}) S^\alpha (u^\kappa u^\lambda \partial_\kappa \partial_\lambda h) \\
&\quad = u^\kappa \partial_\kappa \left\{ (\theta_{;h} - \theta) S^\alpha (\partial_\lambda u^\lambda) \right\} \\
&\qquad + (\theta_h - \theta_{;h;h}) S^\alpha (u^\kappa \partial_\kappa h)(\partial_\lambda u^\lambda) + (\theta - \theta_{;h})(u^\kappa \partial_\kappa S^\alpha)(\partial_\lambda u^\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\theta_{;h} - \theta) S^{\alpha} (\partial_{\kappa} u^{\lambda})(\partial_{\lambda} u^{\kappa}) + (\theta_{;h} - \theta) S^{\alpha} (u^{\kappa} \partial_{\kappa} u^{\lambda})(\partial_{\lambda} h) \\
&+ (\theta_{;h} - \theta) S^{\alpha} (\partial_{\kappa} u^{\kappa})(u^{\lambda} \partial_{\lambda} h) \\
&+ (\theta - \theta_{;h}) q S^{\alpha} (\partial_{\kappa} S^{\kappa}) + (\theta - \theta_{;h}) q_{;h} S^{\alpha} (S^{\kappa} \partial_{\kappa} h) + (\theta - \theta_{;h}) q_{;s} S^{\alpha} S_{\kappa} S^{\kappa},
\end{aligned} \tag{8.2a}
$$

$$
\begin{aligned}
(\theta_{;h} - \theta) u^{\alpha} (S^{\kappa} u^{\lambda} \partial_{\kappa} \partial_{\lambda} h) = {} & u^{\kappa} \partial_{\kappa} \left\{ (\theta_{;h} - \theta) u^{\alpha} (S^{\lambda} \partial_{\lambda} h) \right\} \\
&+ (\theta_{h} - \theta_{;h;h}) u^{\alpha} (u^{\kappa} \partial_{\kappa} h)(S^{\lambda} \partial_{\lambda} h) \\
&+ (\theta - \theta_{;h}) (u^{\kappa} \partial_{\kappa} u^{\alpha})(S^{\lambda} \partial_{\lambda} h) \\
&+ (\theta - \theta_{;h}) u^{\alpha} (u^{\kappa} \partial_{\kappa} S^{\lambda})(\partial_{\lambda} h),
\end{aligned} \tag{8.2b}
$$

$$
\begin{aligned}
(\theta_{;h} - \theta)(\eta^{-1})^{\alpha\lambda} (S^{\kappa} \partial_{\kappa} \partial_{\lambda} h) = {} & u^{\kappa} \partial_{\kappa} \left\{ (\theta - \theta_{;h})(\eta^{-1})^{\alpha\lambda} S^{\beta} (\partial_{\lambda} u_{\beta}) \right\} \\
&+ (\theta_{;h;h} - \theta_{h})(u^{\kappa} \partial_{\kappa} h)(\eta^{-1})^{\alpha\lambda} S^{\beta} (\partial_{\lambda} u_{\beta}) \\
&+ (\theta_{;h} - \theta)(u^{\kappa} \partial_{\kappa} S^{\beta})(\eta^{-1})^{\alpha\lambda} (\partial_{\lambda} u_{\beta}) \\
&+ (\theta - \theta_{;h}) S^{\beta} ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u_{\beta})(u^{\kappa} \partial_{\kappa} h) \\
&+ (\theta - \theta_{;h}) S^{\beta} ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u^{\kappa})(\partial_{\kappa} u_{\beta}) \\
&+ (\theta - \theta_{;h}) q ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} S^{\beta}) S_{\beta} \\
&+ (\theta_{;h} - \theta) q_{;h} ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} h) S_{\kappa} S^{\kappa} \\
&+ (\theta_{;h} - \theta) q_{;s} S^{\alpha} S_{\kappa} S^{\kappa} \\
&+ 2(\theta_{;h} - \theta) q ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} S^{\kappa}) S_{\kappa}.
\end{aligned} \tag{8.2c}
$$

Identities that reveal null-form structure and cancellations. *The following identities hold*[25]*:*

$$
\begin{aligned}
\mathscr{Q}_2 &:= (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} h) \partial_{\lambda} S^{\lambda} + (\theta - \theta_{;h})(\eta^{-1})^{\kappa\lambda} (\partial_{\kappa} h)(\partial_{\lambda} S^{\alpha}) \\
&= (\theta_{;h} - \theta)(\eta^{-1})^{\alpha\kappa} \left\{ (\partial_{\kappa} h)(\partial_{\lambda} S^{\lambda}) - (\partial_{\lambda} h)(\partial_{\kappa} S^{\lambda}) \right\},
\end{aligned} \tag{8.3a}
$$

$$
\begin{aligned}
\mathscr{Q}_4 &:= (\theta_{h} - \theta_{;h;h}) S^{\alpha} (\eta^{-1})^{\kappa\lambda} (\partial_{\kappa} h)(\partial_{\lambda} h) \\
&\quad + (\theta_{h} - \theta_{;h;h}) S^{\alpha} (u^{\kappa} \partial_{\kappa} h)(u^{\lambda} \partial_{\lambda} h) + (\theta_{h} - \theta_{;h;h}) S^{\alpha} (u^{\kappa} \partial_{\kappa} h)(\partial_{\lambda} u^{\lambda}) \\
&= c^{-2} (\theta_{h} - \theta_{;h;h}) S^{\alpha} (g^{-1})^{\kappa\lambda} (\partial_{\kappa} h)(\partial_{\lambda} h),
\end{aligned} \tag{8.3b}
$$

$$
\begin{aligned}
\mathscr{Q}_5 &:= (\theta_{;h;h} - \theta_{h})((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} h)(S^{\lambda} \partial_{\lambda} h) \\
&\quad + (\theta_{;h;h} - \theta_{h})(u^{\kappa} \partial_{\kappa} h)(\eta^{-1})^{\alpha\lambda} S^{\beta} (\partial_{\lambda} u_{\beta}) \\
&= (\theta_{;h;h} - \theta_{h}) S^{\beta} u^{\kappa} (\eta^{-1})^{\alpha\lambda} \left\{ (\partial_{\kappa} h)(\partial_{\lambda} u_{\beta}) - (\partial_{\lambda} h)(\partial_{\kappa} u_{\beta}) \right\} \\
&\quad + (\theta_{;h;h} - \theta_{h}) q ((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} h) S^{\lambda} S_{\lambda},
\end{aligned} \tag{8.3c}
$$

$$
\begin{aligned}
\mathscr{Q}_6 &:= c^{-2} (\theta_{;h} - \theta) S^{\alpha} (u^{\kappa} \partial_{\kappa} h)(u^{\lambda} \partial_{\lambda} h) \\
&\quad + c^{-2} (\theta_{;h} - \theta) S^{\alpha} (\eta^{-1})^{\kappa\lambda} (\partial_{\kappa} h)(\partial_{\lambda} h) + (\theta - \theta_{;h}) S^{\alpha} (\partial_{\kappa} u^{\kappa})(\partial_{\lambda} u^{\lambda}) \\
&= c^{-4} (\theta_{;h} - \theta) S^{\alpha} (g^{-1})^{\kappa\lambda} (\partial_{\kappa} h)(\partial_{\lambda} h),
\end{aligned} \tag{8.3d}
$$

[25] Our labeling of the terms $\mathscr{Q}_2$, $\mathscr{Q}_3$, etc. is tied to the order in which terms appear in our proof of (8.41).

$$\begin{aligned}\mathscr{Q}_7 &:= (\uptheta_{;h} - \uptheta)S^{\alpha}(u^{\kappa}\partial_{\kappa}h)(u^{\lambda}\partial_{\lambda}h) + (\uptheta_{;h} - \uptheta)S^{\alpha}(\upeta^{-1})^{\kappa\lambda}(\partial_{\kappa}h)(\partial_{\lambda}h)\\ &\quad + (\uptheta_{;h} - \uptheta)S^{\alpha}(u^{\kappa}\partial_{\kappa}u^{\lambda})(\partial_{\lambda}h)\\ &= (\uptheta_{;h} - \uptheta)qS^{\alpha}(S^{\kappa}\partial_{\kappa}h), \end{aligned} \tag{8.3e}$$

$$\begin{aligned}\mathscr{Q}_9 &:= (\uptheta_{;h} - \uptheta)(S^{\kappa}\partial_{\kappa}u^{\alpha})(\partial_{\lambda}u^{\lambda}) + (\uptheta_{;h} - \uptheta)(u^{\kappa}\partial_{\kappa}u^{\alpha})(S^{\lambda}\partial_{\lambda}h)\\ &\quad + (\uptheta - \uptheta_{;h})((\upeta^{-1})^{\kappa\lambda}\partial_{\kappa}u^{\alpha})S^{\beta}(\partial_{\lambda}u_{\beta})\\ &= (\uptheta_{;h} - \uptheta)S^{\kappa}\left\{(\partial_{\kappa}u^{\alpha})(\partial_{\lambda}u^{\lambda}) - (\partial_{\lambda}u^{\alpha})(\partial_{\kappa}u^{\lambda})\right\}\\ &\quad + \frac{1}{H}(\uptheta - \uptheta_{;h})\upepsilon^{\kappa\beta\gamma\delta}(\partial_{\kappa}u^{\alpha})S_{\beta}u_{\gamma}\varpi_{\delta} + q(\uptheta_{;h} - \uptheta)(u^{\kappa}\partial_{\kappa}u^{\alpha})S^{\lambda}S_{\lambda}, \end{aligned} \tag{8.3f}$$

$$\begin{aligned}\mathscr{Q}_{11} &:= (\uptheta - \uptheta_{;h})(u^{\kappa}\partial_{\kappa}S^{\alpha})(\partial_{\lambda}u^{\lambda}) + (\uptheta - \uptheta_{;h})S^{\beta}((\upeta^{-1})^{\alpha\lambda}\partial_{\lambda}u^{\kappa})(\partial_{\kappa}u_{\beta})\\ &= (\uptheta_{;h} - \uptheta)S^{\beta}(\upeta^{-1})^{\alpha\kappa}\left\{(\partial_{\kappa}u_{\beta})(\partial_{\lambda}u^{\lambda}) - (\partial_{\lambda}u_{\beta})(\partial_{\kappa}u^{\lambda})\right\}, \end{aligned} \tag{8.3g}$$

$$\begin{aligned}\mathscr{Q}_{12} &:= 2(\uptheta_{;h} - \uptheta)(\partial_{\kappa}u^{\kappa})(\upeta^{-1})^{\alpha\lambda}S^{\beta}(\partial_{\lambda}u_{\beta})\\ &\quad + c^{-2}(\uptheta - \uptheta_{;h})(S^{\kappa}\partial_{\kappa}h)((\upeta^{-1})^{\alpha\lambda}\partial_{\lambda}h)\\ &\quad + (\uptheta_{;h} - \uptheta)(u^{\kappa}\partial_{\kappa}S^{\beta})((\upeta^{-1})^{\alpha\lambda}\partial_{\lambda}u_{\beta})\\ &= (\uptheta_{;h} - \uptheta)(\upeta^{-1})^{\alpha\kappa}S^{\beta}\left\{(\partial_{\kappa}u_{\beta})(\partial_{\lambda}u^{\lambda}) - (\partial_{\lambda}u_{\beta})(\partial_{\kappa}u^{\lambda})\right\}\\ &\quad + c^{-2}(\uptheta_{;h} - \uptheta)(\upeta^{-1})^{\alpha\kappa}S^{\beta}u^{\lambda}\left\{(\partial_{\lambda}u_{\beta})(\partial_{\kappa}h) - (\partial_{\kappa}u_{\beta})(\partial_{\lambda}h)\right\}\\ &\quad + qc^{-2}(\uptheta - \uptheta_{;h})S_{\kappa}S^{\kappa}((\upeta^{-1})^{\alpha\lambda}\partial_{\lambda}h), \end{aligned} \tag{8.3h}$$

$$\begin{aligned}\mathscr{Q}_{13} &:= (\uptheta - \uptheta_{;h})u^{\alpha}(\partial_{\kappa}u^{\kappa})(S^{\lambda}\partial_{\lambda}h) + (\uptheta - \uptheta_{;h})u^{\alpha}(u^{\kappa}\partial_{\kappa}u^{\lambda})S^{\beta}(\partial_{\lambda}u_{\beta})\\ &= (\uptheta_{;h} - \uptheta)u^{\alpha}S^{\beta}u^{\lambda}\left\{(\partial_{\kappa}u^{\kappa})(\partial_{\lambda}u_{\beta}) - (\partial_{\lambda}u^{\kappa})(\partial_{\kappa}u_{\beta})\right\}\\ &\quad + q(\uptheta - \uptheta_{;h})u^{\alpha}S_{\kappa}S^{\kappa}(\partial_{\lambda}u^{\lambda}), \end{aligned} \tag{8.3i}$$

$$\begin{aligned}\mathscr{Q}_{14} &:= (\uptheta - \uptheta_{;h})u^{\alpha}(\partial_{\kappa}u^{\kappa})(S^{\lambda}\partial_{\lambda}h) + (\uptheta - \uptheta_{;h})u^{\alpha}(u^{\kappa}\partial_{\kappa}h)(S^{\lambda}\partial_{\lambda}h)\\ &\quad + (\uptheta - \uptheta_{;h})u^{\alpha}(\upeta^{-1})^{\kappa\lambda}(\partial_{\kappa}h)u_{\beta}(\partial_{\lambda}S^{\beta})\\ &= n(\uptheta - \uptheta_{;h})u^{\alpha}(u^{\kappa}\partial_{\kappa}h)\mathcal{D}\\ &\quad + (\uptheta_{;h} - \uptheta)u^{\alpha}u^{\kappa}\left\{(\partial_{\kappa}h)(\partial_{\lambda}S^{\lambda}) - (\partial_{\lambda}h)(\partial_{\kappa}S^{\lambda})\right\}, \end{aligned} \tag{8.3j}$$

$$\begin{aligned}\mathscr{Q}_{15} &:= (\partial_{\kappa}u^{\kappa})\upepsilon^{\alpha\beta\gamma\delta}(\partial_{\beta}h)u_{\gamma}\varpi_{\delta} + c^{-2}\upepsilon^{\alpha\beta\gamma\delta}(u^{\kappa}\partial_{\kappa}u_{\beta})(\partial_{\gamma}h)\varpi_{\delta}\\ &= c^{-2}q\upepsilon^{\alpha\beta\gamma\delta}S_{\beta}(\partial_{\gamma}h)\varpi_{\delta}, \end{aligned} \tag{8.3k}$$

$$\begin{aligned}\mathscr{Q}_{16} &:= -c^{-2}u^{\alpha}\upepsilon^{\sigma\beta\gamma\delta}(u^{\kappa}\partial_{\kappa}u_{\sigma})u_{\beta}(\partial_{\gamma}h)\varpi_{\delta}\\ &= -c^{-2}qu^{\alpha}\upepsilon^{\kappa\beta\gamma\delta}S_{\kappa}u_{\beta}(\partial_{\gamma}h)\varpi_{\delta}, \end{aligned} \tag{8.3l}$$

$$\begin{aligned}\mathscr{Q}_{18} &:= (\uptheta - \uptheta_{;h})((\upeta^{-1})^{\alpha\kappa}\partial_{\kappa}h)(S^{\lambda}\partial_{\lambda}h)\\ &\quad + (\uptheta_{;h} - \uptheta)((\upeta^{-1})^{\alpha\kappa}\partial_{\kappa}h)u_{\beta}(u^{\lambda}\partial_{\lambda}S^{\beta})\\ &= q(\uptheta - \uptheta_{;h})S_{\kappa}S^{\kappa}((\upeta^{-1})^{\alpha\lambda}\partial_{\lambda}h), \end{aligned} \tag{8.3m}$$

$$\begin{aligned}\mathscr{Q}_{19} &:= (\uptheta_{;h} - \uptheta)u^{\alpha}(u^{\kappa}\partial_{\kappa}u_{\sigma})S^{\sigma}(\partial_{\lambda}u^{\lambda})\\ &\quad + c^{-2}(\uptheta - \uptheta_{;h})u^{\alpha}(S^{\kappa}\partial_{\kappa}h)(u^{\lambda}\partial_{\lambda}h)\\ &= q(\uptheta_{;h} - \uptheta)u^{\alpha}S_{\kappa}S^{\kappa}(\partial_{\lambda}u^{\lambda}), \end{aligned} \tag{8.3n}$$

$$\begin{aligned}\mathscr{Q}_{20} &:= (\theta - \theta_{;h})S^{\beta}((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}u_{\beta})(u^{\kappa}\partial_{\kappa}h) \\ &\quad + (\theta - \theta_{;h})(u^{\kappa}\partial_{\kappa}h)(u^{\lambda}\partial_{\lambda}S^{\alpha}) \\ &= 0, \end{aligned} \tag{8.3o}$$

$$\begin{aligned}\mathscr{Q}_{21} &:= (\theta_{;h} - \theta)u^{\alpha}(u^{\kappa}\partial_{\kappa}u_{\beta})u^{\beta}(S^{\lambda}\partial_{\lambda}h) \\ &= 0. \end{aligned} \tag{8.3p}$$

Proof. We split the proof into many pieces.

• **Proof of** (8.1a): We first use Eq. (2.17) to deduce

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}\partial_{\kappa}u^{\kappa})\varpi_{\delta} &= -\epsilon^{\alpha\beta\gamma\delta}u_{\beta}\left\{\partial_{\gamma}(c^{-2}u^{\kappa}\partial_{\kappa}h)\right\}\varpi_{\delta} \\ &= -\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(c^{-2}u^{\kappa}\partial_{\kappa}\partial_{\gamma}h)\varpi_{\delta} \\ &\quad + 2c^{-3}c_{;h}(u^{\kappa}\partial_{\kappa}h)\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)\varpi_{\delta} \\ &\quad + 2c^{-3}c_{;s}(u^{\kappa}\partial_{\kappa}h)\epsilon^{\alpha\beta\gamma\delta}u_{\beta}S_{\gamma}\varpi_{\delta} \\ &\quad - c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}u^{\kappa})(\partial_{\kappa}h)\varpi_{\delta}. \end{aligned} \tag{8.4}$$

Next, we rewrite the first term on RHS (8.4) as a perfect $u^{\kappa}\partial_{\kappa}$ derivative plus error terms, thereby obtaining, with the help of (2.19), the following identity:

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}\partial_{\kappa}u^{\kappa})\varpi_{\delta} &= -u^{\kappa}\partial_{\kappa}\left\{\epsilon^{\alpha\beta\gamma\delta}c^{-2}u_{\beta}(\partial_{\gamma}h)\varpi_{\delta}\right\} \\ &\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}(u^{\kappa}\partial_{\kappa}u_{\beta})(\partial_{\gamma}h)\varpi_{\delta} \\ &\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)(u^{\kappa}\partial_{\kappa}\varpi_{\delta}) \\ &\quad - c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}u^{\kappa})(\partial_{\kappa}h)\varpi_{\delta} \\ &\quad + 2c^{-3}c_{;s}(u^{\kappa}\partial_{\kappa}h)\epsilon^{\alpha\beta\gamma\delta}u_{\beta}S_{\gamma}\varpi_{\delta}. \end{aligned} \tag{8.5}$$

Using Eq. (7.1) to substitute for the factor $u^{\kappa}\partial_{\kappa}\varpi_{\delta}$ in the third product on RHS (8.5), we deduce

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}\partial_{\kappa}u^{\kappa})\varpi_{\delta} &= -u^{\kappa}\partial_{\kappa}\left\{c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)\varpi_{\delta}\right\} \\ &\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}(u^{\kappa}\partial_{\kappa}u_{\beta})(\partial_{\gamma}h)\varpi_{\delta} \\ &\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)(\varpi^{\kappa}\partial_{\kappa}u_{\delta}) \\ &\quad - c^{-2}(\partial_{\kappa}u^{\kappa})\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)\varpi_{\delta} \\ &\quad + c^{-2}(\theta - \theta_{;h})\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda}u_{\beta}u^{\nu}(\partial_{\gamma}h)(\partial_{\kappa}h)S_{\lambda} \\ &\quad - c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}u^{\kappa})(\partial_{\kappa}h)\varpi_{\delta} \\ &\quad + 2c^{-3}c_{;s}(u^{\kappa}\partial_{\kappa}h)\epsilon^{\alpha\beta\gamma\delta}u_{\beta}S_{\gamma}\varpi_{\delta}. \end{aligned} \tag{8.6}$$

Next, using the identity (4.21), we express the third product on RHS (8.6) as follows:

$$c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)(\varpi^{\kappa}\partial_{\kappa}u_{\delta}) = c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)\varpi^{\kappa}(\partial_{\delta}u_{\kappa}). \tag{8.7}$$

Next, using the identity

$$\begin{aligned}
-\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda} &= (\eta^{-1})^{\lambda\beta}\delta_{\nu}^{\alpha}(\eta^{-1})^{\kappa\gamma} - (\eta^{-1})^{\lambda\beta}\delta_{\nu}^{\gamma}(\eta^{-1})^{\kappa\alpha} \\
&\quad + (\eta^{-1})^{\lambda\gamma}\delta_{\nu}^{\beta}(\eta^{-1})^{\kappa\alpha} - (\eta^{-1})^{\lambda\gamma}\delta_{\nu}^{\alpha}(\eta^{-1})^{\kappa\beta} \\
&\quad + (\eta^{-1})^{\lambda\alpha}\delta_{\nu}^{\gamma}(\eta^{-1})^{\kappa\beta} - (\eta^{-1})^{\lambda\alpha}\delta_{\nu}^{\beta}(\eta^{-1})^{\kappa\gamma}
\end{aligned}$$

and Eqs. (2.20), (2.21), and (4.2), we express the third-from-last product on RHS (8.6) as follows:

$$\begin{aligned}
&c^{-2}(\theta - \theta_{;h})\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda}u_{\beta}u^{\nu}(\partial_{\gamma}h)(\partial_{\kappa}h)S_{\lambda} \\
&\quad = c^{-2}(\theta - \theta_{;h})(S^{\kappa}\partial_{\kappa}h)((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}h) \\
&\qquad + c^{-2}(\theta - \theta_{;h})u^{\alpha}(S^{\kappa}\partial_{\kappa}h)(u^{\lambda}\partial_{\lambda}h) \\
&\qquad + c^{-2}(\theta_{;h} - \theta)S^{\alpha}(u^{\kappa}\partial_{\kappa}h)(u^{\lambda}\partial_{\lambda}h) \\
&\qquad + c^{-2}(\theta_{;h} - \theta)S^{\alpha}(\eta^{-1})^{\kappa\lambda}(\partial_{\kappa}h)(\partial_{\lambda}h). \qquad (8.8)
\end{aligned}$$

Using (8.7) and (8.8) to substitute for the relevant products on RHS (8.6), adding and subtracting $c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}\varpi_{\delta}(\partial_{\kappa}u^{\kappa})(\partial_{\gamma}h)$, and reorganizing the terms, we arrive at the desired identity (8.1a).

- **Proof of** (8.1b): We first use (4.22) to deduce

$$\begin{aligned}
&\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\varpi^{\kappa}\partial_{\kappa}\partial_{\gamma}u_{\delta}) \\
&\quad = u_{\beta}\varpi^{\kappa}\partial_{\kappa}\left\{\frac{1}{H}\varpi^{\alpha}u^{\beta} - \frac{1}{H}u^{\alpha}\varpi^{\beta} - \epsilon^{\alpha\beta\gamma\delta}(\partial_{\gamma}h)u_{\delta} + q\epsilon^{\alpha\beta\gamma\delta}S_{\gamma}u_{\delta}\right\}. \quad (8.9)
\end{aligned}$$

The desired identity (8.1b) now follows from (2.6), (2.20), (4.2), (4.3), (4.5), (4.21), (8.9), and straightforward calculations.

- **Proof of** (8.1c): We first use Eq. (2.22) to substitute for the factor $u^{\kappa}\partial_{\kappa}u_{\beta}$ on LHS (8.1c), thereby obtaining the identity

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}(u^{\kappa}\partial_{\kappa}u_{\beta})(u^{\lambda}\partial_{\lambda}\varpi_{\delta})u_{\gamma} &= -\epsilon^{\alpha\beta\gamma\delta}(\partial_{\beta}h)(u^{\lambda}\partial_{\lambda}\varpi_{\delta})u_{\gamma} \\
&\quad + q\epsilon^{\alpha\beta\gamma\delta}S_{\beta}(u^{\lambda}\partial_{\lambda}\varpi_{\delta})u_{\gamma}. \qquad (8.10)
\end{aligned}$$

We then use Eq. (7.1) to substitute for the two factors of $u^{\lambda}\partial_{\lambda}\varpi_{\delta}$ on RHS (8.10), which yields the identity

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}(u^{\kappa}\partial_{\kappa}u_{\beta})(u^{\lambda}\partial_{\lambda}\varpi_{\delta})u_{\gamma} &= -\epsilon^{\alpha\beta\gamma\delta}(\partial_{\beta}h)u_{\gamma}(\varpi^{\kappa}\partial_{\kappa}u_{\delta}) \\
&\quad + (\partial_{\kappa}u^{\kappa})\epsilon^{\alpha\beta\gamma\delta}(\partial_{\beta}h)u_{\gamma}\varpi_{\delta} \\
&\quad + (\theta_{;h} - \theta)\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda}(\partial_{\beta}h)u^{\nu}(\partial_{\kappa}h)S_{\lambda}u_{\gamma} \\
&\quad + q\epsilon^{\alpha\beta\gamma\delta}S_{\beta}u_{\gamma}(\varpi^{\kappa}\partial_{\kappa}u_{\delta}) \\
&\quad - q(\partial_{\kappa}u^{\kappa})\epsilon^{\alpha\beta\gamma\delta}S_{\beta}u_{\gamma}\varpi_{\delta} \\
&\quad + q(\theta - \theta_{;h})\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda}u^{\nu}(\partial_{\kappa}h)S_{\lambda}S_{\beta}u_{\gamma}. \quad (8.11)
\end{aligned}$$

Next, we use the identity (4.21) to express the first and fourth products on RHS (8.11) as follows:

$$-\epsilon^{\alpha\beta\gamma\delta}(\partial_{\beta}h)(\varpi^{\lambda}\partial_{\lambda}u_{\delta})u_{\gamma} = -\epsilon^{\alpha\beta\gamma\delta}(\partial_{\beta}h)\varpi^{\lambda}(\partial_{\delta}u_{\lambda})u_{\gamma}, \qquad (8.12)$$

$$q\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma(\varpi^\kappa\partial_\kappa u_\delta) = q\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi^\kappa(\partial_\delta u_\kappa). \tag{8.13}$$

We then use the identity

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda} &= (\eta^{-1})^{\lambda\beta}\delta_\nu^\gamma(\eta^{-1})^{\kappa\alpha} - (\eta^{-1})^{\lambda\beta}\delta_\nu^\alpha(\eta^{-1})^{\kappa\gamma}\\ &\quad + (\eta^{-1})^{\lambda\gamma}\delta_\nu^\alpha(\eta^{-1})^{\kappa\beta} - (\eta^{-1})^{\lambda\gamma}\delta_\nu^\beta(\eta^{-1})^{\kappa\alpha}\\ &\quad + (\eta^{-1})^{\lambda\alpha}\delta_\nu^\beta(\eta^{-1})^{\kappa\gamma} - (\eta^{-1})^{\lambda\alpha}\delta_\nu^\gamma(\eta^{-1})^{\kappa\beta}\end{aligned}$$

and Eqs. (2.20) and (2.21) to express the third product on RHS (8.11) as follows:

$$\begin{aligned}&(\theta_{;h}-\theta)\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda}(\partial_\beta h)u^\nu(\partial_\kappa h)S_\lambda u_\gamma\\ &\quad = (\theta-\theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)(S^\lambda\partial_\lambda h)\\ &\qquad + (\theta-\theta_{;h})u^\alpha(u^\kappa\partial_\kappa h)(S^\lambda\partial_\lambda h)\\ &\qquad + (\theta_{;h}-\theta)S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\ &\qquad + (\theta_{;h}-\theta)S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h).\end{aligned} \tag{8.14}$$

Similarly, we express the last product on RHS (8.11) as follows:

$$\begin{aligned}q(\theta-\theta_{;h})\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\delta\nu}{}^{\kappa\lambda}u^\nu(\partial_\kappa h)S_\lambda S_\beta u_\gamma &= q(\theta_{;h}-\theta)((\eta^{-1})^{\kappa\alpha}\partial_\kappa h)S^\lambda S_\lambda\\ &\quad + q(\theta_{;h}-\theta)u^\alpha(u^\kappa\partial_\kappa h)S^\lambda S_\lambda\\ &\quad + q(\theta-\theta_{;h})S^\alpha(S^\kappa\partial_\kappa h).\end{aligned} \tag{8.15}$$

Using (8.12)–(8.13) and (8.14)–(8.15) to substitute for the relevant products on RHS (8.11), we arrive at the desired identity (8.1c).

• **Proof of** (8.1d): (8.1d) follows easily from using Eq. (2.22) to substitute for the factor $u^\kappa\partial_\kappa u_\beta$ on the LHS.

• **Proof of** (8.1e): We first use the identity (4.17) to deduce

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\kappa u_\delta) &= \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa)\\ &\quad + \frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\kappa\delta\theta\lambda}u^\theta\varpi^\lambda u_\beta(\partial_\gamma\varpi^\kappa)\\ &\quad + \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)u_\kappa(\partial_\delta h) - q\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)u_\kappa S_\delta.\end{aligned} \tag{8.16}$$

Next, we note the identity

$$\begin{aligned}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\kappa\delta\theta\lambda} = -\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\kappa\lambda\theta\delta} &= \delta_\kappa^\alpha\delta_\lambda^\beta\delta_\theta^\gamma - \delta_\kappa^\beta\delta_\lambda^\alpha\delta_\theta^\gamma\\ &\quad + \delta_\kappa^\beta\delta_\lambda^\gamma\delta_\theta^\alpha - \delta_\kappa^\alpha\delta_\lambda^\gamma\delta_\theta^\beta + \delta_\kappa^\gamma\delta_\lambda^\alpha\delta_\theta^\beta - \delta_\kappa^\gamma\delta_\lambda^\beta\delta_\theta^\alpha,\end{aligned}$$

which, in view of (2.20), (4.2), and (4.5), allows us to express the second product on RHS (8.16) as follows:

$$\begin{aligned}\frac{1}{H}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\kappa\delta\theta\lambda}u^\theta\varpi^\lambda u_\beta(\partial_\gamma\varpi^\kappa) &= -\frac{1}{H}\varpi^\alpha u_\lambda(u^\kappa\partial_\kappa\varpi^\lambda)\\ &\quad + \frac{1}{H}u^\alpha u_\lambda(\varpi^\kappa\partial_\kappa\varpi^\lambda)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{H}(\varpi^\kappa\partial_\kappa\varpi^\alpha)-\frac{1}{H}\varpi^\alpha(\partial_\kappa\varpi^\kappa)\\
&=\frac{1}{H}\varpi^\alpha\varpi^\lambda(u^\kappa\partial_\kappa u_\lambda)-\frac{1}{H}u^\alpha\varpi^\lambda(\varpi^\kappa\partial_\kappa u_\lambda)\\
&+\frac{1}{H}(\varpi^\kappa\partial_\kappa\varpi^\alpha)-\frac{1}{H}\varpi^\alpha(\partial_\kappa\varpi^\kappa). \qquad (8.17)
\end{aligned}
$$

Using (8.17) to substitute for the second product on RHS (8.16), and using (4.5) to express the third product on RHS (8.16) as $\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)u_\kappa(\partial_\delta h) = -\epsilon^{\alpha\beta\gamma\delta}u_\beta\varpi^\kappa(\partial_\gamma u_\kappa)(\partial_\delta h) = \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa)$ and the last product on RHS (8.16) as $-q\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)u_\kappa S_\delta = q\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma u^\kappa)\varpi_\kappa S_\delta$, we arrive at the desired identity (8.1e).

- **Proof of** (8.1f): (8.1f) follows from definition (2.4) with $V_\delta := \varpi_\delta$.

- **Proof of** (8.1g): (8.1g) is a straightforward consequence of (4.22), (2.20), and (4.2).

- **Proof of** (8.2a): We first use (5.5) to express LHS (8.2a) as follows:

$$
\begin{aligned}
&(\theta-\theta_{;h})S^\alpha((\eta^{-1})^{\kappa\lambda}\partial_\kappa\partial_\lambda h)+(\theta-\theta_{;h})S^\alpha(u^\kappa u^\lambda\partial_\kappa\partial_\lambda h)\\
&\quad=(\theta_{;h}-\theta)S^\alpha(u^\lambda\partial_\lambda\partial_\kappa u^\kappa)+(\theta_{;h}-\theta)S^\alpha(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa)\\
&\qquad+(\theta_{;h}-\theta)S^\alpha(u^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h)+(\theta_{;h}-\theta)S^\alpha(\partial_\kappa u^\kappa)(u^\lambda\partial_\lambda h)\\
&\qquad+(\theta-\theta_{;h})qS^\alpha(\partial_\kappa S^\kappa)+(\theta-\theta_{;h})q_{;h}S^\alpha(S^\kappa\partial_\kappa h)+(\theta-\theta_{;h})q_{;s}S^\alpha S_\kappa S^\kappa.
\end{aligned} \tag{8.18}
$$

Next, with the help of Eq. (2.19), we rewrite the first product on RHS (8.18) as follows:

$$
\begin{aligned}
(\theta_{;h}-\theta)S^\alpha(u^\lambda\partial_\lambda\partial_\kappa u^\kappa) &= u^\kappa\partial_\kappa\left\{(\theta_{;h}-\theta)S^\alpha(\partial_\lambda u^\lambda)\right\}\\
&+(\theta_h-\theta_{;h;h})S^\alpha(u^\kappa\partial_\kappa h)(\partial_\lambda u^\lambda)\\
&+(\theta-\theta_{;h})(u^\kappa\partial_\kappa S^\alpha)(\partial_\lambda u^\lambda).
\end{aligned} \tag{8.19}
$$

Using (8.19) to substitute for the first product on RHS (8.18), we arrive at the desired identity (8.2a).

- **Proof of** (8.2b): (8.2b) is a straightforward consequence of Eq. (2.19).

- **Proof of** (8.2c): We first differentiate Eq. (4.18) with $(\eta^{-1})^{\alpha\lambda}\partial_\lambda$ and then multiply the resulting identity by $(\theta_{;h}-\theta)$ to obtain

$$
\begin{aligned}
(\theta_{;h}-\theta)(\eta^{-1})^{\alpha\lambda}(S^\kappa\partial_\kappa\partial_\lambda h) &= (\theta-\theta_{;h})(\eta^{-1})^{\alpha\lambda}S^\beta(u^\kappa\partial_\kappa\partial_\lambda u_\beta)\\
&+(\theta-\theta_{;h})((\eta^{-1})^{\alpha\lambda}\partial_\lambda S^\kappa)(\partial_\kappa h)\\
&+(\theta-\theta_{;h})((\eta^{-1})^{\alpha\lambda}\partial_\lambda S^\beta)(u^\kappa\partial_\kappa u_\beta)\\
&+(\theta-\theta_{;h})S^\beta((\eta^{-1})^{\alpha\lambda}\partial_\lambda u^\kappa)(\partial_\kappa u_\beta)\\
&+(\theta_{;h}-\theta)q_{;h}((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)S_\kappa S^\kappa\\
&+(\theta_{;h}-\theta)q_{;s}S^\alpha S_\kappa S^\kappa\\
&+2(\theta_{;h}-\theta)q((\eta^{-1})^{\alpha\lambda}\partial_\lambda S^\kappa)S_\kappa.
\end{aligned} \tag{8.20}
$$

Next, with the help of Eq. (2.19), we rewrite the first product on RHS (8.20) as follows:

$$\begin{aligned}&(\theta - \theta_{;h})(\eta^{-1})^{\alpha\lambda} S^{\beta}(u^{\kappa}\partial_{\kappa}\partial_{\lambda}u_{\beta})\\ &\quad = u^{\kappa}\partial_{\kappa}\left\{(\theta - \theta_{;h})(\eta^{-1})^{\alpha\lambda} S^{\beta}(\partial_{\lambda}u_{\beta})\right\}\\ &\qquad + (\theta_{;h;h} - \theta_{h})(u^{\kappa}\partial_{\kappa}h)(\eta^{-1})^{\alpha\lambda} S^{\beta}(\partial_{\lambda}u_{\beta})\\ &\qquad + (\theta_{;h} - \theta)(u^{\kappa}\partial_{\kappa}S^{\beta})(\eta^{-1})^{\alpha\lambda}(\partial_{\lambda}u_{\beta}).\end{aligned} \tag{8.21}$$

Next, we use Eqs. (2.22) and (4.4) to express the sum of the second and third products on RHS (8.20) as follows:

$$\begin{aligned}&(\theta - \theta_{;h})((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}S^{\kappa})(\partial_{\kappa}h) + (\theta - \theta_{;h})((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}S^{\beta})(u^{\kappa}\partial_{\kappa}u_{\beta})\\ &\quad = (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}S^{\beta})u_{\beta}(u^{\kappa}\partial_{\kappa}h) + (\theta - \theta_{;h})q((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}S^{\beta})S_{\beta}\\ &\quad = (\theta - \theta_{;h})S^{\beta}((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}u_{\beta})(u^{\kappa}\partial_{\kappa}h) + (\theta - \theta_{;h})q((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}S^{\beta})S_{\beta}.\end{aligned} \tag{8.22}$$

Using (8.21) to substitute for the first product on RHS (8.20), and using (8.22) to substitute for the second and third products on RHS (8.20), we arrive at the desired identity (8.2c).

• **Proof of** (8.3a): We simply use (4.1) to express the second product on LHS (8.3a) as follows:

$$(\theta - \theta_{;h})(\eta^{-1})^{\kappa\lambda}(\partial_{\kappa}h)(\partial_{\lambda}S^{\alpha}) = (\theta - \theta_{;h})(\eta^{-1})^{\alpha\kappa}(\partial_{\lambda}h)(\partial_{\kappa}S^{\lambda}).$$

• **Proof of** (8.3b): We use Eq. (2.17) to substitute for the last factor $\partial_{\lambda}u^{\lambda}$ on LHS (8.3b) and then appeal to Eq. (2.13b).

• **Proof of** (8.3c): We first use (4.18) to express the first product on LHS (8.3c) as follows:

$$\begin{aligned}&(\theta_{;h;h} - \theta_{h})((\eta^{-1})^{\alpha\kappa}\partial_{\kappa}h)(S^{\lambda}\partial_{\lambda}h)\\ &\quad = (\theta_{h} - \theta_{;h;h})((\eta^{-1})^{\alpha\kappa}\partial_{\kappa}h)(u^{\lambda}\partial_{\lambda}u_{\beta})S^{\beta}\\ &\qquad + (\theta_{;h;h} - \theta_{h})q((\eta^{-1})^{\alpha\kappa}\partial_{\kappa}h)S^{\lambda}S_{\lambda}.\end{aligned} \tag{8.23}$$

Using (8.23) to substitute for the first product on LHS (8.3c), we arrive at the desired identity.

• **Proof of** (8.3d): To prove (8.3d), we first use Eq. (2.17) to express the last product on LHS (8.3d) as follows:

$$(\theta - \theta_{;h})S^{\alpha}(\partial_{\kappa}u^{\kappa})(\partial_{\lambda}u^{\lambda}) = c^{-4}(\theta - \theta_{;h})S^{\alpha}(u^{\kappa}\partial_{\kappa}h)(u^{\lambda}\partial_{\lambda}h). \tag{8.24}$$

Using (8.24) to substitute for the last product on LHS (8.3d) and appealing to Eq. (2.13b), we arrive at the desired identity.

• **Proof of** (8.3e): We first use (2.22) to substitute for the factor $u^{\kappa}\partial_{\kappa}u^{\lambda}$ in the last product on LHS (8.3e), thereby obtaining the following identity:

$$(\theta_{;h} - \theta)S^{\alpha}(u^{\kappa}\partial_{\kappa}u^{\lambda})(\partial_{\lambda}h) = (\theta - \theta_{;h})S^{\alpha}(\eta^{-1})^{\kappa\lambda}(\partial_{\kappa}h)(\partial_{\lambda}h)$$

$$+ (\theta - \theta_{;h}) S^{\alpha} (u^{\kappa} \partial_{\kappa} h)(u^{\lambda} \partial_{\lambda} h) \\ + (\theta_{;h} - \theta) q S^{\alpha} (S^{\kappa} \partial_{\kappa} h). \tag{8.25}$$

Using (8.25) to substitute for the last product on LHS (8.3e), we arrive at the desired identity.

• **Proof of** (8.3f): We first use (4.1), (4.4), and the first equality in (6.1) to express the last product on LHS (8.3f) as follows:

$$\begin{aligned}(\theta - \theta_{;h})((\eta^{-1})^{\kappa\lambda} \partial_{\kappa} u^{\alpha}) S^{\beta} (\partial_{\lambda} u_{\beta}) &= (\theta_{;h} - \theta)(\partial_{\kappa} u^{\alpha})(u^{\beta} \partial_{\beta} S^{\kappa}) \\ &= (\theta - \theta_{;h})(\partial_{\kappa} u^{\alpha})(S^{\lambda} \partial_{\lambda} u^{\kappa}) \\ &\quad + \frac{1}{H}(\theta - \theta_{;h}) \epsilon^{\kappa\beta\gamma\delta} (\partial_{\kappa} u^{\alpha}) S_{\beta} u_{\gamma} \varpi_{\delta} \\ &\quad + (\theta - \theta_{;h})(u^{\kappa} \partial_{\kappa} u^{\alpha})(S^{\lambda} \partial_{\lambda} h) \\ &\quad + q(\theta_{;h} - \theta)(u^{\kappa} \partial_{\kappa} u^{\alpha}) S^{\lambda} S_{\lambda}. \end{aligned} \tag{8.26}$$

Using (8.26) to substitute for the last product on LHS (8.3f), we arrive at the desired identity.

• **Proof of** (8.3g): We use (4.1) and (4.4) to express the first product on LHS (8.3g) as follows:

$$\begin{aligned}(\theta - \theta_{;h})(u^{\kappa} \partial_{\kappa} S^{\alpha})(\partial_{\lambda} u^{\lambda}) &= (\theta - \theta_{;h})(u^{\kappa} (\eta^{-1})^{\alpha\beta} \partial_{\beta} S_{\kappa})(\partial_{\lambda} u^{\lambda}) \\ &= (\theta_{;h} - \theta) S^{\beta} ((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} u_{\beta})(\partial_{\lambda} u^{\lambda}). \end{aligned} \tag{8.27}$$

Using (8.27) to substitute for the first product on LHS (8.3g), we conclude the desired identity.

• **Proof of** (8.3h): To prove (8.3h), we first note the following identity, which we derive below:

$$\begin{aligned} & c^{-2}(\theta - \theta_{;h})(S^{\kappa} \partial_{\kappa} h)((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} h) \\ &\quad + (\theta_{;h} - \theta)(u^{\kappa} \partial_{\kappa} S^{\beta})((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u_{\beta}) \\ &= (\theta - \theta_{;h}) S^{\kappa} (\partial_{\beta} u_{\kappa})((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u^{\beta}) \\ &\quad + c^{-2}(\theta_{;h} - \theta)(u^{\lambda} \partial_{\lambda} u_{\beta}) S^{\beta} ((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} h) \\ &\quad + c^{-2}(\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} u_{\beta}) S^{\beta} (u^{\lambda} \partial_{\lambda} h) \\ &\quad + (\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} u_{\beta}) S^{\beta} (\partial_{\lambda} u^{\lambda}) + q c^{-2} (\theta - \theta_{;h}) S^{\beta} S_{\beta} ((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} h). \end{aligned} \tag{8.28}$$

Using (8.28) to substitute for the sum of the second and third products on LHS (8.3h), we conclude the desired identity (8.3h).

It remains for us to prove (8.28). To proceed, we first use (4.1) and (4.4) to express the second product on LHS (8.28) as follows:

$$\begin{aligned} & (\theta_{;h} - \theta)(u^{\kappa} \partial_{\kappa} S^{\beta})((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u_{\beta}) \\ &\quad = (\theta_{;h} - \theta)(u^{\kappa} \partial_{\beta} S_{\kappa})((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u^{\beta}) \\ &\quad = (\theta - \theta_{;h})(S^{\beta} \partial_{\lambda} u_{\beta})((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} u^{\lambda}) \end{aligned}$$

$$= (\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(\partial_\lambda u^\lambda)$$
$$+ (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(\partial_\lambda u^\lambda)$$
$$+ (\theta - \theta_{;h})(S^\beta\partial_\lambda u_\beta)((\eta^{-1})^{\alpha\kappa}\partial_\kappa u^\lambda), \tag{8.29}$$

where to obtain the last equality, we have added and subtracted

$$(\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(\partial_\lambda u^\lambda).$$

Next, we use Eq. (2.17) to substitute for the factor $\partial_\lambda u^\lambda$ in the first product on RHS (8.29), which allows us to express the product as follows:

$$(\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(\partial_\lambda u^\lambda)$$
$$= c^{-2}(\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(u^\lambda\partial_\lambda h)$$
$$= c^{-2}(\theta_{;h} - \theta)(u^\lambda\partial_\lambda u_\beta)S^\beta((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)$$
$$+ c^{-2}(\theta - \theta_{;h})(u^\lambda\partial_\lambda u_\beta)S^\beta((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)$$
$$+ c^{-2}(\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(u^\lambda\partial_\lambda h), \tag{8.30}$$

where to obtain the last equality, we have added and subtracted

$$c^{-2}(\theta - \theta_{;h})(u^\lambda\partial_\lambda u_\beta)S^\beta((\eta^{-1})^{\alpha\kappa}\partial_\kappa h).$$

Next, we use Eq. (4.18) to express the first product on RHS (8.30) as follows:

$$c^{-2}(\theta_{;h} - \theta)(u^\lambda\partial_\lambda u_\beta)S^\beta((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)$$
$$= c^{-2}(\theta - \theta_{;h})(S^\kappa\partial_\kappa h)((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)$$
$$+ qc^{-2}(\theta_{;h} - \theta)S_\kappa S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda h). \tag{8.31}$$

Combining (8.29)–(8.31), we find that

$$c^{-2}(\theta - \theta_{;h})(S^\kappa\partial_\kappa h)((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)$$
$$= (\theta_{;h} - \theta)(u^\kappa\partial_\kappa S^\beta)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta)$$
$$+ (\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(\partial_\lambda u^\lambda) + (\theta_{;h} - \theta)(S^\beta\partial_\lambda u_\beta)((\eta^{-1})^{\alpha\kappa}\partial_\kappa u^\lambda)$$
$$+ c^{-2}(\theta_{;h} - \theta)(u^\lambda\partial_\lambda u_\beta)S^\beta((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)$$
$$+ c^{-2}(\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(u^\lambda\partial_\lambda h)$$
$$+ qc^{-2}(\theta - \theta_{;h})S_\kappa S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda h). \tag{8.32}$$

Using (8.32) to substitute for the first product on LHS (8.28), we deduce

$$c^{-2}(\theta - \theta_{;h})(S^\kappa\partial_\kappa h)((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)$$
$$+ (\theta_{;h} - \theta)(u^\kappa\partial_\kappa S^\beta)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta)$$
$$= 2(\theta_{;h} - \theta)(u^\kappa\partial_\kappa S^\beta)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta)$$
$$+ (\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(\partial_\lambda u^\lambda) + (\theta_{;h} - \theta)(S^\beta\partial_\lambda u_\beta)(\eta^{-1})^{\alpha\kappa}(\partial_\kappa u^\lambda)$$
$$+ c^{-2}(\theta_{;h} - \theta)(u^\lambda\partial_\lambda u_\beta)S^\beta((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)$$
$$+ c^{-2}(\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa u_\beta)S^\beta(u^\lambda\partial_\lambda h)$$
$$+ qc^{-2}(\theta - \theta_{;h})S_\kappa S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda h). \tag{8.33}$$

Next, we use (4.1) and (4.4) to express the first product on RHS (8.33) as

$$\begin{aligned} &2(\theta_{;h} - \theta)(u^\kappa \partial_\kappa S^\beta)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta) \\ &\quad = 2(\theta_{;h} - \theta)(u^\kappa \partial_\beta S_\kappa)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u^\beta) \\ &\quad = 2(\theta - \theta_{;h})(S^\kappa \partial_\beta u_\kappa)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u^\beta). \end{aligned} \tag{8.34}$$

Using (8.34) to substitute for the first product on RHS (8.33), we arrive at the desired identity (8.28). This completes the proof of (8.3h).

• **Proof of** (8.3i): We use the identity (4.18) to substitute for the factor $S^\lambda \partial_\lambda h$ on LHS (8.3i), thus obtaining

$$\begin{aligned} (\theta - \theta_{;h})u^\alpha(\partial_\kappa u^\kappa)(S^\lambda \partial_\lambda h) = (\theta_{;h} - \theta)u^\alpha(\partial_\kappa u^\kappa)S^\beta(u^\lambda \partial_\lambda u_\beta) \\ + (\theta - \theta_{;h})q u^\alpha(\partial_\kappa u^\kappa)S^\lambda S_\lambda. \end{aligned} \tag{8.35}$$

Using (8.35) to substitute for the first product on LHS (8.3i), we arrive at the desired identity.

• **Proof of** (8.3j): We first use Eq. (4.1) to express the last product on LHS (8.3j) as follows:

$$\begin{aligned} &(\theta - \theta_{;h})u^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)u_\beta(\partial_\lambda S^\beta) \\ &\quad = (\theta - \theta_{;h})u^\alpha(\partial_\kappa h)(u^\beta \partial_\beta S^\kappa) \\ &\quad = (\theta - \theta_{;h})u^\alpha(u^\kappa \partial_\kappa h)(\partial_\lambda S^\lambda) \\ &\qquad + (\theta_{;h} - \theta)u^\alpha u^\lambda \left\{(\partial_\lambda h)(\partial_\kappa S^\kappa) - (\partial_\kappa h)(\partial_\lambda S^\kappa)\right\}, \end{aligned} \tag{8.36}$$

where to obtain the second equality in (8.36), we added and subtracted $(\theta_{;h} - \theta)u^\alpha(u^\kappa \partial_\kappa h)(\partial_\lambda S^\lambda)$. Next, we solve for $\partial_\lambda S^\lambda$ in terms of the remaining terms in definition (2.16b) and then use the resulting identity to algebraically substitute for the factor $\partial_\lambda S^\lambda$ in the first product on RHS (8.36), which yields the identity

$$\begin{aligned} (\theta - \theta_{;h})u^\alpha(u^\kappa \partial_\kappa h)(\partial_\lambda S^\lambda) = n(\theta - \theta_{;h})u^\alpha(u^\kappa \partial_\kappa h)\mathcal{D} \\ + (\theta_{;h} - \theta)u^\alpha(u^\kappa \partial_\kappa h)(S^\lambda \partial_\lambda h) \\ + c^{-2}(\theta - \theta_{;h})u^\alpha(u^\kappa \partial_\kappa h)(S^\lambda \partial_\lambda h). \end{aligned} \tag{8.37}$$

Next, we use Eq. (2.17) to substitute for the factor $u^\kappa \partial_\kappa h$ in the last product on RHS (8.37), which yields the identity

$$\begin{aligned} (\theta - \theta_{;h})u^\alpha(u^\kappa \partial_\kappa h)(\partial_\lambda S^\lambda) = n(\theta - \theta_{;h})u^\alpha(u^\kappa \partial_\kappa h)\mathcal{D} \\ + (\theta_{;h} - \theta)u^\alpha(u^\kappa \partial_\kappa h)(S^\lambda \partial_\lambda h) \\ + (\theta_{;h} - \theta)u^\alpha(\partial_\kappa u^\kappa)(S^\lambda \partial_\lambda h). \end{aligned} \tag{8.38}$$

Substituting RHS (8.38) for the first product on RHS (8.36) and then using the resulting identity to substitute for the last product on LHS (8.3j), we arrive at the desired identity (8.3j).

• **Proof of** (8.3k): We first use Eq. (2.22) to substitute for the factor of $u^\kappa \partial_\kappa u_\beta$ in the second product on LHS (8.3k), which yields the identity

$$\begin{aligned}
&(\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma \varpi_\delta + c^{-2}\epsilon^{\alpha\beta\gamma\delta}(u^\kappa\partial_\kappa u_\beta)(\partial_\gamma h)\varpi_\delta \\
&\quad = (\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma \varpi_\delta - c^{-2}(u^\kappa\partial_\kappa h)\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi_\delta \\
&\qquad + c^{-2}q\epsilon^{\alpha\beta\gamma\delta}S_\beta(\partial_\gamma h)\varpi_\delta. \qquad (8.39)
\end{aligned}$$

Using Eq. (2.17) to substitute for the factor $\partial_\kappa u^\kappa$ in the first product on RHS (8.39) and taking into account the antisymmetry of ϵ, we see that the first and second products on RHS (8.39) cancel, which yields the desired identity (8.3k).

• **Proof of** (8.3l): We simply use Eq. (2.22) to substitute for the factor $u^\kappa \partial_\kappa u_\sigma$ on LHS (8.3l).

• **Proof of** (8.3m): We simply multiply Eq. (4.19) by $(\theta_{;h} - \theta)(\eta^{-1})^{\alpha\kappa}\partial_\kappa h$.

• **Proof of** (8.3n): We use Eq. (4.18) to substitute for the factor $(u^\kappa\partial_\kappa u_\sigma)S^\sigma$ in the first product on LHS (8.3n) and Eq. (2.17) to substitute for the factor $u^\lambda \partial_\lambda h$ in the second product on LHS (8.3n).

• **Proof of** (8.3o): We simply use Eq. (2.24) to substitute for the factor $u^\lambda\partial_\lambda S^\alpha$ in the second product on LHS (8.3o).

• **Proof of** (8.3p): (8.3p) follows from (4.3). □

8.2. The Transport-div-curl System

Armed with Lemma 8.1, we now derive the main result of this section.

Proposition 8.2 (The transport-div-curl system for the vorticity). *Assume that (h, s, u^α) is a C^3 solution to (2.17)–(2.19) + (2.20). Then the divergence of the vorticity vectorfield ϖ^α defined in (2.5) verifies the following identity:*

$$\partial_\alpha \varpi^\alpha = -\varpi^\kappa \partial_\kappa h + 2q\varpi^\kappa S_\kappa. \qquad (8.40)$$

Moreover, the rectangular components $\mathcal{C}^\alpha$ of the modified vorticity of the vorticity, which is defined in (2.16a), verify the following transport equations:

$$\begin{aligned}
u^\kappa\partial_\kappa \mathcal{C}^\alpha &= \mathcal{C}^\kappa\partial_\kappa u^\alpha - 2(\partial_\kappa u^\kappa)\mathcal{C}^\alpha + u^\alpha(u^\kappa\partial_\kappa u_\lambda)\mathcal{C}^\lambda \\
&\quad - 2\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa) \\
&\quad + (\theta_{;h} - \theta)\left\{(\eta^{-1})^{\alpha\kappa} + 2u^\alpha u^\kappa\right\}\left\{(\partial_\kappa h)(\partial_\lambda S^\lambda) - (\partial_\lambda h)(\partial_\kappa S^\lambda)\right\} \\
&\quad + n(\theta - \theta_{;h})u^\alpha(u^\kappa\partial_\kappa h)\mathcal{D} \\
&\quad + (\theta - \theta_{;h})qS^\alpha\partial_\kappa S^\kappa + (\theta_{;h} - \theta)q((\eta^{-1})^{\alpha\lambda}\partial_\lambda S^\kappa)S_\kappa \\
&\quad + \mathfrak{Q}_{(\mathcal{C}^\alpha)} + \mathfrak{L}_{(\mathcal{C}^\alpha)}, \qquad (8.41)
\end{aligned}$$

where $\mathfrak{Q}_{(\mathcal{C}^\alpha)}$ is the linear combination of ***null forms*** *defined by*

$$\begin{aligned}
\mathfrak{Q}_{(\mathcal{C}^\alpha)} &:= -c^{-2}\epsilon^{\kappa\beta\gamma\delta}(\partial_\kappa u^\alpha)u_\beta(\partial_\gamma h)\varpi_\delta \\
&\quad + (c^{-2} + 2)\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa) \\
&\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta\varpi_\delta\left\{(\partial_\kappa u^\kappa)(\partial_\gamma h) - (\partial_\gamma u^\kappa)(\partial_\kappa h)\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \left\{(\theta_{;h;h} - \theta_h) + c^{-2}(\theta - \theta_{;h})\right\} (\eta^{-1})^{\alpha\lambda} S^{\beta} u^{\kappa} \times \\
&\quad \{(\partial_\kappa h)(\partial_\lambda u_\beta) - (\partial_\lambda h)(\partial_\kappa u_\beta)\} \\
&+ (\theta_{;h} - \theta) S^{\kappa} u^{\lambda} \{(\partial_\kappa u^\alpha)(\partial_\lambda h) - (\partial_\lambda u^\alpha)(\partial_\kappa h)\} \\
&+ (\theta_{;h} - \theta) \left\{(\eta^{-1})^{\alpha\kappa} + u^\alpha u^\kappa\right\} S^{\beta} \times \\
&\quad \left\{(\partial_\kappa u_\beta)(\partial_\lambda u^\lambda) - (\partial_\lambda u_\beta)(\partial_\kappa u^\lambda)\right\} \\
&+ (\theta_{;h} - \theta) S^{\alpha} \left\{(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) - (\partial_\kappa u^\kappa)(\partial_\lambda u^\lambda)\right\} \\
&+ (\theta_{;h} - \theta) S^{\kappa} \left\{(\partial_\kappa u^\alpha)(\partial_\lambda u^\lambda) - (\partial_\lambda u^\alpha)(\partial_\kappa u^\lambda)\right\} \\
&+ S^{\alpha} \left\{c^{-2}(\theta_h - \theta_{;h;h}) + c^{-4}(\theta_{;h} - \theta)\right\} (g^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h), \qquad (8.42)
\end{aligned}
$$

and $\mathfrak{L}_{(\mathcal{C}^\alpha)}$, which is at most linear in the derivatives of the solution variables, is defined by

$$
\begin{aligned}
\mathfrak{L}_{(\mathcal{C}^\alpha)} :=\ & \frac{2q}{H} \varpi^\kappa S_\kappa \varpi^\alpha - \frac{2}{H} \varpi^\alpha (\varpi^\kappa \partial_\kappa h) \\
&+ 2c^{-3} c_{;s} (u^\kappa \partial_\kappa h) \epsilon^{\alpha\beta\gamma\delta} u_\beta S_\gamma \varpi_\delta \\
&- 2q \epsilon^{\alpha\beta\gamma\delta} u_\beta S_\gamma \varpi^\kappa (\partial_\delta u_\kappa) - q(\partial_\kappa u^\kappa) \epsilon^{\alpha\beta\gamma\delta} S_\beta u_\gamma \varpi_\delta \\
&+ \frac{1}{H} (\theta - \theta_{;h}) \epsilon^{\kappa\beta\gamma\delta} (\partial_\kappa u^\alpha) S_\beta u_\gamma \varpi_\delta + c^{-2} q \epsilon^{\alpha\beta\gamma\delta} S_\beta (\partial_\gamma h) \varpi_\delta \\
&- c^{-2} q u^\alpha \epsilon^{\kappa\beta\gamma\delta} S_\kappa u_\beta (\partial_\gamma h) \varpi_\delta \\
&+ q(\theta_{;h} - \theta) S_\kappa S^\kappa (u^\lambda \partial_\lambda u^\alpha) \\
&+ q(\theta_{;h} - \theta) u^\alpha S_\kappa S^\kappa (u^\lambda \partial_\lambda h) + (\theta_{;h;s} - \theta_{;s}) u^\alpha S_\kappa S^\kappa (u^\lambda \partial_\lambda h) \\
&+ (\theta_{;s} - \theta_{;h;s}) S^\alpha (S^\kappa \partial_\kappa h) + (\theta - \theta_{;h}) q_{;h} S^\alpha (S^\kappa \partial_\kappa h) \\
&+ q(\theta_{;h;h} - \theta_h) S_\kappa S^\kappa ((\eta^{-1})^{\alpha\lambda} \partial_\lambda h) + (\theta_{;h;s} - \theta_{;s}) S_\kappa S^\kappa ((\eta^{-1})^{\alpha\lambda} \partial_\lambda h) \\
&+ q c^{-2} (\theta - \theta_{;h}) S_\kappa S^\kappa ((\eta^{-1})^{\alpha\lambda} \partial_\lambda h) + (\theta_{;h} - \theta) q_{;h} S_\kappa S^\kappa ((\eta^{-1})^{\alpha\lambda} \partial_\lambda h). \qquad (8.43)
\end{aligned}
$$

Proof. We split the proof into several pieces.

• **Proof of** (8.40): First, from definition (2.5) and the antisymmetry of $\epsilon^{\kappa\lambda\gamma\delta}$, we deduce

$$\partial_\kappa \varpi^\kappa = -\epsilon^{\kappa\lambda\gamma\delta} (\partial_\kappa u_\lambda) \partial_\gamma (H u_\delta). \qquad (8.44)$$

Next, using (4.16), we deduce that

$$\text{RHS } (8.44) = \varpi^\lambda (u^\kappa \partial_\kappa u_\lambda) - \varpi^\kappa u^\lambda (\partial_\kappa u_\lambda) - \theta \epsilon^{\kappa\lambda\gamma\delta} (\partial_\kappa u_\lambda) S_\gamma u_\delta. \qquad (8.45)$$

Using (4.3), we see that the second product on RHS (8.45) vanishes. Moreover, using Eq. (2.22) and the identity (4.2), we can express the first product on RHS (8.45) as follows:

$$\varpi^\lambda (u^\kappa \partial_\kappa u_\lambda) = -\varpi^\kappa \partial_\kappa h + q \varpi^\kappa S_\kappa. \qquad (8.46)$$

In addition, using definition (2.7) and the identity (4.23), we can express the last product on RHS (8.45) as follows:

$$-\theta \epsilon^{\kappa\lambda\gamma\delta} (\partial_\kappa u_\lambda) S_\gamma u_\delta = q \varpi^\kappa S_\kappa. \qquad (8.47)$$

Combining these calculations, we arrive at the desired identity (8.40).

• **Proof of** (8.41): The proof is a series of lengthy calculations in which we observe many cancellations. We start by using (8.1a)–(8.1g) to substitute for all of the terms on the third through seventh lines of RHS (7.4) except for the term $-\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa)$ from the fifth line, which we leave as is. We also use (8.40) to express the fourth product on RHS (8.1e) as $\frac{1}{H}\varpi^\alpha(\partial_\kappa\varpi^\kappa) = -\frac{1}{H}\varpi^\alpha(\varpi^\kappa\partial_\kappa h) + \frac{2q}{H}\varpi^\alpha\varpi^\kappa S_\kappa$, and we use (8.2a)–(8.2c) to substitute for the four products (which depend on the second derivatives of h) on the sixth-to-last and fifth-to-last lines of RHS (7.4), thereby obtaining the following equation (where at this stage in the argument, we have simply performed a term-by-term substitution and have not yet organized the terms):

$$
\begin{aligned}
u^\kappa\partial_\kappa \mathrm{vort}^\alpha(\varpi) &= \mathrm{vort}^\kappa(\varpi)\partial_\kappa u^\alpha - (\partial_\kappa u^\kappa)\mathrm{vort}^\alpha(\varpi)\\
&\quad + u^\alpha(u^\kappa\partial_\kappa u_\beta)\mathrm{vort}^\beta(\varpi)\\
&\quad - u^\kappa\partial_\kappa\left\{c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi_\delta\right\}\\
&\quad - 2(\partial_\kappa u^\kappa)c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi_\delta\\
&\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}(u^\kappa\partial_\kappa u_\beta)(\partial_\gamma h)\varpi_\delta\\
&\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa)\\
&\quad + c^{-2}(\theta-\theta_{;h})(S^\kappa\partial_\kappa h)((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)\\
&\quad + c^{-2}(\theta-\theta_{;h})u^\alpha(S^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&\quad + c^{-2}(\theta_{;h}-\theta)S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&\quad + c^{-2}(\theta_{;h}-\theta)S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h)\\
&\quad + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta\left\{(\partial_\kappa u^\kappa)(\partial_\gamma h) - (\partial_\gamma u^\kappa)(\partial_\kappa h)\right\}\varpi_\delta\\
&\quad + 2c^{-3}c_{;s}(u^\kappa\partial_\kappa h)\epsilon^{\alpha\beta\gamma\delta}u_\beta S_\gamma\varpi_\delta\\
&\quad + \frac{1}{H}(\varpi^\kappa\partial_\kappa\varpi^\alpha) - \frac{1}{H}(\varpi^\kappa\partial_\kappa h)\varpi^\alpha\\
&\quad - \frac{1}{H}u^\alpha\varpi^\lambda(\varpi^\kappa\partial_\kappa u_\lambda) + \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa)\\
&\quad - q\epsilon^{\alpha\beta\gamma\delta}u_\beta S_\gamma\varpi^\kappa(\partial_\delta u_\kappa)\\
&\quad - \epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma\varpi^\kappa(\partial_\delta u_\kappa) + (\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma\varpi_\delta\\
&\quad + (\theta-\theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)(S^\lambda\partial_\lambda h)\\
&\quad + (\theta-\theta_{;h})u^\alpha(u^\kappa\partial_\kappa h)(S^\lambda\partial_\lambda h)\\
&\quad + (\theta_{;h}-\theta)S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&\quad + (\theta_{;h}-\theta)S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h)\\
&\quad + q\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi^\kappa(\partial_\delta u_\kappa)\\
&\quad - q(\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi_\delta\\
&\quad + q(\theta_{;h}-\theta)((\eta^{-1})^{\kappa\alpha}\partial_\kappa h)S^\lambda S_\lambda
\end{aligned}
$$

$$
\begin{aligned}
&+ q(\theta_{;h} - \theta)u^\alpha(u^\kappa\partial_\kappa h)S^\lambda S_\lambda + q(\theta - \theta_{;h})S^\alpha(S^\kappa\partial_\kappa h) \\
&- \epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma\varpi^\lambda(\partial_\delta u_\lambda) + q\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi^\lambda(\partial_\delta u_\lambda) \\
&- \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa) \\
&- \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa) - \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa) \\
&- \frac{1}{H}(\varpi^\kappa\partial_\kappa\varpi^\alpha) - \frac{1}{H}\varpi^\alpha(\varpi^\kappa\partial_\kappa h) + \frac{2q}{H}\varpi^\alpha\varpi^\kappa S_\kappa \\
&- \frac{1}{H}\varpi^\alpha\varpi^\lambda(u^\kappa\partial_\kappa u_\lambda) + \frac{1}{H}u^\alpha\varpi^\lambda(\varpi^\kappa\partial_\kappa u_\lambda) \\
&- q\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma u^\kappa)\varpi_\kappa S_\delta \\
&- (\partial_\kappa u^\kappa)\mathrm{vort}^\alpha(\varpi) \\
&+ \frac{1}{H}\varpi^\alpha\varpi^\lambda(u^\kappa\partial_\kappa u_\lambda) \\
&+ (\theta_h - \theta_{;h;h})S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h) \\
&+ (\theta_h - \theta_{;h;h})S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h) \\
&+ (\theta_{;h;h} - \theta_h)u^\alpha(S^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h) \\
&+ (\theta_{;h;h} - \theta_h)((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)(S^\lambda\partial_\lambda h) \\
&+ (\theta_{;s} - \theta_{;h;s})S^\alpha(S^\kappa\partial_\kappa h) + (\theta_{;h;s} - \theta_{;s})u^\alpha S_\kappa S^\kappa(u^\lambda\partial_\lambda h) \\
&+ (\theta_{;h;s} - \theta_{;s})S_\kappa S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda h) \\
&+ (\theta - \theta_{;h})S^\alpha(\partial_\kappa u^\kappa)(u^\lambda\partial_\lambda h) \\
&+ (\theta_{;h} - \theta)(S^\kappa\partial_\kappa u^\alpha)(u^\lambda\partial_\lambda h) \\
&+ u^\kappa\partial_\kappa\left\{(\theta_{;h} - \theta)S^\alpha(\partial_\lambda u^\lambda)\right\} \\
&+ (\theta_h - \theta_{;h;h})S^\alpha(u^\kappa\partial_\kappa h)(\partial_\lambda u^\lambda) \\
&+ (\theta - \theta_{;h})(u^\kappa\partial_\kappa S^\alpha)(\partial_\lambda u^\lambda) \\
&+ (\theta_{;h} - \theta)S^\alpha(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) \\
&+ (\theta_{;h} - \theta)S^\alpha(u^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h) + (\theta_{;h} - \theta)S^\alpha(\partial_\kappa u^\kappa)(u^\lambda\partial_\lambda h) \\
&+ (\theta - \theta_{;h})qS^\alpha(\partial_\kappa S^\kappa) + (\theta - \theta_{;h})q_{;h}S^\alpha(S^\kappa\partial_\kappa h) \\
&+ (\theta - \theta_{;h})q_{;s}S^\alpha S_\kappa S^\kappa \\
&+ u^\kappa\partial_\kappa\left\{(\theta_{;h} - \theta)u^\alpha(S^\lambda\partial_\lambda h)\right\} \\
&+ (\theta_h - \theta_{;h;h})u^\alpha(u^\kappa\partial_\kappa h)(S^\lambda\partial_\lambda h) \\
&+ (\theta - \theta_{;h})(u^\kappa\partial_\kappa u^\alpha)(S^\lambda\partial_\lambda h) \\
&+ (\theta - \theta_{;h})u^\alpha(u^\kappa\partial_\kappa S^\lambda)(\partial_\lambda h) \\
&+ u^\kappa\partial_\kappa\left\{(\theta - \theta_{;h})(\eta^{-1})^{\alpha\lambda}S^\beta(\partial_\lambda u_\beta)\right\} \\
&+ (\theta_{;h;h} - \theta_h)(u^\kappa\partial_\kappa h)S^\beta((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta) \\
&+ (\theta_{;h} - \theta)(u^\kappa\partial_\kappa S^\beta)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta) \\
&+ (\theta - \theta_{;h})S^\beta((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta)(u^\kappa\partial_\kappa h)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\theta - \theta_{;h}) S^{\beta} ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u^{\kappa})(\partial_{\kappa} u_{\beta}) \\
&+ (\theta - \theta_{;h}) q ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} S^{\beta}) S_{\beta} \\
&+ (\theta_{;h} - \theta) q_{;h} ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} h) S_{\kappa} S^{\kappa} \\
&+ (\theta_{;h} - \theta) q_{;s} S^{\alpha} S_{\kappa} S^{\kappa} + 2(\theta_{;h} - \theta) q ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} S^{\kappa}) S_{\kappa} \\
&+ (\theta - \theta_{;h}) (\eta^{-1})^{\kappa\lambda} (\partial_{\kappa} h)(\partial_{\lambda} S^{\alpha}) \\
&+ (\theta - \theta_{;h}) (u^{\kappa} \partial_{\kappa} h)(u^{\lambda} \partial_{\lambda} S^{\alpha}) \\
&+ (\theta_{;h} - \theta) u^{\alpha} (u^{\kappa} \partial_{\kappa} h)(\partial_{\lambda} S^{\lambda}) \\
&+ (\theta_{;h} - \theta) ((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} h)(\partial_{\lambda} S^{\lambda}) \\
&+ (\theta_{;h} - \theta) ((\eta^{-1})^{\alpha\kappa} \partial_{\kappa} h) u_{\beta} (u^{\lambda} \partial_{\lambda} S^{\beta}) \\
&+ (\theta - \theta_{;h}) u^{\alpha} (\eta^{-1})^{\kappa\lambda} (\partial_{\kappa} h) u_{\beta} (\partial_{\lambda} S^{\beta}).
\end{aligned} \tag{8.48}
$$

Next, we bring the four perfect-derivative terms $u^{\kappa}\partial_{\kappa}\{\cdots\}$ on RHS (8.48) over to the left-hand side, which yields the equation

$$
\begin{aligned}
u^{\kappa} \partial_{\kappa} \Big\{ &\mathrm{vort}^{\alpha}(\varpi) + c^{-2} \epsilon^{\alpha\beta\gamma\delta} u_{\beta} (\partial_{\gamma} h) \varpi_{\delta} + (\theta - \theta_{;h}) S^{\alpha} (\partial_{\lambda} u^{\lambda}) \\
&+ (\theta - \theta_{;h}) u^{\alpha} (S^{\lambda} \partial_{\lambda} h) + (\theta_{;h} - \theta) (\eta^{-1})^{\alpha\lambda} S^{\beta} (\partial_{\lambda} u_{\beta}) \Big\} \\
&= \mathrm{vort}^{\kappa}(\varpi) \partial_{\kappa} u^{\alpha} - 2(\partial_{\kappa} u^{\kappa}) \mathrm{vort}^{\alpha}(\varpi) + u^{\alpha} (u^{\kappa} \partial_{\kappa} u_{\beta}) \mathrm{vort}^{\beta}(\varpi) + \cdots,
\end{aligned} \tag{8.49}
$$

where the terms $\cdots$ do not involve $\mathrm{vort}(\varpi)$. Next, we solve for $\mathrm{vort}(\varpi)$ in terms of the remaining terms in definition (2.16a) and then use the resulting identity to algebraically substitute for the four instances of $\mathrm{vort}(\varpi)$ in Eq. (8.49) (note in particular that the terms in braces on LHS (8.49) are equal to $\mathcal{C}^{\alpha}$). In total, this yields the following equation, where we have placed the terms generated by the algebraic substitution on the first through tenth lines of RHS (8.50):

$$
\begin{aligned}
u^{\kappa} \partial_{\kappa} \mathcal{C}^{\alpha} = \; &\mathcal{C}^{\kappa} \partial_{\kappa} u^{\alpha} - 2(\partial_{\kappa} u^{\kappa}) \mathcal{C}^{\alpha} + u^{\alpha} (u^{\kappa} \partial_{\kappa} u_{\beta}) \mathcal{C}^{\beta} \\
&- c^{-2} \epsilon^{\kappa\beta\gamma\delta} (\partial_{\kappa} u^{\alpha}) u_{\beta} (\partial_{\gamma} h) \varpi_{\delta} + (\theta_{;h} - \theta)(S^{\kappa} \partial_{\kappa} u^{\alpha})(\partial_{\lambda} u^{\lambda}) \\
&+ (\theta_{;h} - \theta)(u^{\kappa} \partial_{\kappa} u^{\alpha})(S^{\lambda} \partial_{\lambda} h) + (\theta - \theta_{;h})(\eta^{-1})^{\kappa\lambda} (\partial_{\kappa} u^{\alpha}) S^{\beta} (\partial_{\lambda} u_{\beta}) \\
&+ 2(\partial_{\kappa} u^{\kappa}) c^{-2} \epsilon^{\alpha\beta\gamma\delta} u_{\beta} (\partial_{\gamma} h) \varpi_{\delta} + 2(\theta - \theta_{;h}) S^{\alpha} (\partial_{\kappa} u^{\kappa})(\partial_{\lambda} u^{\lambda}) \\
&+ 2(\theta - \theta_{;h}) u^{\alpha} (\partial_{\kappa} u^{\kappa})(S^{\lambda} \partial_{\lambda} h) \\
&+ 2(\theta_{;h} - \theta)(\partial_{\kappa} u^{\kappa}) S^{\beta} ((\eta^{-1})^{\alpha\lambda} \partial_{\lambda} u_{\beta}) \\
&- u^{\alpha} (u^{\kappa} \partial_{\kappa} u_{\sigma}) c^{-2} \epsilon^{\sigma\beta\gamma\delta} u_{\beta} (\partial_{\gamma} h) \varpi_{\delta} \\
&+ (\theta_{;h} - \theta) u^{\alpha} (u^{\kappa} \partial_{\kappa} u_{\sigma}) S^{\sigma} (\partial_{\lambda} u^{\lambda}) \\
&+ (\theta_{;h} - \theta) u^{\alpha} (u^{\kappa} \partial_{\kappa} u_{\beta}) u^{\beta} (S^{\lambda} \partial_{\lambda} h) \\
&+ (\theta - \theta_{;h}) u^{\alpha} (u^{\kappa} \partial_{\kappa} u^{\lambda}) S^{\beta} (\partial_{\lambda} u_{\beta}) \\
&- 2(\partial_{\kappa} u^{\kappa}) c^{-2} \epsilon^{\alpha\beta\gamma\delta} u_{\beta} (\partial_{\gamma} h) \varpi_{\delta} \\
&+ c^{-2} \epsilon^{\alpha\beta\gamma\delta} (u^{\kappa} \partial_{\kappa} u_{\beta})(\partial_{\gamma} h) \varpi_{\delta}
\end{aligned}
$$

$$
\begin{aligned}
&+ c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa)\\
&+ c^{-2}(\theta - \theta_{;h})(S^\kappa\partial_\kappa h)((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)\\
&+ c^{-2}(\theta - \theta_{;h})u^\alpha(S^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&+ c^{-2}(\theta_{;h} - \theta)S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&+ c^{-2}(\theta_{;h} - \theta)S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h)\\
&+ c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta\left\{(\partial_\kappa u^\kappa)(\partial_\gamma h) - (\partial_\gamma u^\kappa)(\partial_\kappa h)\right\}\varpi_\delta\\
&+ 2c^{-3}c_{;s}(u^\kappa\partial_\kappa h)\epsilon^{\alpha\beta\gamma\delta}u_\beta S_\gamma\varpi_\delta\\
&+ \frac{1}{H}(\varpi^\kappa\partial_\kappa\varpi^\alpha) - \frac{1}{H}\varpi^\alpha(\varpi^\kappa\partial_\kappa h)\\
&- \frac{1}{H}u^\alpha\varpi^\lambda(\varpi^\kappa\partial_\kappa u_\lambda) + \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa)\\
&- q\epsilon^{\alpha\beta\gamma\delta}u_\beta S_\gamma\varpi^\kappa(\partial_\delta u_\kappa)\\
&- \epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma\varpi^\kappa(\partial_\delta u_\kappa) + (\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma\varpi_\delta\\
&+ (\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)(S^\lambda\partial_\lambda h) + (\theta - \theta_{;h})u^\alpha(u^\kappa\partial_\kappa h)(S^\lambda\partial_\lambda h)\\
&+ (\theta_{;h} - \theta)S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h) + (\theta_{;h} - \theta)S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h)\\
&+ q\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi^\kappa(\partial_\delta u_\kappa)\\
&- q(\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi_\delta\\
&+ q(\theta_{;h} - \theta)((\eta^{-1})^{\kappa\alpha}\partial_\kappa h)S^\lambda S_\lambda + q(\theta_{;h} - \theta)u^\alpha(u^\kappa\partial_\kappa h)S^\lambda S_\lambda\\
&+ q(\theta - \theta_{;h})S^\alpha(S^\kappa\partial_\kappa h)\\
&- \epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma\varpi^\lambda(\partial_\delta u_\lambda) + q\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi^\lambda(\partial_\delta u_\lambda)\\
&- \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa)\\
&- \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma\varpi^\kappa)(\partial_\delta u_\kappa) - \epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa)\\
&- \frac{1}{H}(\varpi^\kappa\partial_\kappa\varpi^\alpha) - \frac{1}{H}\varpi^\alpha(\varpi^\kappa\partial_\kappa h) + \frac{2q}{H}\varpi^\alpha\varpi^\kappa S_\kappa\\
&- \frac{1}{H}\varpi^\alpha\varpi^\lambda(u^\kappa\partial_\kappa u_\lambda) + \frac{1}{H}u^\alpha\varpi^\lambda(\varpi^\kappa\partial_\kappa u_\lambda)\\
&- q\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma u^\kappa)\varpi_\kappa S_\delta\\
&+ \frac{1}{H}\varpi^\alpha\varpi^\lambda(u^\kappa\partial_\kappa u_\lambda)\\
&+ (\theta_h - \theta_{;h;h})S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h)\\
&+ (\theta_h - \theta_{;h;h})S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&+ (\theta_{;h;h} - \theta_h)u^\alpha(S^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h)\\
&+ (\theta_{;h;h} - \theta_h)((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)(S^\lambda\partial_\lambda h)\\
&+ (\theta_{;s} - \theta_{;h;s})S^\alpha(S^\kappa\partial_\kappa h) + (\theta_{;h;s} - \theta_{;s})u^\alpha S_\kappa S^\kappa(u^\lambda\partial_\lambda h)\\
&+ (\theta_{;h;s} - \theta_{;s})S_\kappa S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\theta - \theta_{;h}) S^\alpha (\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) + (\theta_{;h} - \theta)(S^\kappa \partial_\kappa u^\alpha)(u^\lambda \partial_\lambda h) \\
&+ (\theta_h - \theta_{;h;h}) S^\alpha (u^\kappa \partial_\kappa h)(\partial_\lambda u^\lambda) + (\theta - \theta_{;h})(u^\kappa \partial_\kappa S^\alpha)(\partial_\lambda u^\lambda) \\
&+ (\theta_{;h} - \theta) S^\alpha (\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) + (\theta_{;h} - \theta) S^\alpha (u^\kappa \partial_\kappa u^\lambda)(\partial_\lambda h) \\
&+ (\theta_{;h} - \theta) S^\alpha (\partial_\kappa u^\kappa)(u^\lambda \partial_\lambda h) \\
&+ (\theta - \theta_{;h}) q S^\alpha (\partial_\kappa S^\kappa) + (\theta - \theta_{;h}) q_{;h} S^\alpha (S^\kappa \partial_\kappa h) \\
&+ (\theta - \theta_{;h}) q_{;s} S^\alpha S_\kappa S^\kappa \\
&+ (\theta_h - \theta_{;h;h}) u^\alpha (u^\kappa \partial_\kappa h)(S^\lambda \partial_\lambda h) + (\theta - \theta_{;h})(u^\kappa \partial_\kappa u^\alpha)(S^\lambda \partial_\lambda h) \\
&+ (\theta - \theta_{;h}) u^\alpha (u^\kappa \partial_\kappa S^\lambda)(\partial_\lambda h) \\
&+ (\theta_{;h;h} - \theta_h)(u^\kappa \partial_\kappa h) S^\beta ((\eta^{-1})^{\alpha\lambda} \partial_\lambda u_\beta) \\
&+ (\theta_{;h} - \theta)(u^\kappa \partial_\kappa S^\beta)((\eta^{-1})^{\alpha\lambda} \partial_\lambda u_\beta) \\
&+ (\theta - \theta_{;h}) S^\beta ((\eta^{-1})^{\alpha\lambda} \partial_\lambda u_\beta)(u^\kappa \partial_\kappa h) \\
&+ (\theta - \theta_{;h}) S^\beta ((\eta^{-1})^{\alpha\lambda} \partial_\lambda u^\kappa)(\partial_\kappa u_\beta) \\
&+ (\theta - \theta_{;h}) q ((\eta^{-1})^{\alpha\lambda} \partial_\lambda S^\beta) S_\beta + (\theta_{;h} - \theta) q_{;h} ((\eta^{-1})^{\alpha\lambda} \partial_\lambda h) S_\kappa S^\kappa \\
&+ (\theta_{;h} - \theta) q_{;s} S^\alpha S_\kappa S^\kappa + 2(\theta_{;h} - \theta) q ((\eta^{-1})^{\alpha\lambda} \partial_\lambda S^\kappa) S_\kappa \\
&+ (\theta - \theta_{;h})(\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda S^\alpha) + (\theta - \theta_{;h})(u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda S^\alpha) \\
&+ (\theta_{;h} - \theta) u^\alpha (u^\kappa \partial_\kappa h)(\partial_\lambda S^\lambda) + (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa} \partial_\kappa h)(\partial_\lambda S^\lambda) \\
&+ (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa} \partial_\kappa h) u_\beta (u^\lambda \partial_\lambda S^\beta) \\
&+ (\theta - \theta_{;h}) u^\alpha (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h) u_\beta (\partial_\lambda S^\beta). \qquad (8.50)
\end{aligned}
$$

Next, we reorganize the terms on RHS (8.50) to obtain the equation

$$
u^\kappa \partial_\kappa \mathcal{C}^\alpha = \mathcal{C}^\kappa \partial_\kappa u^\alpha - 2(\partial_\kappa u^\kappa)\mathcal{C}^\alpha + u^\alpha (u^\kappa \partial_\kappa u_\beta)\mathcal{C}^\beta + \sum_{i=1}^{21} \mathscr{Q}_i + \mathscr{L}, \qquad (8.51)
$$

where

$$
\mathscr{Q}_1 := -2\epsilon^{\alpha\beta\gamma\delta} u_\beta (\partial_\gamma \varpi^\kappa)(\partial_\delta u_\kappa), \qquad (8.52)
$$

$$
\mathscr{Q}_2 := (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa} \partial_\kappa h) \partial_\lambda S^\lambda + (\theta - \theta_{;h})(\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda S^\alpha), \qquad (8.53)
$$

$$
\mathscr{Q}_3 := (\theta_{;h} - \theta) u^\alpha (u^\kappa \partial_\kappa h)(\partial_\lambda S^\lambda) + (\theta - \theta_{;h}) u^\alpha (u^\kappa \partial_\kappa S^\lambda)(\partial_\lambda h), \qquad (8.54)
$$

$$
\begin{aligned}
\mathscr{Q}_4 :=& (\theta_h - \theta_{;h;h}) S^\alpha (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h) \\
&+ (\theta_h - \theta_{;h;h}) S^\alpha (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) + (\theta_h - \theta_{;h;h}) S^\alpha (u^\kappa \partial_\kappa h)(\partial_\lambda u^\lambda),
\end{aligned} \qquad (8.55)
$$

$$
\begin{aligned}
\mathscr{Q}_5 :=& (\theta_{;h;h} - \theta_h)(\eta^{-1})^{\alpha\kappa} (\partial_\kappa h)(S^\lambda \partial_\lambda h) \\
&+ (\theta_{;h;h} - \theta_h)(u^\kappa \partial_\kappa h) S^\beta ((\eta^{-1})^{\alpha\lambda} \partial_\lambda u_\beta),
\end{aligned} \qquad (8.56)
$$

$$
\begin{aligned}
\mathscr{Q}_6 :=& c^{-2} (\theta_{;h} - \theta) S^\alpha (u^\kappa \partial_\kappa h)(u^\lambda \partial_\lambda h) \\
&+ c^{-2} (\theta_{;h} - \theta) S^\alpha (\eta^{-1})^{\kappa\lambda} (\partial_\kappa h)(\partial_\lambda h) + (\theta - \theta_{;h}) S^\alpha (\partial_\kappa u^\kappa)(\partial_\lambda u^\lambda),
\end{aligned} \qquad (8.57)
$$

$$\mathscr{Q}_7 := (\theta_{;h} - \theta)S^\alpha(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h) + (\theta_{;h} - \theta)S^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)(\partial_\lambda h) + (\theta_{;h} - \theta)S^\alpha(u^\kappa\partial_\kappa u^\lambda)(\partial_\lambda h), \quad (8.58)$$

$$\mathscr{Q}_8 := (\theta_{;h} - \theta)S^\alpha(\partial_\kappa u^\lambda)(\partial_\lambda u^\kappa) + (\theta - \theta_{;h})S^\alpha(\partial_\kappa u^\kappa)(\partial_\lambda u^\lambda), \quad (8.59)$$

$$\mathscr{Q}_9 := (\theta_{;h} - \theta)(S^\kappa\partial_\kappa u^\alpha)(\partial_\lambda u^\lambda) + (\theta_{;h} - \theta)(u^\kappa\partial_\kappa u^\alpha)(S^\lambda\partial_\lambda h) + (\theta - \theta_{;h})((\eta^{-1})^{\kappa\lambda}\partial_\kappa u^\alpha)S^\beta(\partial_\lambda u_\beta), \quad (8.60)$$

$$\mathscr{Q}_{10} := (\theta_{;h} - \theta)(S^\kappa\partial_\kappa u^\alpha)(u^\lambda\partial_\lambda h) + (\theta - \theta_{;h})(u^\kappa\partial_\kappa u^\alpha)(S^\lambda\partial_\lambda h), \quad (8.61)$$

$$\mathscr{Q}_{11} := (\theta - \theta_{;h})(u^\kappa\partial_\kappa S^\alpha)(\partial_\lambda u^\lambda) + (\theta - \theta_{;h})S^\beta((\eta^{-1})^{\alpha\lambda}\partial_\lambda u^\kappa)(\partial_\kappa u_\beta), \quad (8.62)$$

$$\mathscr{Q}_{12} := 2(\theta_{;h} - \theta)(\partial_\kappa u^\kappa)(\eta^{-1})^{\alpha\lambda}S^\beta(\partial_\lambda u_\beta) + c^{-2}(\theta - \theta_{;h})(S^\kappa\partial_\kappa h)((\eta^{-1})^{\alpha\lambda}\partial_\lambda h) + (\theta_{;h} - \theta)(u^\kappa\partial_\kappa S^\beta)((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta), \quad (8.63)$$

$$\mathscr{Q}_{13} := (\theta - \theta_{;h})u^\alpha(\partial_\kappa u^\kappa)(S^\lambda\partial_\lambda h) + (\theta - \theta_{;h})u^\alpha(u^\kappa\partial_\kappa u^\lambda)S^\beta(\partial_\lambda u_\beta), \quad (8.64)$$

$$\mathscr{Q}_{14} := (\theta - \theta_{;h})u^\alpha(\partial_\kappa u^\kappa)(S^\lambda\partial_\lambda h) + (\theta - \theta_{;h})u^\alpha(u^\kappa\partial_\kappa h)(S^\lambda\partial_\lambda h) + (\theta - \theta_{;h})u^\alpha(\eta^{-1})^{\kappa\lambda}(\partial_\kappa h)u_\beta(\partial_\lambda S^\beta), \quad (8.65)$$

$$\mathscr{Q}_{15} := (\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}(\partial_\beta h)u_\gamma\varpi_\delta + c^{-2}\epsilon^{\alpha\beta\gamma\delta}(u^\kappa\partial_\kappa u_\beta)(\partial_\gamma h)\varpi_\delta, \quad (8.66)$$

$$\mathscr{Q}_{16} := -c^{-2}u^\alpha\epsilon^{\sigma\beta\gamma\delta}(u^\kappa\partial_\kappa u_\sigma)u_\beta(\partial_\gamma h)\varpi_\delta, \quad (8.67)$$

$$\mathscr{Q}_{17} := -c^{-2}\epsilon^{\kappa\beta\gamma\delta}(\partial_\kappa u^\alpha)u_\beta(\partial_\gamma h)\varpi_\delta + (c^{-2} + 2)\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma h)\varpi^\kappa(\partial_\delta u_\kappa) + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_\beta\varpi_\delta\left\{(\partial_\kappa u^\kappa)(\partial_\gamma h) - (\partial_\gamma u^\kappa)(\partial_\kappa h)\right\}, \quad (8.68)$$

$$\mathscr{Q}_{18} := (\theta - \theta_{;h})((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)(S^\lambda\partial_\lambda h) + (\theta_{;h} - \theta)((\eta^{-1})^{\alpha\kappa}\partial_\kappa h)u_\beta(u^\lambda\partial_\lambda S^\beta), \quad (8.69)$$

$$\mathscr{Q}_{19} := (\theta_{;h} - \theta)u^\alpha(u^\kappa\partial_\kappa u_\sigma)S^\sigma(\partial_\lambda u^\lambda) + c^{-2}(\theta - \theta_{;h})u^\alpha(S^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda h), \quad (8.70)$$

$$\mathscr{Q}_{20} := (\theta - \theta_{;h})S^\beta((\eta^{-1})^{\alpha\lambda}\partial_\lambda u_\beta)(u^\kappa\partial_\kappa h) + (\theta - \theta_{;h})(u^\kappa\partial_\kappa h)(u^\lambda\partial_\lambda S^\alpha), \quad (8.71)$$

$$\mathscr{Q}_{21} := (\theta_{;h} - \theta)u^\alpha(u^\kappa\partial_\kappa u_\beta)(u^\beta S^\lambda\partial_\lambda h), \quad (8.72)$$

and

$$\mathscr{L} := -\frac{2}{H}\varpi^\alpha(\varpi^\kappa\partial_\kappa h) + \frac{2q}{H}\varpi^\alpha\varpi^\kappa S_\kappa + 2c^{-3}c_{;s}(u^\kappa\partial_\kappa h)\epsilon^{\alpha\beta\gamma\delta}u_\beta S_\gamma\varpi_\delta - q\epsilon^{\alpha\beta\gamma\delta}u_\beta S_\gamma\varpi^\kappa(\partial_\delta u_\kappa) + 2q\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi^\kappa(\partial_\delta u_\kappa) - q(\partial_\kappa u^\kappa)\epsilon^{\alpha\beta\gamma\delta}S_\beta u_\gamma\varpi_\delta$$

$$
\begin{aligned}
&+ q(\theta_{;h} - \theta)((\eta^{-1})^{\kappa\alpha}\partial_\kappa h)S^\lambda S_\lambda + q(\theta_{;h} - \theta)u^\alpha(u^\kappa\partial_\kappa h)S^\lambda S_\lambda \\
&+ q(\theta - \theta_{;h})S^\alpha(S^\kappa\partial_\kappa h) \\
&- q\epsilon^{\alpha\beta\gamma\delta}u_\beta(\partial_\gamma u^\kappa)\varpi_\kappa S_\delta \\
&+ (\theta_{;s} - \theta_{;h;s})S^\alpha(S^\kappa\partial_\kappa h) + (\theta_{;h;s} - \theta_{;s})u^\alpha S_\kappa S^\kappa(u^\lambda\partial_\lambda h) \\
&+ (\theta_{;h;s} - \theta_{;s})S_\kappa S^\kappa((\eta^{-1})^{\alpha\lambda}\partial_\lambda h) \\
&+ (\theta - \theta_{;h})qS^\alpha(\partial_\kappa S^\kappa) + (\theta - \theta_{;h})q_{;h}S^\alpha(S^\kappa\partial_\kappa h) + (\theta - \theta_{;h})q_{;s}S^\alpha S_\kappa S^\kappa \\
&+ (\theta - \theta_{;h})q((\eta^{-1})^{\alpha\lambda}\partial_\lambda S^\beta)S_\beta + (\theta_{;h} - \theta)q_{;h}((\eta^{-1})^{\alpha\lambda}\partial_\lambda h)S_\kappa S^\kappa \\
&+ (\theta_{;h} - \theta)q_{;s}S^\alpha S_\kappa S^\kappa + 2(\theta_{;h} - \theta)q((\eta^{-1})^{\alpha\lambda}\partial_\lambda S^\kappa)S_\kappa. \qquad (8.73)
\end{aligned}
$$

Note that the terms on RHSs (8.52)–(8.72) are precisely quadratic in the first-order derivatives of the solution variables $(h, u^\alpha, \varpi^\alpha, S^\alpha)_{\alpha=0,1,2,3}$ while the terms on RHS (8.73) are at most linear in the derivatives of the solution variables. We will now show that $\mathscr{Q}_1$, $\mathscr{Q}_2$, $\cdots$, $\mathscr{Q}_{21}$ can be expressed as null forms or terms that are at most linear in the derivatives of the solution variables. To this end, we simply use (8.3a)–(8.3p) to algebraically substitute for $\mathscr{Q}_2$, $\mathscr{Q}_4$, $\mathscr{Q}_5$, $\mathscr{Q}_6$, $\mathscr{Q}_7$, $\mathscr{Q}_9$, $\mathscr{Q}_{11}$, $\mathscr{Q}_{12}$, $\mathscr{Q}_{13}$, $\mathscr{Q}_{14}$, $\mathscr{Q}_{15}$, $\mathscr{Q}_{16}$, $\mathscr{Q}_{18}$, $\mathscr{Q}_{19}$, $\mathscr{Q}_{20}$, and $\mathscr{Q}_{21}$ (we do not substitute for $\mathscr{Q}_1$, $\mathscr{Q}_3$, $\mathscr{Q}_8$, $\mathscr{Q}_{10}$, and $\mathscr{Q}_{17}$ since these terms are already manifestly linear combinations of null forms). Following this substitution, there are only two kinds of terms on RHS (8.51): null forms and terms that are at most linear in the derivatives of the solution variables. We now place all null forms on RHS (8.42) except for null forms that involve the derivatives of ϖ or S; these null forms we place directly on RHS (8.41). We then place all terms that are linear in $\mathcal{C}$, linear in $\mathcal{D}$, linear in the first-order derivatives of ϖ, or linear in the first-order derivatives of S directly on RHS (8.41). Finally, we place all remaining terms, which are at most linear in the derivatives of the solution variables and do not depend on the derivatives of ϖ or S, on RHS (8.43). This completes the proof of the proposition. □

9. Local Well-Posedness with Additional Regularity for the Vorticity and Entropy

Our main goal in this section is to prove Theorem 9.12, which is a local well-posedness result for the relativistic Euler equations based on our new formulation of the equations, that is, based on the equations of Theorem 3.1. The main new feature of Theorem 9.12 compared to standard local well-posedness results for the relativistic Euler equations (see Theorem 9.10 for a statement of standard local well-posedness) is that it yields an extra degree of differentiability for the vorticity and the entropy, assuming that the initial vorticity and entropy enjoy the same extra differentiability. We stress that this gain in regularity holds even though the logarithmic enthalpy and four-velocity do not generally enjoy the same gain. As we described in Sect. 1.2, this extra regularity for the vorticity and the entropy is essential for the study of shocks in more than one spatial dimension.

For convenience, instead of proving local well-posedness for the relativistic Euler equations on the standard Minkowski spacetime background, we instead consider the spacetime background $(\mathbb{R}\times\mathbb{T}^3, \eta)$, where the "spatial manifold" $\mathbb{T}^3$ is the standard three-dimensional torus and, relative to standard coordinates on $\mathbb{R}\times\mathbb{T}^3$, $\eta_{\alpha\beta} :=$ diag$(-1,1,1,1)$ is the standard Minkowski metric. Thus, strictly speaking, in this section, η denotes a tensor on a different manifold compared to the rest of the paper, but this minor change has no substantial bearing on the discussion. In particular, the relativistic Euler equations on $(\mathbb{R}\times\mathbb{T}^3, \eta)$ take the same form that they take in Theorem 3.1. The advantage of the compact spatial topology is that it allows for a simplified approach to some technical aspects of the proof of local well-posedness. However, the arguments that we give in this section feature all of the main ideas needed to prove local well-posedness on the standard Minkowski spacetime background (in which the spacetime manifold is diffeomorphic to $\mathbb{R}^{1+3}$).

9.1. Notation, Norms, and Basic Tools from Analysis

9.1.1. Notation. Throughout this section, $\{x^\alpha\}_{\alpha=0,1,2,3}$ denote standard rectangular coordinates on $\mathbb{R}\times\mathbb{T}^3$, where $\{x^a\}_{a=1,2,3}$ are standard local coordinates on $\mathbb{T}^3$, and we often use the alternate notation $t := x^0$. Note that even though $\{x^a\}_{a=1,2,3}$ are only locally defined on $\mathbb{T}^3$, the coordinate partial derivative vectorfields $\{\partial_a\}_{a=1,2,3}$ can be extended to a smooth global frame on $\mathbb{T}^3$; by a slight abuse of notation, we will denote the globally defined "spatial" frame by $\{\partial_a\}_{a=1,2,3}$, and the corresponding globally defined "spacetime frame" by $\{\partial_\alpha\}_{\alpha=0,1,2,3}$. Also, we often use the alternate partial derivative notation $\partial_t := \partial_0$.

$$\Sigma_t := \{(t,x) \mid x \in \mathbb{T}^3\} \tag{9.1}$$

denotes the standard flat constant-time hypersurface.

Throughout Sect. 9, we use the same conventions for lowering and raising indices stated in Sect. 2.1, i.e., we lower and raise indices with the Minkowski metric and its inverse. Note that for Latin "spatial" indices, this is equivalent to lowering and raising via the Euclidean metric $\delta_{ij} =$ diag$(1,1,1)$ and its inverse $\delta^{ij} =$ diag$(1,1,1)$. Finally, we note that we sometimes identify the Euclidean metric or its inverse with the Kronecker delta.

To each "spatial multi-index" $\vec{I} = (\iota_1,\iota_2,\iota_3)$, where the ι_a are non-negative integers, we associate the spatial differential operator $\partial_{\vec{I}} := \partial_1^{\iota_1}\partial_2^{\iota_2}\partial_3^{\iota_3}$. Note that $\partial_{\vec{I}}$ is an operator of order $|\vec{I}| := \iota_1+\iota_2+\iota_3$.

If V is a spacetime vectorfield or a one-form, then $\underline{V}$ denotes the η-orthogonal projection of V onto Σ_t, that is, the "spatial part" of V. For example, $\underline{\varpi}$ is the vectorfield on Σ_t with rectangular components $\underline{\varpi}^i := \varpi^i$ for $i = 1,2,3$. Moreover, we use the notation

$$^{(3)}\mathrm{curl}^i(W) := \varepsilon^{ijk}\partial_j W_k \tag{9.2}$$

to denote the standard Euclidean curl operator acting on one-forms on Σ_t, where ϵ^{ijk} is the fully antisymmetric symbol normalized by $\epsilon^{123} = 1$.

9.1.2. Norms.

Definition 9.1 (*Lebesgue and Sobolev norms*). We define the following Lebesgue norms for scalar functions f:

$$\|f\|_{L^\infty(\mathbb{T}^3)} := \text{ess sup}_{x\in\mathbb{T}^3}|f(x)|, \tag{9.3}$$

$$\|f\|_{L^2(\mathbb{T}^3)} := \left\{\int_{\mathbb{T}^3} f^2(x)\,\mathrm{d}x\right\}^{1/2}, \tag{9.4}$$

where in the rest of Sect. 9, $\mathrm{d}x := \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$ denotes the standard volume form on $\mathbb{T}^3$ induced by the Euclidean metric $\operatorname{diag}(1,1,1)$.

Remark 9.2 (*Extending the definitions of the norms from* $\mathbb{T}^3$ *to* Σ_t). In our proof of local well-posedness, we will use norms in which the manifold $\mathbb{T}^3$ from Definition 9.1 is replaced with the constant time slice $\Sigma_t = \{t\}\times\mathbb{T}^3$, which is diffeomorphic to $\mathbb{T}^3$. We will not explicitly define these norms along Σ_t since their definitions are obvious analogs of the ones appearing in Definition 9.1. For example, $\|f\|_{L^2(\Sigma_t)} := \left\{\int_{\Sigma_t} f^2(t,x)\,\mathrm{d}x\right\}^{1/2}$, which is also equal to $\left\{\int_{\mathbb{T}^3} f^2(t,x)\,\mathrm{d}x\right\}^{1/2}$. Here, we are using that the volume form induced by the Minkowski metric on Σ_t equals $\mathrm{d}x$. Similar remarks apply to other norms on $\mathbb{T}^3$ introduced later in this subsubsection.

We define the following Sobolev norms for integers $r \geq 0$:

$$\|f\|_{H^r(\mathbb{T}^3)} := \left\{\sum_{|\vec{I}|\leq r}\|\partial_{\vec{I}}f\|^2_{L^2(\mathbb{R}^3)}\right\}^{1/2}, \tag{9.5a}$$

$$\|f\|_{\dot{H}^r(\mathbb{T}^3)} := \left\{\sum_{|\vec{I}|= r}\|\partial_{\vec{I}}f\|^2_{L^2(\mathbb{R}^3)}\right\}^{1/2}. \tag{9.5b}$$

If $r\in\mathbb{R}$ is not an integer, then we define[26]

$$\|f\|_{H^r(\mathbb{T}^3)} := \left\{\sum_{(k_1,k_2,k_3)\in\mathbb{Z}^3}(1+|k|^2)^r\left|\hat{f}(k_1,k_2,k_3)\right|^2\right\}^{1/2}, \tag{9.6}$$

where $\hat{f}(k_1,k_2,k_3) := \int_{\mathbb{T}^3} f(x)e^{-2\pi i\sum_{a=1}^3 x^a k_a}\,\mathrm{d}x$ is the spatial Fourier transform of f and $|k|^2 := \sum_{a=1}^3 k_a^2$.

If $U = (U_1,\ldots,U_m)$ is an array of scalar-valued functions and $\|\cdot\|$ denotes any of the norms introduced in this subsubsection, then we define

$$\|U\| := \sum_{a=1}^m \|U_a\|. \tag{9.7}$$

[26] As is well known, when r is an integer, RHS (9.6) defines a norm that is equivalent to the norm defined in (9.5a).

Definition 9.3 (*Some additional function spaces*). If $\mathfrak{B}$ is a Banach space with norm $\|\cdot\|_{\mathfrak{B}}$ and $r \geq 0$ is an integer, then $C^r([0,T],\mathfrak{B})$ denotes the space of r-times continuously differentiable functions from $[0,T]$ to $\mathfrak{B}$. We omit the superscript when $r=0$. We denote the corresponding norm of an element f of this space by $\|f\|_{C^r([0,T],\mathfrak{B})} := \max_{t\in[0,T]} \sum_{k=0}^{r} \|f^{(k)}(t)\|_{\mathfrak{B}}$, where $f^{(k)}$ denotes the kth derivative of f with respect to t.

$L^\infty([0,T],\mathfrak{B})$ denotes the space of functions from $[0,T]$ to $\mathfrak{B}$ that are essentially bounded over the interval $[0,T]$. We denote the corresponding norm of an element f of this space by $\|f\|_{L^\infty([0,T],\mathfrak{B})} := \text{ess sup}_{t\in[0,T]}\|f(t)\|_{\mathfrak{B}}$.

$C^r(\mathbb{T}^3)$ denotes the space of functions on $\mathbb{T}^3$ that are r-times continuously differentiable. We omit the superscript when $r=0$. We denote the corresponding norm of an element f of this space by

$$\|f\|_{C^r(\mathbb{T}^3)} := \sum_{|\vec{I}|\leq r} \max_{x\in\mathbb{T}^3} |\partial_{\vec{I}} f(x)|.$$

We now fix, for the rest of Sect. 9, an integer N subject to

$$N \geq 3. \tag{9.8}$$

9.1.3. Basic Analytical Tools. In our analysis, we will rely on the following standard results; see, e.g., [1,26,38] for proofs.

Lemma 9.4 (Sobolev embedding, product, difference, and interpolation estimates). *If $r > 3/2$, then $H^r(\mathbb{T}^3)$ continuously embeds into $C(\mathbb{T}^3)$, and there exists a constant $C_r > 0$ such that the following estimate holds for $v \in H^r(\mathbb{T}^3)$:*

$$\|v\|_{C(\mathbb{T}^3)} \leq C_r \|v\|_{H^r(\mathbb{T}^3)}. \tag{9.9}$$

Let $r \geq 0$ be an integer and let $v := (v_1,\ldots,v_A)$ and $w := (w_1,\ldots,w_B)$ be finite-dimensional arrays of real-valued functions on $\mathbb{T}^3$ such that $v_a \in \dot{H}^r(\mathbb{T}^3)\cap C(\mathbb{T}^3)$ for $1\leq a\leq A$ and $w_b \in C(\mathbb{T}^3)$ $1 \leq b \leq B$. Let

$$\mathcal{I}_r := \left\{(\vec{I}_1,\ldots,\vec{I}_A) \mid \sum_{a=1}^{A} |\vec{I}_a| = r\right\}. \tag{9.10}$$

Assume that $w(\mathbb{T}^3) \subset int\mathscr{K}$, where $\mathscr{K}$ is a compact subset of $\mathbb{R}^B$, and let f *be a smooth real-valued function on an open subset of $\mathbb{R}^B$ containing $\mathscr{K}$. Then the following estimate holds:*

$$\max_{(\vec{I}_1,\ldots,\vec{I}_A)\in\mathcal{I}_r} \left\| \mathrm{f}(w) \prod_{a=1}^{A} \partial_{\vec{I}_a} v_a \right\|_{L^2(\mathbb{T}^3)} \leq C_{\mathrm{f},\mathscr{K},r} \sum_{a=1}^{A} \|v_a\|_{\dot{H}^r(\mathbb{T}^3)} \prod_{b\neq a} \|v_b\|_{C(\mathbb{T}^3)}. \tag{9.11}$$

Moreover, under the same assumptions stated in the previous paragraph, if $(\vec{I}_1,\ldots,\vec{I}_A) \in \mathcal{I}_r$, then the map $(v,w) \to \mathrm{f}(w)\prod_{a=1}^{A} \partial_{\vec{I}_a} v_a$ is continuous from $\left(\dot{H}^r(\mathbb{T}^3)\cap C(\mathbb{T}^3)\right)^A \times \left(C(\mathbb{T}^3)\right)^B$ to $L^2(\mathbb{T}^3)$. In particular, let $\delta = \delta_w > 0$

be such that the following holds[27]*:if* $d(p, w(\mathbb{T}^3)) < \delta$, $d(q, w(\mathbb{T}^3)) < \delta$, *and* $d(p,q) < \delta$, *where* d *is the standard Euclidean distance function on* $\mathbb{R}^B$, *then the straight-line segment joining* p *to* q *is contained in int* $\mathscr{K}$. *Then if* (v, w) *and* $(\widetilde{v}, \widetilde{w})$ *are two array pairs of the type described in the previous paragraph such that* $\|w - \widetilde{w}\|_{C(\mathbb{T}^3)} \leq \delta$, *and if* $r > 3/2$, *then the following estimate holds (where the function* f *is assumed to be the same in both appearances on LHS* (9.12) *and* $\mathcal{I}_r$ *is defined by* (9.10)*):*

$$\max_{(\vec{I}_1,\ldots,\vec{I}_A)\in\mathcal{I}_r} \left\| \mathrm{f}(w) \prod_{a=1}^{A} \partial_{\vec{I}_a} v_a - \mathrm{f}(\widetilde{w}) \prod_{a=1}^{A} \partial_{\vec{I}_a} \widetilde{v}_a \right\|_{L^2(\mathbb{T}^3)} \leq C_{\mathrm{f},\mathscr{K},\|v\|_{H^r(\mathbb{T}^3)},\|\widetilde{v}\|_{H^r(\mathbb{T}^3)},A,r} \left\{ \|v - \widetilde{v}\|_{H^r(\mathbb{T}^3)} + \|w - \widetilde{w}\|_{C(\mathbb{T}^3)} \right\}. \tag{9.12}$$

Furthermore, if $r > 3/2$ *and* $v_a \in H^r(\mathbb{T}^3)$ *for* $a = 1, 2$, *then* $v_1 v_2 \in H^r(\mathbb{T}^3)$, *and there exists a constant* $C_r > 0$ *such that*

$$\|v_1 v_2\|_{H^r(\mathbb{T}^3)} \leq C_r \|v_1\|_{H^r(\mathbb{T}^3)} \|v_2\|_{H^r(\mathbb{T}^3)}, \tag{9.13}$$

and function multiplication $(v_1, v_2) \to v_1 v_2$ *is a continuous map from* $H^r(\mathbb{T}^3) \times H^r(\mathbb{T}^3)$ *to* $H^r(\mathbb{T}^3)$.

Finally, if $0 \leq s \leq r$ *and* $v \in H^r(\mathbb{T}^3)$, *then there exists a constant* $C_{r,s} > 0$ *such that*

$$\|v\|_{H^s(\mathbb{T}^3)} \leq C_{r,s} \|v\|_{L^2(\mathbb{T}^3)}^{1-\frac{s}{r}} \|v\|_{H^r(\mathbb{T}^3)}^{\frac{s}{r}}. \tag{9.14}$$

Remark 9.5 (*The same estimates hold along* Σ_t). All of the results of Lemma 9.4 hold verbatim if we replace $\mathbb{T}^3$ by Σ_t throughout.

9.1.4. An L^2-in-time Continuity Result for Transport Equations. We will use the following simple technical result in our proof of local well-posedness.

Lemma 9.6 (An L^2-in-time continuity result for transport equations). *Let* $T > 0$. *Assume that* $\mathscr{F} \in L^\infty\big([0,T], L^2(\mathbb{T}^3)\big)$, *and let* f *be the solution to the inhomogeneous transport equation initial value problem*

$$u^\alpha \partial_\alpha f = \mathscr{F}, \tag{9.15}$$

$$f|_{\Sigma_0} := \mathring{f} \in L^2(\Sigma_0). \tag{9.16}$$

Assume further that $u^\alpha \in L^\infty\big([0,T], C^1(\mathbb{T}^3)\big)$ *for* $\alpha = 0, 1, 2, 3$. *Then*

$$f \in C\big([0,T], L^2(\mathbb{T}^3)\big). \tag{9.17}$$

Proof. We will prove right continuity at $t = 0$; continuity at any other time $t \in (0, T]$ could be proved using similar arguments. More precisely, we will show that

$$\lim_{t \downarrow 0} \left\| f(t, \cdot) - \mathring{f} \right\|_{L^2(\mathbb{T}^3)} = 0. \tag{9.18}$$

[27] Such a $\delta > 0$ exists due to the compactness of $w(\mathbb{T}^3)$ and $\mathscr{K}$, where the compactness of $w(\mathbb{T}^3)$ follows from the assumption that the v_a are continuous.

To proceed, we let $\{\mathring{f}_k\}_{k=1}^{\infty} \subset C^{\infty}(\mathbb{T}^3)$ be a sequence of smooth functions such that

$$\|\mathring{f} - \mathring{f}_k\|_{L^2(\Sigma_0)} \leq \frac{1}{k}. \tag{9.19}$$

Note that

$$u^{\alpha}\partial_{\alpha}(f - \mathring{f}_k) = -u^a\partial_a\mathring{f}_k + \mathscr{F}. \tag{9.20}$$

Hence, a standard integration by parts argument based on the divergence identity

$$\begin{aligned}\partial_t\left\{(f - \mathring{f}_k)\right\}^2 = &\left\{\partial_a\left(\frac{u^a}{u^0}\right)\right\}(f - \mathring{f}_k)^2 \\ &+ 2\frac{(f - \mathring{f}_k)}{u^0}\left\{-u^a\partial_a\mathring{f}_k + \mathscr{F}\right\} \\ &- \partial_a\left\{\left(\frac{u^a}{u^0}\right)(f - \mathring{f}_k)^2\right\}\end{aligned} \tag{9.21}$$

yields that for $0 \leq t \leq T$, we have

$$\begin{aligned}\|f - \mathring{f}_k\|^2_{L^2(\Sigma_t)} = &\|\mathring{f} - \mathring{f}_k\|^2_{L^2(\Sigma_0)} \\ &+ \int_{\tau=0}^{t}\int_{\Sigma_\tau}\left\{\partial_a\left(\frac{u^a}{u^0}\right)\right\}(f - \mathring{f}_k)^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &+ 2\int_{\tau=0}^{t}\int_{\Sigma_\tau}\frac{(f - \mathring{f}_k)}{u^0}\left\{-u^a\partial_a\mathring{f}_k + \mathscr{F}\right\}\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{9.22}$$

In particular, from (9.19), (9.22), our assumptions on $\mathscr{F}$ and u^{α}, and Young's inequality, we find that if $0 \leq t \leq T$, then there is a constant C_T (independent of k) such that

$$\begin{aligned}\|f - \mathring{f}_k\|^2_{L^2(\Sigma_t)} \leq &\frac{1}{k^2} + C_T\int_{\tau=0}^{t}\left\{1 + \|\mathring{f}_k\|^2_{H^1(\Sigma_0)}\right\}\mathrm{d}\tau \\ &+ C_T\int_{\tau=0}^{t}\|f - \mathring{f}_k\|^2_{L^2(\Sigma_\tau)}\,\mathrm{d}\tau.\end{aligned} \tag{9.23}$$

From (9.23) and Gronwall's inequality, we deduce (allowing C_T to vary from line to line in the rest of the proof) that if $0 \leq t \leq T$, then the following inequality holds:

$$\|f - \mathring{f}_k\|^2_{L^2(\Sigma_t)} \leq \left\{\frac{1}{k^2} + C_T t\left(1 + \|\mathring{f}_k\|^2_{H^1(\Sigma_0)}\right)\right\}\exp(C_T t). \tag{9.24}$$

From (9.24), (9.19), and the triangle inequality, it follows that

$$\lim_{t\to 0^+}\sup_{0\leq\tau\leq t}\|f - \mathring{f}\|_{L^2(\Sigma_\tau)} \leq \frac{2}{k}. \tag{9.25}$$

Finally, allowing $k \to \infty$ in (9.25), we conclude (9.18). We have therefore proved the lemma. □

9.2. The Regime of Hyperbolicity

Our proof of well-posedness relies on a standard assumption, namely that the solution lies in the interior of the region of state space where the equations are hyperbolic without degeneracy. This notion is precisely captured by the next definition.

Definition 9.7 (*Regime of hyperbolicity*). We define the regime of hyperbolicity $\mathcal{H}$ to be the following subset of state-space:

$$\mathcal{H} := \left\{ (h, s, u^1, u^2, u^3) \in \mathbb{R}^5 \mid 0 < c(h, s) \leq 1 \right\}. \tag{9.26}$$

9.3. Standard Local Well-Posedness

Our principal goal in this subsection is to state Theorem 9.12, which is our main local well-posedness result exhibiting the gain in regularity for the vorticity and entropy. Most aspects of the theorem are standard. We summarize these standard aspects in Theorem 9.10, which will serve as a precursor to our proof of Theorem 9.12.

Remark 9.8 (*Some non-standard aspects of Theorem* 9.12). One of the non-standard aspects of Theorem 9.12 is that it shows the continuous time dependence of the top-order derivatives of ϖ and s in the norm $\| \cdot \|_{L^2(\Sigma_t)}$. The proof relies on some results that are not easy to locate in the literature, tied in part to the fact that the required estimates are of elliptic–hyperbolic type. In our proof of Theorem 9.12, we will show how to obtain these top-order time-continuity results. A second non-standard aspect of Theorem 9.12 is that the transport-div-curl systems [specifically (3.9a)–(3.9b) and (3.11a)–(3.11b)] leading to the gain in regularity for ϖ and s involve *spacetime* divergence and curl operators. Hence, additional arguments are needed to obtain the needed *spatial* elliptic estimates along Σ_t; the key ingredients in this vein are provided by the identity (9.34) and Lemma 9.20.

Remark 9.9 (*The "fundamental" initial data*). In the rest of Sect. 9, we view $\mathring{h} := h|_{\Sigma_0}$, $\mathring{s} := s|_{\Sigma_0}$, and $\mathring{u}^i := u^i|_{\Sigma_0}$ to be the "fundamental" initial data in the following sense: with the help of the relativistic Euler equations (2.17)–(2.19) + (2.20), along Σ_0, all of the other quantities that are relevant for our analysis can be expressed in term of the fundamental initial data; see Lemma 9.17.

Theorem 9.10 (Standard local Well-Posedness). *Let $\mathring{h} := h|_{\Sigma_0}$, $\mathring{s} := s|_{\Sigma_0}$, and $\mathring{u}^i := u^i|_{\Sigma_0}$ be initial data*[28] *for the relativistic Euler equations* (2.17)–(2.19) + (2.20). *Assume that for some integer $N \geq 3$, we have*

$$\mathring{h},\ \mathring{s},\ \mathring{u}^i \in H^N(\Sigma_0). \tag{9.27}$$

Assume moreover that there is a compact subset $\mathfrak{K} \subset int\mathcal{H}$ (where $int\mathcal{H}$ is the interior of $\mathcal{H}$) such that for all $p \in \Sigma_0$, we have

$$(\mathring{h}(p), \mathring{s}(p), \mathring{u}^1(p), \mathring{u}^2(p), \mathring{u}^3(p)) \in int\mathfrak{K}.$$

[28]The datum $u^0|_{\Sigma_0}$ is determined from the other data by virtue of the constraint (2.20).

Then there exists a time $T > 0$ depending only on[29] $\mathfrak{K}$, $\|\mathring{h}\|_{H^3(\Sigma_0)}$, $\|\mathring{s}\|_{H^3(\Sigma_0)}$, *and* $\|\mathring{u}^i\|_{H^3(\Sigma_0)}$*, such that a unique classical solution* $(h, s, u^\alpha, \varpi^\alpha)$ *exists on the slab* $[0,T] \times \mathbb{T}^3$ *and satisfies* $(h(p), s(p), u^1(p), u^2(p), u^3(p)) \in int\mathfrak{K}$ *for* $p \in [0,T] \times \mathbb{T}^3$*. Moreover, the solution depends continuously on the initial data,*[30] *and its components relative to the standard coordinates enjoy the following regularity properties:*

$$h,\, s,\, u^\alpha \in C\big([0,T], H^N(\mathbb{T}^3)\big), \tag{9.28a}$$

$$S^\alpha,\, \varpi^\alpha \in C\big([0,T], H^{N-1}(\mathbb{T}^3)\big). \tag{9.28b}$$

Proof. (Discussion of the proof). Theorem 9.10 is standard. Readers can consult, for example, [31] for detailed proofs in the case of the relativistic Euler equations on a family of conformally flat[31] spacetimes. The main step in the proof is deriving a priori energy estimates for linearized versions of a first-order formulation of the equations, such as (2.17)–(2.19) + (2.20). For a first-order formulation that is equivalent (for C^1 solutions) to (2.17)–(2.19) + (2.20), this step was carried out in detail in [31] using the method of energy currents, a technique that originated in the context of the relativistic Euler equations in Christodoulou's foundational work [4] on shock formation. □

Remark 9.11 (C^∞ *data give rise to* C^∞ *solutions*). In view of the Sobolev embedding result (9.9), we see that Theorem 9.10 implies that C^∞ initial data give rise to (local-in-time) C^∞ solutions.

We now state our main local well-posedness theorem. Its proof is located in Sect. 9.7.

Theorem 9.12 (Local well-posedness with improved regularity for the entropy and vorticity). *Assume the hypotheses of Theorem* 9.10*, but in addition to* (9.27)*, assume also that the initial vorticity and entropy enjoy one extra degree of Sobolev regularity. That is, assume that for some integer* $N \geq 3$ *and* $i = 1, 2, 3$*, we have*

$$\mathring{h},\, \mathring{u}^i \in H^N(\Sigma_0), \tag{9.29a}$$

$$\mathring{s},\, \mathring{\varpi}^i \in H^N(\Sigma_0), \tag{9.29b}$$

where ϖ *is defined in* (2.5) *and* $\mathring{\varpi}^i := \varpi|^i_{\Sigma_0}$.

Then the conclusions of Theorem 9.10 *hold, and the solution's components relative to standard coordinates enjoy the following regularity properties for*

[29] In fact, using additional arguments not presented here, one can show that for any fixed real number $r > 5/2$, the time of existence can be controlled by a function of $\mathfrak{K}$, $\|\mathring{h}\|_{H^r(\Sigma_0)}$, $\|\mathring{s}\|_{H^r(\Sigma_0)}$, and $\|\mathring{u}^i\|_{H^r(\Sigma_0)}$. Of course, if the initial data enjoy additional Sobolev regularity, then the additional regularity persists in the solution during its classical lifespan.

[30] In particular, there is a $\big(H^3(\Sigma_0)\big)^5$-neighborhood of $(\mathring{h}, \mathring{s}, \mathring{u}^i)$ such that all data in the neighborhood launch solutions that exist on the same slab $[0,T] \times \mathbb{T}^3$ and, assuming also that the data belong to $\big(H^N(\Sigma_0)\big)^5$, enjoy the regularity properties stated in the theorem.

[31] More precisely, in [31], the spacetime metrics are scalar function multiples of the Minkowski metric on $\mathbb{R}^{1+3}$.

$\alpha = 0, 1, 2, 3$, *where* $T > 0$ *is the same time from Theorem* 9.10*:*

$$h,\, u^{\alpha} \in C\big([0,T], H^N(\mathbb{T}^3)\big), \tag{9.30a}$$

$$s \in C\big([0,T], H^{N+1}(\mathbb{T}^3)\big), \qquad S^{\alpha},\, \varpi^{\alpha} \in C\big([0,T], H^N(\mathbb{T}^3)\big). \tag{9.30b}$$

In particular, according to (9.30b)*, the additional regularity of the entropy and vorticity featured in the initial data is propagated by the flow of the equations. Moreover, the solution depends continuously on the initial data relative the norms corresponding to* (9.30a)*–*(9.30b)*.*

9.4. A New Inverse Riemannian Metric and the Classification of Various Combinations of Solution Variables

In our proof of Theorem 9.12, when controlling the top-order derivatives of the vorticity and entropy, we will rely on "geometrically sharp" elliptic estimates in which the precise details of the principal coefficients of the elliptic operators are important for our arguments. Due to the quasilinear nature of the relativistic Euler equations, these precise elliptic estimates involve the inverse Riemannian metric G^{-1} from the next definition. In particular, we will need to use G^{-1}-based norms when proving that the top-order derivatives of ϖ and S are continuous in time with values in $L^2(\mathbb{T}^3)$ [these facts are contained within the statement (9.30b)]; the role of G^{-1} in our analysis will become clear in Sect. 9.7.

Definition 9.13 (*An inverse Riemannian metric on* Σ_t). On each Σ_t, we define the inverse Riemannian metric G^{-1} as follows:

$$(G^{-1})^{ij} := \delta^{ij} - \frac{u^i u^j}{(u^0)^2}, \tag{9.31}$$

where $\delta^{ij} := \mathrm{diag}(1,1,1)$ is the standard Kronecker delta.

Remark 9.14 From the relation $\eta_{\alpha\beta} u^{\alpha} u^{\beta} = -1$, one can easily show that G^{-1} is Riemannian, that is, of signature $(+,+,+)$.

In proving that the solution depends continuously on the initial data, we will use a modified version of Kato's framework [17–19]. His framework was designed to handle hyperbolic systems, while our formulation of the relativistic Euler equations is elliptic–hyperbolic. For this reason, we find it convenient to divide the solution variables into various classes, which we provide in the next definition. Roughly, we will handle the "hyperbolic quantities" using Kato's framework, and to handle the remaining quantities, we will use elliptic estimates and algebraic relationships to control them in terms of the hyperbolic quantities.

Definition 9.15 (*Classification of various combinations of solution variables*). We define the *hyperbolic quantities* $\mathbf{H}$, the *elliptic quantities* $\mathbf{E}$, and the *algebraic quantities* $\mathbf{A_H}$, $\mathbf{A_{H,E}}$, and $\mathbf{A}$ as follows, where the Euclidean curl operator ${}^{(3)}\mathrm{curl}$ is defined in (9.2):

$$\mathbf{H} := (h, s, u^a, \partial_a h, \partial_a u^b, \varpi^a, S^a, \mathcal{C}^a, \mathcal{D})_{a,b=1,2,3}, \tag{9.32a}$$

$$\mathbf{E} := (\partial_a \varpi_b, \partial_a S_b)_{a,b=1,2,3}, \tag{9.32b}$$

$$\mathbf{A_H} := \left(u^0 - 1, \varpi^0, S^0, \mathcal{C}^0, \partial_t h, \partial_t u^\alpha, \partial_a u^0, \partial_t s\right)_{\alpha=0,1,2,3;a=1,2,3} \cup \left((G^{-1})^{cd}\partial_c \varpi_d, (G^{-1})^{cd}\partial_c S_d, {}^{(3)}\mathrm{curl}^a(\underline{\varpi}), {}^{(3)}\mathrm{curl}^a(\underline{S})\right)_{a=1,2,3}, \tag{9.32c}$$

$$\mathbf{A_{H,E}} := (\partial_t \varpi_\alpha, \partial_a \varpi_0, \partial_t S_\alpha, \partial_a S_0, \partial_b \varpi^b, \partial_b S^b)_{\alpha=0,1,2,3;a=1,2,3}, \tag{9.32d}$$

$$\mathbf{A} := \mathbf{A_H} \cup \mathbf{A_{H,E}}. \tag{9.32e}$$

Some remarks are in order.

- The point of introducing the algebraic quantities $\mathbf{A}$ is that, by virtue of the relativistic Euler equations, they can be algebraically expressed in terms of $\mathbf{H}$ and $\mathbf{E}$ (and thus are redundant); see Lemma 9.17. We stress that in (9.32c), it is crucial that the inverse metric G^{-1} is the one from Definition 9.13; the proof of (9.33a) will clarify that it is essential that the inverse metric is precisely G^{-1}.
- The elliptic quantities $\mathbf{E}$ can be controlled (in appropriate Sobolev norms) in terms of $\mathbf{H}$ via elliptic estimates; see Lemma 9.20 and its proof.
- The hyperbolic quantities $\mathbf{H}$ solve evolution equations with source terms that depend on $\mathbf{H}$ and $\mathbf{E}$. In view of the previous point, we see that one can bound the source terms (in appropriate Sobolev norms) in terms of $\mathbf{H}$. This will allow us to derive a closed system of energy inequalities that can be used to estimate $\mathbf{H}$. In view of the previous two points, we see that the estimates for $\mathbf{H}$ imply corresponding estimates for $\mathbf{E}$ and $\mathbf{A}$.

Remark 9.16 (*The hyperbolic quantities verify first-order hyperbolic equations*). In our proof of local well-posedness, we will use the fact that the hyperbolic quantities $\mathbf{H}$ solve first-order hyperbolic equations. More precisely, the elements h, s, and u^a of (9.32a) satisfy the first-order hyperbolic system (2.17)–(2.19) + (2.20), the elements $\partial_a h$ and $\partial_a u^b$ satisfy hyperbolic equations obtained by taking one spatial derivative of the Eqs. (2.17)–(2.19) + (2.20), and S^a, ϖ^a, $\mathcal{C}^a$, and $\mathcal{D}$ respectively satisfy the (spatial components of the) transport Eqs. (3.7), (3.8), (3.11b), and (3.9a); it is in this sense that we consider the variables $\mathbf{H}$ to be "hyperbolic."

Lemma 9.17 (Expressions for the algebraic quantities in terms of the hyperbolic and elliptic quantities). *Assume that* (h, s, u^α) *is a smooth solution to* (2.17)–(2.19) + (2.20). *Then we can express*

$$\mathbf{A_H} = \mathrm{f}(\mathbf{H}), \tag{9.33a}$$

$$\mathbf{A_{H,E}} = \mathrm{f}(\mathbf{H}, \mathbf{E}), \tag{9.33b}$$

$$\mathbf{A} = \mathrm{f}(\mathbf{H}, \mathbf{E}), \tag{9.33c}$$

where in (9.33a)–(9.33c), f *is a schematically denoted smooth function that satisfies* $\mathrm{f}(0) = 0$ *and that is allowed to vary from line to line.*

Moreover, let $\vec{I}$ be a spatial multi-index with $|\vec{I}| \geq 1$. Then

$$(G^{-1})^{ab}\partial_a\partial_{\vec{I}}\varpi_b,\ (G^{-1})^{ab}\partial_a\partial_{\vec{I}}S_b,\ {}^{(3)}\mathrm{curl}^i(\partial_{\vec{I}}\underline{\varpi}) = \sum_{\substack{M=1 \\ |\vec{J}_1|+\cdots+|\vec{J}_M|=|\vec{I}|}}^{|\vec{I}|} \mathrm{f}_{\vec{J}_1,\ldots,\vec{J}_M}(\mathbf{H}) \prod_{m=1}^{M} \partial_{\vec{J}_m}\mathbf{H}. \tag{9.34}$$

where $\mathrm{f}_{\vec{J}_1,\ldots,\vec{J}_M}$ are schematically denoted smooth functions (not necessarily vanishing at 0) and $\prod_{m=1}^{M}\partial_{\vec{J}_m}\mathbf{H}$ schematically denotes an order M monomial in the derivatives of the elements of $\mathbf{H}$.

Proof. Throughout this proof, f is a smooth function that can vary from line to line and satisfies $\mathrm{f}(0) = 0$ (except that the functions $\mathrm{f}_{\vec{J}_1,\ldots,\vec{J}_M}$ on RHS (9.34) do not necessarily satisfy $\mathrm{f}_{\vec{J}_1,\ldots,\vec{J}_M}(0) = 0$). Moreover, $\mathbf{H}$ and $\mathbf{E}$ are as defined in (9.32a) and (9.32b).

We first prove (9.33a). We must show that the elements of (9.32c) can be written as smooth functions of the elements of (9.32a) that vanish at 0. We first note that by the normalization condition $\eta_{\kappa\lambda}u^\kappa u^\lambda = -1$, $u^0 - 1$ is a smooth function of the spatial components of u that vanishes when $u^1 = u^2 = u^3 = 0$. From this fact and the identity $u^\kappa S_\kappa = 0$ [see (2.21)], we deduce that S^0 is a smooth function of the spatial components of u and S that vanishes at 0. A similar result holds for ϖ^0 by virtue of (4.2). Next, we note that, in view of the above discussion and the discussion surrounding Eq. (2.28), we can solve for the time derivatives of h, s, and u^α in terms of their spatial derivatives. Thus far, we have shown that $u^0-1, \varpi^0, S^0, \partial_t h, \partial_t u^\alpha, \partial_a u^0, \partial_t s$ can be expressed as $\mathrm{f}(\mathbf{H})$. In the rest of the proof, we will use these facts without explicitly mentioning them every time. Next, we use definitions (2.4) and (2.16a) to deduce that $u^\kappa \mathcal{C}_\kappa = \mathrm{f}(\mathbf{H})$. Using this equation to algebraically solve for $\mathcal{C}^0$, we deduce that $\mathcal{C}^0 = \mathrm{f}(\mathbf{H})$, as desired. We will now show that $(G^{-1})^{cd}\partial_c S_d = \mathrm{f}(\mathbf{H})$. To begin, we use definition (2.16b) to deduce that $\partial_i S^i = \partial_\alpha S^\alpha - \partial_t S^0 = n\mathcal{D} - S^\kappa\partial_\kappa h + c^{-2}S^\kappa\partial_\kappa h - \partial_t S^0 = \mathrm{f}(\mathbf{H}) - \partial_t S^0$. Next, using the identity $\partial_t = \frac{u^\kappa\partial_\kappa}{u^0} - \frac{u^i\partial_i}{u^0}$ and the evolution equation (3.7) with $\alpha = 0$, we find that $\partial_t S^0 = \mathrm{f}(\mathbf{H}) - \frac{u^i\partial_i S^0}{u^0}$. Moreover, using (2.21), we find that $S^0 = \frac{S_j u^j}{u^0}$, from which we deduce that $\frac{u^i\partial_i S^0}{u^0} = \mathrm{f}(\mathbf{H}) + \frac{u^i u^j \partial_i S_j}{(u^0)^2}$. Combining the above calculations, we find that $\partial_i S^i - \frac{u^i u^j\partial_i S_j}{(u^0)^2} = \mathrm{f}(\mathbf{H})$ which, in view of definition (9.31), yields the desired relation $(G^{-1})^{cd}\partial_c S_d = \mathrm{f}(\mathbf{H})$. The relation $(G^{-1})^{cd}\partial_c\varpi_d = \mathrm{f}(\mathbf{H})$ can be proved using a similar argument based on Eqs. (3.8) and (3.11a), and we omit the details. To show that ${}^{(3)}\mathrm{curl}^a(\underline{\varpi}) = \mathrm{f}(\mathbf{H})$, we first note that by definition (9.2), it suffices to show that $\partial_i\varpi_j - \partial_j\varpi_i = \mathrm{f}(\mathbf{H})$ for $i, j = 1, 2, 3$. To proceed, we use (4.10) with $V := \varpi$ [which is applicable in view of (4.2)], definition (2.16a), and the transport Eq. (3.8) to deduce that $\partial_i\varpi_j - \partial_j\varpi_i = \epsilon_{ij\gamma\delta}u^\gamma \mathrm{vort}^\delta(\varpi) + u_j u^\kappa\partial_\kappa\varpi_i - u_i u^\kappa\partial_\kappa\varpi_j + \mathrm{f}(\mathbf{H}) = \mathrm{f}(\mathbf{H})$, which is the desired result. The fact that ${}^{(3)}\mathrm{curl}^a(\underline{S}) = 0 = \mathrm{f}(\mathbf{H})$ is a trivial consequence of the symmetry property (4.1) and definition (9.2). We have therefore proved (9.33a).

We now prove (9.33b). We must show that elements of (9.32d) can be written as smooth functions of the elements of (9.32a) and the elements of (9.32b) that vanish at 0. To handle $\partial_t \varpi_i$,we use the identity $\partial_t = \frac{u^\kappa \partial_\kappa}{u^0} - \frac{u^j \partial_j}{u^0}$ and the transport Eq. (3.8) to deduce that $\partial_t \varpi_i = \frac{u^\kappa \partial_\kappa \varpi_i}{u^0} + \mathrm{f}(\mathbf{H}, \mathbf{E}) = \mathrm{f}(\mathbf{H}, \mathbf{E})$ as desired. To handle $\partial_t \varpi_0$, we simply use (4.2) to obtain the identity $\varpi^0 = \frac{\varpi_j u^j}{u^0}$, differentiate this identity with respect to ∂_t, and then use the already proven facts that ϖ_j, $u^\alpha - \delta_0^\alpha$, and their time derivatives are equal to $\mathrm{f}(\mathbf{H}, \mathbf{E})$. Similarly, by differentiating the identity $\varpi^0 = \frac{\varpi_j u^j}{u^0}$ with ∂_a, we conclude that $\partial_a \varpi_0 = \mathrm{f}(\mathbf{H}, \mathbf{E})$. The relations $\partial_t S_\alpha = \mathrm{f}(\mathbf{H}, \mathbf{E})$ and $\partial_a S_0 = \mathrm{f}(\mathbf{H}, \mathbf{E})$ can be proved using a similar argument based on Eqs. (2.21) and (3.7), and we omit the details. The facts that $\partial_b \varpi^b = \mathrm{f}(\mathbf{H}, \mathbf{E})$ and $\partial_b S^b = \mathrm{f}(\mathbf{H}, \mathbf{E})$ follow trivially from the definitions. We have therefore proved (9.33b). Equation (9.33c) then follows from definition (9.32e) and (9.33a)–(9.33b).

To prove (9.34), we first note that definition (9.32c) and (9.33a) imply that $(G^{-1})^{ab} \partial_a \varpi_b$, $(G^{-1})^{ab} \partial_a S_b$, and ${}^{(3)}\mathrm{curl}^i(\underline{\varpi})$ are all of the form $\mathrm{f}(\mathbf{H})$. Hence, (9.34) follows from the Leibniz and chain rules and the definition (9.32a) of $\mathbf{H}$. □

9.5. Elliptic Estimates and the Corresponding Energies

In this subsection, we construct the energies that we will use to control the top-order derivatives of the vorticity and entropy; see Definition 9.19. The proof that the energies are coercive relies on elliptic estimates; see the proof of Lemma 9.20. We start by defining a bilinear form on the relevant Hilbert space of functions. Lemma 9.20 shows that the bilinear form induces a norm on the Hilbert space.

Definition 9.18 (*A new Hilbert space inner product*). Let $(\underline{\varpi}, \underline{S})$ denote the array of spatial components of the vorticity and entropy gradient (i.e., the η-orthogonally projection of (ϖ, S) onto Σ_t, as in Sect. 9.1.1). Let $\alpha > 0$ be a parameter and let $M^{-1}(t, \cdot)$ be an inverse Riemannian metric on Σ_t. We define the following bilinear form on the corresponding Hilbert space $\left(H^N(\Sigma_t)\right)^3 \times \left(H^N(\Sigma_t)\right)^3$:

$$
\begin{aligned}
&\left\langle (\underline{\varpi}, \underline{S}), \left(\underline{\widetilde{\varpi}}, \underline{\widetilde{S}}\right) \right\rangle_{M^{-1};\alpha}(t) \\
&\quad := \alpha \sum_{|\vec{I}|=N-1} \int_{\Sigma_t} \left\{(M^{-1})^{ab} \partial_a \partial_{\vec{I}} \varpi_b\right\} \left\{(M^{-1})^{cd} \partial_c \partial_{\vec{I}} \widetilde{\varpi}_d\right\} \mathrm{d}x \\
&\qquad + \alpha \sum_{|\vec{I}|=N-1} \int_{\Sigma_t} \left\{(M^{-1})^{ab} \partial_a \partial_{\vec{I}} S_b\right\} \left\{(M^{-1})^{cd} \partial_c \partial_{\vec{I}} \widetilde{S}_d\right\} \mathrm{d}x \\
&\qquad + \alpha \sum_{|\vec{I}|=N-1} \int_{\Sigma_t} (M^{-1})^{ab} (M^{-1})^{cd} \epsilon_{aci} \epsilon_{bdj} {}^{(3)}\mathrm{curl}^i(\partial_{\vec{I}} \underline{\varpi}) {}^{(3)}\mathrm{curl}^j(\partial_{\vec{I}} \underline{\widetilde{\varpi}}) \,\mathrm{d}x \\
&\qquad + \alpha \sum_{|\vec{I}|=N-1} \int_{\Sigma_t} (M^{-1})^{ab} (M^{-1})^{cd} \epsilon_{aci} \epsilon_{bdj} {}^{(3)}\mathrm{curl}^i(\partial_{\vec{I}} \underline{S}) {}^{(3)}\mathrm{curl}^j(\partial_{\vec{I}} \underline{\widetilde{S}}) \,\mathrm{d}x
\end{aligned}
$$

$$+\sum_{|\vec{I}|\le N-1}\int_{\Sigma_t}\delta^{ab}(\partial_{\vec{I}}\varpi_a)(\partial_{\vec{I}}\widetilde{\varpi}_b)\,dx+\sum_{|\vec{I}|\le N-1}\int_{\Sigma_t}\delta^{ab}(\partial_{\vec{I}}S_a)(\partial_{\vec{I}}\widetilde{S}_b)\,dx, \tag{9.35}$$

where δ^{ab} is the standard Kronecker delta and ϵ_{abc} is the fully antisymmetric symbol normalized by $\epsilon_{123}=1$.

We now define the family of energies that we will use to control the top-order derivatives of the vorticity and entropy.

Definition 9.19 (*"Elliptic" energy*). Let $N\ge 3$ be an integer, let $\alpha>0$ be a parameter (below we will choose it to be sufficiently small), and let $M^{-1}(t,\cdot)$ be a C^1 inverse Riemannian metric on Σ_t. We define the square of the "elliptic" energy $\mathbb{E}_{N;M^{-1};\alpha}[(\underline{\varpi},\underline{S})]=\mathbb{E}_{N;M^{-1};\alpha}[(\underline{\varpi},\underline{S})](t)\ge 0$ as follows:

$$\mathbb{E}^2_{N;M^{-1};\alpha}[(\underline{\varpi},\underline{S})](t):=\langle(\underline{\varpi},\underline{S})\,,(\underline{\varpi},\underline{S})\rangle_{M^{-1};\alpha}(t). \tag{9.36}$$

In the next lemma, with the help of elliptic estimates, we exhibit the coercivity of $\mathbb{E}_{N;M^{-1};\alpha}[(\underline{\varpi},\underline{S})](t)$. The lemma shows in particular that if $\alpha>0$ is sufficiently small (depending on the inverse Riemannian metric M^{-1}), then the bilinear form from Definition 9.18 is a Hilbert space inner product.

Lemma 9.20 (Energy-norm comparison estimate based on elliptic estimates). *Let $T>0$, and let $M^{-1}=M^{-1}(t,x)$ be an inverse Riemannian metric defined for $(t,x)\in[0,T]\times\mathbb{T}^3$. Let λ be the infimum of the eigenvalues of the 3×3 matrix $(M^{-1})^{ij}(t,x)$ over $(t,x)\in[0,T]\times\mathbb{T}^3$, and let Λ be the supremum of the eigenvalues of the 3×3 matrix $(M^{-1})^{ij}(t,x)$ over $(t,x)\in[0,T]\times\mathbb{T}^3$, and assume that $0<\lambda\le\Lambda<\infty$. Let $\mathbb{E}_{N;M^{-1};\alpha}[(\underline{\varpi},\underline{S})]$ be as in Definition 9.19. There exist a small constant $\alpha_*>0$ and a large constant $C>0$ such that α_*^{-1} and C depend continuously in an increasing fashion on (i) $\max_{i,j=1,2,3}\|(M^{-1})^{ij}\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)}$; (ii) Λ; and (iii) λ^{-1}, such that the following comparison estimates hold for $t\in[0,T]$:*

$$\mathbb{E}_{N;M^{-1};\alpha_*}[(\underline{\varpi},\underline{S})](t)\le C\sum_{a=1}^{3}\|\varpi^a\|_{H^N(\Sigma_t)}+C\sum_{a=1}^{3}\|S_a\|_{H^N(\Sigma_t)}, \tag{9.37a}$$

$$\sum_{a=1}^{3}\|\varpi^a\|_{H^N(\Sigma_t)}+\sum_{a=1}^{3}\|S_a\|_{H^N(\Sigma_t)}\le C\mathbb{E}_{N;M^{-1};\alpha_*}[(\underline{\varpi},\underline{S})](t). \tag{9.37b}$$

Proof. We prove only (9.37b) since (9.37a) can be proved using similar but simpler arguments. Throughout the proof, $C>0$ denotes a constant with the dependence-properties stated in the lemma. To proceed, we note the following divergence identity for one-forms V on Σ_t, which can be directly verified:

$$(M^{-1})^{ab}(M^{-1})^{cd}(\partial_a V_b)(\partial_c V_d)$$
$$+\overbrace{\frac{1}{2}(M^{-1})^{ab}(M^{-1})^{cd}\epsilon_{aci}\epsilon_{bdj}{}^{(3)}\mathrm{curl}^i(V)^{(3)}\mathrm{curl}^j(V)}^{\frac{1}{2}(M^{-1})^{ab}(M^{-1})^{cd}(\partial_a V_c-\partial_c V_a)(\partial_b V_d-\partial_d V_b)}$$

$$
\begin{aligned}
&= (M^{-1})^{ab}(M^{-1})^{cd}(\partial_a V_c)(\partial_b V_d) \\
&\quad + \frac{1}{2}\left\{\partial_a\left[(M^{-1})^{ab}(M^{-1})^{cd}\right]\right\}\left[V_c\partial_b V_d + V_c\partial_d V_b\right] \\
&\quad + \frac{1}{2}\left\{\partial_c\left[(M^{-1})^{ab}(M^{-1})^{cd}\right]\right\}\left[V_a\partial_b V_d + V_a\partial_d V_b\right] \\
&\quad - \frac{1}{2}\left\{\partial_b\left[(M^{-1})^{ab}(M^{-1})^{cd}\right]\right\}\left[V_a\partial_c V_d + V_c\partial_a V_d\right] \\
&\quad - \frac{1}{2}\left\{\partial_d\left[(M^{-1})^{ab}(M^{-1})^{cd}\right]\right\}\left[V_a\partial_c V_b + V_c\partial_a V_b\right] \\
&\quad + \frac{1}{2}\partial_b\left\{(M^{-1})^{ab}(M^{-1})^{cd}\left[V_a\partial_c V_d + V_c\partial_a V_d\right]\right\} \\
&\quad + \frac{1}{2}\partial_d\left\{(M^{-1})^{ab}(M^{-1})^{cd}\left[V_a\partial_c V_b + V_c\partial_a V_b\right]\right\} \\
&\quad - \frac{1}{2}\partial_a\left\{(M^{-1})^{ab}(M^{-1})^{cd}\left[V_c\partial_b V_d + V_c\partial_d V_b\right]\right\} \\
&\quad - \frac{1}{2}\partial_c\left\{(M^{-1})^{ab}(M^{-1})^{cd}\left[V_a\partial_b V_d + V_a\partial_d V_b\right]\right\}. \qquad (9.38)
\end{aligned}
$$

We now integrate (9.38) over Σ_t with respect to dx and note that the integrals of the last four (perfect spatial derivative) terms on the right-hand side vanish. In view of our assumptions on the eigenvalues of $(M^{-1})^{ij}(t,\cdot)$, we see that the integral of the first term $(M^{-1})^{ab}(M^{-1})^{cd}(\partial_a V_c)(\partial_b V_d)$ on RHS (9.38) is $\geq \lambda^2 \sum_{a,b=1}^{3} \|\partial_a V_b\|_{L^2(\Sigma_t)}^2$. Also using Young's inequality, we see that the integrals of the second through fifth terms on RHS (9.38) (in which a derivative falls on M^{-1}) are collectively bounded from below by $\geq -\frac{\lambda^2}{2}\sum_{a,b=1}^{3}\|\partial_a V_b\|_{L^2(\Sigma_t)}^2 - \frac{C}{\lambda^2}\sum_{a=1}^{3}\|V_a\|_{L^2(\Sigma_t)}^2$. It follows that the integral of (9.38) is bounded from below by

$$
\geq \frac{\lambda^2}{2}\sum_{a,b=1}^{3}\|\partial_a V_b\|_{L^2(\Sigma_t)}^2 - \frac{C}{\lambda^2}\sum_{a=1}^{3}\|V_a\|_{L^2(\Sigma_t)}^2 .
$$

The desired estimate (9.37b) now follows from these considerations with $\underline{\varpi}$ and $\underline{S}$ in the role of V, and definitions (9.35) and (9.36), where $\alpha := \alpha_* > 0$ is chosen so that $\alpha_* \frac{C}{\lambda^2} = \frac{1}{2}$, and $\frac{C}{\lambda^2}$ is the (absolute value of the) coefficient from the previous inequality. We clarify that, by our conventions, factors of $\frac{1}{\lambda^2}$ can be absorbed into the constant C on RHS (9.37b). □

In the next lemma, we show that some Sobolev norms of the elliptic variables **E** can be bounded by corresponding Sobolev norms of the hyperbolic variables **H**. We also derive related estimates for the difference of two solutions. The main ingredients in the proofs are the elliptic estimates provided by Lemma 9.20.

Lemma 9.21 (Controlling Sobolev norms of the elliptic variables in terms of Sobolev norms of the hyperbolic variables).

(A) Let $\mathring{h} := h|_{\Sigma_0}$, $\mathring{s} := s|_{\Sigma_0}$, *and* $\mathring{u}^i := u^i|_{\Sigma_0}$ *be initial data for the relativistic Euler equations* (2.17)–(2.19) + (2.20), *let* $\mathring{\varpi}^i := \varpi^i|_{\Sigma_0}$, *and let* (h, s, u^α)

be the solution provided by Theorem 9.10. In particular, let $N \geq 3$ be an integer, let $[0,T] \times \mathbb{T}^3$ be the slab of existence provided by the theorem, and let $\mathfrak{K}$ be the set featured in Theorem 9.10. Assume in addition that the rectangular components of the initial data are elements of $C^\infty(\Sigma_0)$, and note that by Theorem 9.10 and the Sobolev embedding result (9.9), the rectangular components of the solution belong to $C^\infty([0,T] \times \mathbb{T}^3)$. Let $\mathbf{E}$ and $\mathbf{H}$ be the corresponding elliptic and hyperbolic variables as defined in Definition 9.15. Then there exists a constant $C > 0$, depending only on:

1. N
2. $\mathfrak{K}$
3. $\|\mathring{h}\|_{H^N(\Sigma_0)} + \|\mathring{s}\|_{H^{N+1}(\Sigma_0)} + \sum_{a=1}^3 \|\mathring{u}^a\|_{H^N(\Sigma_0)} + \sum_{a=1}^3 \|\mathring{\varpi}^a\|_{H^N(\Sigma_0)}$
4.

$$\|h\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \|s\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|u^a\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|S^a\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|\varpi^a\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)},$$

such that the following estimate holds for $t \in [0,T]$:

$$\|\mathbf{E}\|_{H^{N-1}(\Sigma_t)} \leq C\|\mathbf{H}\|_{H^{N-1}(\Sigma_t)}. \tag{9.39}$$

(B) *For $i = 1,2$, let $(h_{(i)}, s_{(i)}, u_{(i)})$ be classical solutions to the relativistic Euler equations (2.17)–(2.19) + (2.20) that have the properties stated in part (A). Assume that the slab of existence $[0,T] \times \mathbb{T}^3$ is the same for both solutions and that the set $\mathfrak{K}$ is the same for both solutions, that is, that there exists a compact set $\mathfrak{K} \subset int\mathcal{H}$ such that for $i = 1,2$, we have $(h_{(i)}, s_{(i)}, u^1_{(i)}, u^2_{(i)}, u^3_{(i)})([0,T] \times \mathbb{T}^3) \subset int\mathfrak{K}$. Let $\mathbf{E}_{(i)}$ and $\mathbf{H}_{(i)}$ be the corresponding elliptic and hyperbolic variables as defined in Definition 9.15. Then there exist constants $\delta > 0$ and $C > 0$, depending only on:*

1. N
2. $\mathfrak{K}$
3.

$$\sum_{i=1}^2 \left\{ \|\mathring{h}_{(i)}\|_{H^N(\Sigma_0)} + \|\mathring{s}_{(i)}\|_{H^{N+1}(\Sigma_0)} + \sum_{a=1}^3 \|\mathring{u}^a_{(i)}\|_{H^N(\Sigma_0)} + \sum_{a=1}^3 \|\mathring{\varpi}^a_{(i)}\|_{H^N(\Sigma_0)} \right\}$$

4.

$$\sum_{i=1}^2 \left\{ \|h_{(i)}\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \|s_{(i)}\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|u^a_{(i)}\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|S^a_{(i)}\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|\varpi^a_{(i)}\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} \right\}$$

5.

$$\sum_{i=1}^{2}\Big\{\|h_{(i)}\|_{C\left([0,T],H^N(\mathbb{T}^3)\right)}+\sum_{a=1}^{3}\|u^a_{(i)}\|_{C\left([0,T],H^N(\mathbb{T}^3)\right)} + \|s_{(i)}\|_{C\left([0,T],H^{N+1}(\mathbb{T}^3)\right)}+\sum_{a=1}^{3}\|\varpi^a_{(i)}\|_{C\left([0,T],H^N(\mathbb{T}^3)\right)}\Big\},$$

such that if $\|\mathbf{H}_{(1)}-\mathbf{H}_{(2)}\|_{C(\Sigma_t)}\leq\delta$*, then the following estimate holds for* $t\in[0,T]$*:*

$$\|\mathbf{E}_{(1)}-\mathbf{E}_{(2)}\|_{H^{N-1}(\Sigma_t)}\leq C\|\mathbf{H}_{(1)}-\mathbf{H}_{(2)}\|_{H^{N-1}(\Sigma_t)}. \tag{9.40}$$

Proof. Throughout this proof, C denotes a constant with the dependence-properties stated in the lemma. We begin by establishing (9.39). Invoking definitions (9.32a), (9.32b), (9.35), and (9.36), using the fact that ${}^{(3)}\mathrm{curl}(\underline{S})=0$ [see (4.1)], and using the estimate (9.37b) with $M^{-1}:=G^{-1}$ and with $\alpha_*>0$ as in the statement of Lemma 9.20 (where G^{-1} is defined in Definition 9.13, and we stress that the proof of (9.37b) relied on elliptic estimates), we find that

$$\begin{aligned}\|\mathbf{E}\|_{H^{N-1}(\Sigma_t)}&\leq C\mathbb{E}_{N;G^{-1};\alpha_*}[(\underline{\varpi},\underline{S})](t)\\&\leq C\sum_{|\vec{I}|=N-1}\left\|(G^{-1})^{ab}\partial_a\partial_{\vec{I}}\varpi_b\right\|_{L^2(\Sigma_t)}\\&\quad+C\sum_{|\vec{I}|=N-1}\left\|(G^{-1})^{ab}\partial_a\partial_{\vec{I}}S_b\right\|_{L^2(\Sigma_t)}\\&\quad+C\sum_{|\vec{I}|=N-1}\sum_{a=1}^{3}\left\|{}^{(3)}\mathrm{curl}^a(\partial_{\vec{I}}\underline{\varpi})\right\|_{L^2(\Sigma_t)}\\&\quad+C\|\mathbf{H}\|_{H^{N-1}(\Sigma_t)}.\end{aligned}\tag{9.41}$$

Next, using (9.34), we see that the terms $(G^{-1})^{ab}\partial_a\partial_{\vec{I}}\varpi_b$, $(G^{-1})^{ab}\partial_a\partial_{\vec{I}}S_b$, and ${}^{(3)}\mathrm{curl}^a(\partial_{\vec{I}}\underline{\varpi})$ on RHS (9.41) are smooth functions of $\mathbf{H}$ and its spatial derivatives. Thus, using inequality (9.11) to bound RHS (9.34) in the norm $\|\cdot\|_{L^2(\Sigma_t)}$, we arrive at the desired estimate (9.39). We stress that RHS (9.11) is *linear* in the order r derivatives of the solution; this is the reason that RHS (9.39) is linear in $\|\mathbf{H}\|_{H^{N-1}(\Sigma_t)}$.

We now prove (9.40). For $i=1,2$, we let $G^{-1}_{(i)}$ denote the inverse Riemannian metric corresponding to the ith solution, that is, the inverse Riemannian metric whose rectangular components are formed by evaluating RHS (9.31) at the solution corresponding to the labeling index i. To proceed, we use definitions (9.32a), (9.32b), (9.35), and (9.36), the fact that ${}^{(3)}\mathrm{curl}(\underline{S}_{(1)})={}^{(3)}\mathrm{curl}(\underline{S}_{(2)})=0$ [see (4.1)], and the comparison estimate (9.37b) with $M^{-1}:=G^{-1}_{(1)}$ and with $\alpha_*>0$ as in the statement of Lemma 9.20 to deduce that

$$\begin{aligned}&\|\mathbf{E}_{(1)}-\mathbf{E}_{(2)}\|_{H^{N-1}(\Sigma_t)}\\&\quad\leq C\mathbb{E}_{N;G^{-1}_{(1)};\alpha_*}[(\underline{\varpi}_{(1)}-\underline{\varpi}_{(2)},\underline{S}_{(1)}-\underline{S}_{(2)})](t)\end{aligned}$$

$$\leq C \sum_{|\vec{I}|=N-1} \left\| (G^{-1}_{(1)})^{ab} \partial_a \partial_{\vec{I}} \left(\varpi_{(1)b} - \varpi_{(2)b} \right) \right\|_{L^2(\Sigma_t)}$$
$$+ C \sum_{|\vec{I}|=N-1} \left\| (G^{-1}_{(1)})^{ab} \partial_a \partial_{\vec{I}} \left(S_{(1)b} - S_{(2)b} \right) \right\|_{L^2(\Sigma_t)}$$
$$+ C \sum_{|\vec{I}|=N-1} \sum_{a=1}^{3} \left\| {}^{(3)}\mathrm{curl}^a \left(\partial_{\vec{I}} (\underline{\varpi}_{(1)} - \underline{\varpi}_{(2)}) \right) \right\|_{L^2(\Sigma_t)}$$
$$+ C \|\mathbf{H}_{(1)} - \mathbf{H}_{(2)}\|_{H^{N-1}(\Sigma_t)}. \tag{9.42}$$

Next, using the triangle inequality, we find that

$$\text{RHS } (9.42)$$
$$\leq C \sum_{|\vec{I}|=N-1} \left\| (G^{-1}_{(1)})^{ab} \partial_a \partial_{\vec{I}} \varpi_{(1)b} - (G^{-1}_{(2)})^{ab} \partial_a \partial_{\vec{I}} \varpi_{(2)b} \right\|_{L^2(\Sigma_t)}$$
$$+ C \sum_{|\vec{I}|=N-1} \left\| (G^{-1}_{(2)})^{ab} - (G^{-1}_{(1)})^{ab} \right\|_{C(\Sigma_t)} \left\| \partial_a \partial_{\vec{I}} \varpi_{(2)b} \right\|_{L^2(\Sigma_t)}$$
$$+ C \sum_{|\vec{I}|=N-1} \left\| (G^{-1}_{(1)})^{ab} \partial_a \partial_{\vec{I}} S_{(1)b} - (G^{-1}_{(2)})^{ab} \partial_a \partial_{\vec{I}} S_{(2)b} \right\|_{L^2(\Sigma_t)}$$
$$+ C \sum_{|\vec{I}|=N-1} \left\| (G^{-1}_{(2)})^{ab} - (G^{-1}_{(1)})^{ab} \right\|_{C(\Sigma_t)} \left\| \partial_a \partial_{\vec{I}} S_{(2)b} \right\|_{L^2(\Sigma_t)}$$
$$+ C \sum_{|\vec{I}|=N-1} \sum_{a=1}^{3} \left\| {}^{(3)}\mathrm{curl}^a (\partial_{\vec{I}} \underline{\varpi}_{(1)}) - {}^{(3)}\mathrm{curl}^a (\partial_{\vec{I}} \underline{\varpi}_{(2)}) \right\|_{L^2(\Sigma_t)}$$
$$+ C \|\mathbf{H}_{(1)} - \mathbf{H}_{(2)}\|_{H^{N-1}(\Sigma_t)}. \tag{9.43}$$

Using the assumed bounds $\sum_{|\vec{I}|=N-1} \sum_{a,b=1}^{3} \left\| \partial_a \partial_{\vec{I}} \varpi_{(2)b} \right\|_{L^2(\Sigma_t)} \leq C$ and $\sum_{|\vec{I}|=N-1} \sum_{a,b=1}^{3} \left\| \partial_a \partial_{\vec{I}} S_{(2)b} \right\|_{L^2(\Sigma_t)} \leq C$, (9.34), (9.9), and (9.12) (where the hypotheses needed to invoke (9.12) are satisfied if $\|\mathbf{H}_{(1)} - \mathbf{H}_{(2)}\|_{C(\Sigma_t)}$ is sufficiently small), we see that the terms on the first, third, and fifth lines of RHS (9.43) are $\leq C\|\mathbf{H}_{(1)} - \mathbf{H}_{(2)}\|_{H^{N-1}(\Sigma_t)}$ as desired. To handle the terms on the second and fourth lines of RHS (9.43), we use the assumed bounds

$$\sum_{|\vec{I}|=N-1} \sum_{a,b=1}^{3} \left\| \partial_a \partial_{\vec{I}} \varpi_{(2)b} \right\|_{L^2(\Sigma_t)} \leq C, \quad \sum_{|\vec{I}|=N-1} \sum_{a,b=1}^{3} \left\| \partial_a \partial_{\vec{I}} S_{(2)b} \right\|_{L^2(\Sigma_t)} \leq C,$$

the mean value theorem estimate $\left| (G^{-1}_{(2)})^{ab} - (G^{-1}_{(1)})^{ab} \right| \leq C \left| \mathbf{H}_{(1)} - \mathbf{H}_{(2)} \right|$ (where we are using that RHS (9.31) can be viewed as a smooth function of (u^1, u^2, u^3)), and the Sobolev embedding result (9.9) to deduce that the terms on the second and fourth lines of RHS (9.43) are $\leq C\|\mathbf{H}_{(1)} - \mathbf{H}_{(2)}\|_{C(\Sigma_t)} \leq C\|\mathbf{H}_{(1)} - \mathbf{H}_{(2)}\|_{H^{N-1}(\Sigma_t)}$ as desired. We have therefore proved (9.40). □

9.6. Energies for the Wave Equations via the Vectorfield Multiplier Method

In this subsection, we derive a priori estimates for our new formulation of the relativistic Euler equations. The main result is provided by the next proposition. The proposition shows in particular that the vorticity and entropy are one degree more differentiable compared to the standard estimates that follow from first-order formulations of the equations. The main analytic tools in the proof of the proposition are the elliptic estimates from Sect. 9.5 and the vectorfield method for wave equations (see Sect. 9.6.1).

Proposition 9.22 (A priori estimates for solutions to the relativistic Euler equations). *Let $\mathring{h} := h|_{\Sigma_0}$, $\mathring{s} := s|_{\Sigma_0}$, and $\mathring{u}^i := u^i|_{\Sigma_0}$ be initial data for the relativistic Euler equations (2.17)–(2.19) + (2.20) obeying the assumptions of Theorem 9.10, and let $(h, s, u^0, u^1, u^2, u^3)$ be the corresponding solution. In particular, let $N \geq 3$ be an integer, let $[0,T] \times \mathbb{T}^3$ be the slab of existence provided by the theorem, and let $\mathfrak{K}$ be the set featured in theorem. Assume in addition that the components of the initial data relative to standard coordinates belong to $C^\infty(\mathbb{T}^3)$ and note that by Remark 9.11, the solution components belong to $C^\infty([0,T] \times \mathbb{T}^3)$. Let ϖ be the vorticity (see Definition 2.2), and let $\mathring{\varpi}^i := \varpi^i|_{\Sigma_0}$ be its initial spatial components.*

Then there exists a constant $C > 0$, depending only on:

1. N
2. $\mathfrak{K}$
3. $\|\mathring{h}\|_{H^N(\Sigma_0)} + \sum_{a=1}^3 \|\mathring{u}^a\|_{H^N(\Sigma_0)} + \|\mathring{s}\|_{H^{N+1}(\Sigma_0)} + \sum_{a=1}^3 \|\mathring{\varpi}^a\|_{H^N(\Sigma_0)}$
4. $$\|h\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|u^a\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \|s\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|S^a\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)} + \sum_{a=1}^3 \|\varpi^a\|_{C\left([0,T],C^1(\mathbb{T}^3)\right)}$$

such that for $t \in [0,T]$, the components of the solution relative to the standard coordinates verify the following estimates:

$$\begin{aligned} &\|h\|_{H^N(\Sigma_t)} + \sum_{a=0}^3 \|u^\alpha - \delta_0^\alpha\|_{H^N(\Sigma_t)} + \|s\|_{H^{N+1}(\Sigma_t)} \\ &\quad + \sum_{\alpha=0}^3 \|S^\alpha\|_{H^N(\Sigma_t)} + \sum_{\alpha=0}^3 \|\varpi^\alpha\|_{H^N(\Sigma_t)} \\ &\leq C\exp(Ct) \leq C\exp(CT) := C_*, \end{aligned} \tag{9.44}$$

where δ_0^α is the Kronecker delta.

The proof of Proposition 9.22 is located in Sect. 9.6.4. We will first derive some preliminary results. We start by noting that we can rewrite the spatial components of (3.1), (3.3), (3.7), (3.8), (3.9a), and (3.11b) in concise form as follows, where f denotes a smooth function of its arguments that is free to vary from line to line and that satisfies $\mathrm{f}(0) = 0$, $\underline{V}$ denotes η-orthogonal projection of V onto constant-time hypersurfaces (see Sect. 9.1.1), and the hyperbolic variables $\mathbf{H}$ and the elliptic variables $\mathbf{E}$ are as in Definition 9.15:

$$\square_g h = \mathrm{f}(\mathbf{H}), \tag{9.45a}$$

$$\square_g \underline{u} = \mathrm{f}(\mathbf{H}), \tag{9.45b}$$

$$u^\alpha \partial_\alpha \underline{S} = \mathrm{f}(\mathbf{H}), \tag{9.45c}$$

$$u^\alpha \partial_\alpha \underline{\varpi} = \mathrm{f}(\mathbf{H}), \tag{9.45d}$$

$$u^\alpha \partial_\alpha \mathcal{D} = \mathrm{f}(\mathbf{H}, \mathbf{E}), \tag{9.45e}$$

$$u^\alpha \partial_\alpha \underline{\mathcal{C}} = \mathrm{f}(\mathbf{H}, \mathbf{E}). \tag{9.45f}$$

The crux of the proof of Proposition 9.22 is to derive energy estimates for the covariant wave equations (9.45a) and (9.45b), energy estimates for the transport equations (9.45c), (9.45d), (9.45e), and (9.45f), and elliptic estimates to handle the terms $\mathbf{E}$ on RHSs (9.45e) and (9.45f). We have already derived the necessary elliptic estimates in Sect. 9.5. In the next three subsections, we will outline the energy estimates, which are standard.

9.6.1. Energy Estimates for Covariant Wave Equations. The wave operator in (9.45a) and (9.45b) is with respect to the acoustical metric g introduced in Definition 2.6. These are covariant wave equations for the scalar quantities h and u^α. Estimates for such equations can be derived by using the well-known vectorfield multiplier method[32] for wave equations, which we outline in this subsubsection.

Let φ be any element of $\{h, u^1, u^2, u^3\}$ (in practice, we will not need to derive separate energy estimates for u^0 since estimates for u^0 can be obtained as a consequence of the estimates for the spatial components of u and the normalization condition $\eta_{\kappa\lambda} u^\kappa u^\lambda = -1$). We start by defining the energy-momentum tensor associated to a scalar function φ:

$$T_{\alpha\beta} = T_{\alpha\beta}[\varphi] := (\partial_\alpha \varphi)(\partial_\beta \varphi) - \frac{1}{2} g_{\alpha\beta} (g^{-1})^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi). \tag{9.46}$$

A crucial property of $T_{\alpha\beta}$ is that it satisfies the *dominant energy condition*: $T_{\alpha\beta} X^\alpha Y^\beta \geq 0$ whenever the vectorfields X and Y are future-directed[33] and timelike[34] with respect to g. In practice, the dominant energy condition allows one to construct energies that are *coercive* along causal (with respect to g) hypersurfaces;[35] see Eq. (9.56) below for the energy that we use in deriving a priori estimates for h and u.

[32] In deriving a priori estimates, in addition to the multiplier method, we will use only the simplest version of the vectorfield commutator method. Specifically, we will commute the equations only with the coordinate spatial derivative operators $\partial_{\vec{I}}$.
[33] By a "future-directed" vectorfield X, we mean that $X^0 > 0$.
[34] X is defined to be timelike with respect to g if $g_{\alpha\beta} X^\alpha X^\beta < 0$.
[35] By a "causal hypersurface," we mean a hypersurface whose future-directed unit normal is either timelike with respect to g or null with respect to g at each point.

Next, for any vectorfield X (soon to be employed in the role of a "multiplier vectorfield"), we let ${}^{(X)}\pi$ be its deformation tensor relative to g, which takes the following form relative to arbitrary coordinates:

$$ {}^{(X)}\pi_{\alpha\beta} := g_{\beta\mu}\nabla_\alpha X^\mu + g_{\alpha\mu}\nabla_\beta X^\mu. \tag{9.47} $$

In (9.47) and in the rest of this subsubsection, ∇ is the covariant derivative induced by g. Next, we define the *energy current* vectorfield corresponding to X as follows:

$$ {}^{(X)}J^\alpha = {}^{(X)}J^\alpha[\varphi] := (g^{-1})^{\alpha\mu} T_{\mu\beta}[\varphi] X^\beta - X^\alpha \varphi^2. \tag{9.48} $$

From straightforward computations, we derive the following identity:

$$ \begin{aligned} \nabla_\alpha {}^{(X)}J^\alpha = (\Box_g \varphi)\, X^\alpha \partial_\alpha \varphi + \frac{1}{2}(g^{-1})^{\alpha\gamma}(g^{-1})^{\beta\delta} T_{\alpha\beta} {}^{(X)}\pi_{\gamma\delta} \\ - (\nabla_\alpha X^\alpha)\varphi^2 - 2\varphi(X^\alpha \partial_\alpha \varphi). \end{aligned} \tag{9.49} $$

Applying the divergence theorem on the spacetime slab $[0,T] \times \mathbb{T}^3$ and using (9.49), we deduce the following identity:

$$ \begin{aligned} &\int_{\Sigma_t} g_{\alpha\beta} {}^{(X)}J^\alpha[\varphi] \hat{N}^\beta \, \mathrm{d}\mu_{\underline{g}} \\ &= \int_{\Sigma_0} g_{\alpha\beta} {}^{(X)}J^\alpha[\varphi] \hat{N}^\beta \, \mathrm{d}\mu_{\underline{g}} \\ &\quad - \int_{[0,t]\times\mathbb{T}^3} \left\{ (\Box_g \varphi)\, X^\alpha \partial_\alpha \varphi + \frac{1}{2}(g^{-1})^{\alpha\gamma}(g^{-1})^{\beta\delta} T_{\alpha\beta} {}^{(X)}\pi_{\gamma\delta} \right\} \mathrm{d}\mu_g \\ &\quad + \int_{[0,t]\times\mathbb{T}^3} \left\{ (\nabla_\alpha X^\alpha)\varphi^2 + 2\varphi(X^\alpha \partial_\alpha \varphi) \right\} \mathrm{d}\mu_g. \end{aligned} \tag{9.50} $$

In (9.50), $\mathrm{d}\mu_g$ is the volume form that g induces on [0,t] $\times \mathbb{T}^3$, $\hat{N}$ is the future-directed unit normal to Σ_t with respect to the metric g, and $\mathrm{d}\mu_{\underline{g}}$ is the volume form that $\underline{g}$ induces on Σ_t, where $\underline{g}$ is the first fundamental form of Σ_t, that is, $\underline{g}_{ij} := g_{ij}$ for $1 \leq i,j \leq 3$. We also note that relative to the standard coordinates, $\hat{N}^\alpha = -\frac{(g^{-1})^{\alpha 0}}{\sqrt{|(g^{-1})^{00}|}}$, $\mathrm{d}\mu_g = \sqrt{|\mathrm{det} g|}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \mathrm{d}x^0$, and $\mathrm{d}\mu_{\underline{g}} = \sqrt{\mathrm{det}\underline{g}}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 = \sqrt{|(g^{-1})^{00}|}\sqrt{|\mathrm{det} g|}\, \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$, where the last equality is a basic linear algebraic identity. Note that $\hat{N}$ is future-directed and timelike with respect to g, and that we used the fact that $(g^{-1})^{00} < 0$ (which is a simple consequence of the formula (2.13b) and our assumption that $0 < c \leq 1$).

From the above discussion, it follows that along any spacelike (with respect[36] to g) hypersurface with future-directed unit normal $\hat{N}$, we can construct a positive-definite energy density $g_{\alpha\beta} {}^{(X)}J^\alpha[\varphi] \hat{N}^\beta$ using any multiplier vectorfield X that is future-directed and timelike with respect to g. For the basic a priori estimates of interest to us, we will apply the above constructions along Σ_t with $X := u$, which is future-directed timelike with respect to g. As we described in Footnote 18, we cannot generally use $X := \partial_t$ because $g(\partial_t, \partial_t) > 0$

[36] A hypersurface is spacelike with respect to g if, at each point, its unit normal is timelike with respect to g.

can occur when $\sum_{a=1}^{3}|u^a|$ is large; in contrast, note that by (2.13a) and the normalization condition $\eta_{\kappa\lambda}u^\kappa u^\lambda = -1$, we have $g_{\kappa\lambda}u^\kappa u^\lambda = -1$. Thus, we define the following energy (where $\hat{N}^\alpha = -\frac{(g^{-1})^{\alpha 0}}{\sqrt{|(g^{-1})^{00}|}}$):

$$E_{\mathrm{wave}}(t) = E_{\mathrm{wave}}[\varphi](t) := \int_{\Sigma_t} g_{\alpha\beta}\,{}^{(u)}J^\alpha[\varphi]\hat{N}^\beta\,\mathrm{d}\mu_{\underline{g}}. \tag{9.51}$$

From (9.50), definition (9.51), and the standard expansion[37] of covariant derivatives in terms of partial derivatives and Christoffel symbols (which in particular can be used to derive the identity ${}^{(u)}\pi_{\alpha\beta} = u^\kappa\partial_\kappa g_{\alpha\beta} + g_{\alpha\kappa}\partial_\beta u^\kappa + g_{\beta\kappa}\partial_\alpha u^\kappa$), we deduce the following energy identity relative to the standard coordinates:

$$\begin{aligned}
E_{\mathrm{wave}}[\varphi](t) = E_{\mathrm{wave}}[\varphi](0) &- \int_{[0,t]\times\mathbb{T}^3} (\square_g\varphi)u^\kappa\partial_\kappa\varphi\,\mathrm{d}\mu_g \\
&- \frac{1}{2}\int_{[0,t]\times\mathbb{T}^3} (g^{-1})^{\alpha\gamma}(g^{-1})^{\beta\delta}T_{\alpha\beta}[\varphi]u^\kappa\partial_\kappa g_{\gamma\delta}\,\mathrm{d}\mu_g \\
&- \int_{[0,t]\times\mathbb{T}^3} (g^{-1})^{\beta\delta}T_{\alpha\beta}[\varphi]\partial_\delta u^\alpha\,\mathrm{d}\mu_g \\
&+ \int_{[0,t]\times\mathbb{T}^3} \left\{(\partial_\kappa u^\kappa)\varphi^2 + \Gamma_{\kappa\ \lambda}^{\ \kappa}u^\lambda\varphi^2 + 2\varphi u^\kappa\partial_\kappa\varphi\right\}\mathrm{d}\mu_g.
\end{aligned} \tag{9.52}$$

On RHS (9.52),

$$\Gamma_{\alpha\ \beta}^{\ \gamma} := \frac{1}{2}(g^{-1})^{\gamma\delta}\left\{\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right\} \tag{9.53}$$

are the Christoffel symbols of g relative to the standard coordinates. Note that by (2.13a)–(2.13b) we have that

$$\Gamma_{\alpha\ \beta}^{\ \gamma} = \mathrm{f}(h, s, u, \partial h, S, \partial u), \tag{9.54}$$

where f is a smooth function (depending on α, β, and γ).

Next, with the help of (2.13a)–(2.13b) and the normalization condition $\eta_{\kappa\lambda}u^\kappa u^\lambda = -1$, we compute that

$$\begin{aligned}
&g_{\alpha\beta}\,{}^{(u)}J^\alpha[\varphi]\hat{N}^\beta \\
&\quad = \left\{c^2 T_{0\beta}[\varphi]u^\beta + (1-c^2)u^0 T_{\alpha\beta}[\varphi]u^\alpha u^\beta + u^0\varphi^2\right\}\frac{1}{\sqrt{|(g^{-1})^{00}|}} \\
&\quad = \frac{\frac{1}{2}u^0\left\{c^2(\partial_t\varphi)^2 + c^2\delta^{ab}(\partial_a\varphi)\partial_b\varphi + (1-c^2)(u^\alpha\partial_\alpha\varphi)^2\right\}}{\sqrt{|(g^{-1})^{00}|}} \\
&\qquad + \frac{\left\{c^2(\partial_t\varphi)u^a\partial_a\varphi + u^0\varphi^2\right\}}{\sqrt{|(g^{-1})^{00}|}},
\end{aligned} \tag{9.55}$$

where δ^{ab} is the Kronecker delta. From (9.51) and (9.55), it follows that

[37]For example, $\nabla_\alpha X^\beta = \partial_\alpha X^\beta + \Gamma_{\alpha\ \gamma}^{\ \beta}X^\gamma$, where $\Gamma_{\alpha\ \gamma}^{\ \beta}$ is defined by (9.53).

$$\begin{aligned}
&E_{\text{wave}}[\varphi](t)\\
&\quad = \frac{1}{2}\int_{\Sigma_t} u^0 \left\{c^2(\partial_t\varphi)^2 + c^2\delta^{ab}(\partial_a\varphi)\partial_b\varphi + (1-c^2)(u^\alpha\partial_\alpha\varphi)^2\right\} \frac{\mathrm{d}\mu_{\underline{g}}}{\sqrt{|(g^{-1})^{00}|}}\\
&\qquad + \int_{\Sigma_t}\left\{c^2(\partial_t\varphi)u^a\partial_a\varphi + u^0\varphi^2\right\} \frac{\mathrm{d}\mu_{\underline{g}}}{\sqrt{|(g^{-1})^{00}|}}. \qquad (9.56)
\end{aligned}$$

The energy $E_{\text{wave}}[\varphi](t)$ will yield L^2 control of φ and its first derivatives. In Sect. 9.6.3, we will establish the coerciveness $E_{\text{wave}}[\varphi](t)$. To obtain L^2 control of the higher-order spatial derivatives of φ, one can use energies of the form $E_{\text{wave}}[\partial_{\vec{I}}\varphi]$, where $\vec{I}$ is a spatial multi-index.

9.6.2. Energy Estimates for Transport Equations. One can derive energy estimates for transport equations of the form $u^\alpha\partial_\alpha\varphi = f$ by relying on the following energy:

$$E_{\text{transport}}[\varphi](t) := \int_{\Sigma_t} \varphi^2\,\mathrm{d}x, \qquad (9.57)$$

as in the proof of Lemma 9.6. The analog of the wave equation energy identity (9.52) is the following integral identity, whose simple proof follows from the ideas featured in the proof of Lemma 9.6:

$$\begin{aligned}
E_{\text{transport}}[\varphi](t) = E_{\text{transport}}[\varphi](0) &+ \int_0^t\int_{\Sigma_\tau}\left\{\partial_a\left(\frac{u^a}{u^0}\right)\right\}\varphi^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&+ 2\int_0^t\int_{\Sigma_\tau}\varphi\frac{u^\alpha\partial_\alpha\varphi}{u^0}\,\mathrm{d}x\,\mathrm{d}\tau. \qquad (9.58)
\end{aligned}$$

To control the higher-order derivatives of φ, one can rely on energies of the form $E_{\text{transport}}[\partial_{\vec{I}}\varphi]$. We mention that the argument we have sketched here relies on the basic fact that $u^0 > 0$, which allows us to divide by u^0 on RHS (9.58); for the relativistic Euler equations, this fact follows from the normalization condition $\eta_{\kappa\lambda}u^\kappa u^\lambda = -1$ and the fact that u is future-directed.

9.6.3. Comparison of the Energies with the Sobolev Norm. The coerciveness properties of the wave equation energy $E_{\text{wave}}[\varphi](t)$ constructed in Sect. 9.6.1 are tied to the metric g; see (9.51). In order to obtain our results, we need $E_{\text{wave}}[\varphi](t)$ to be uniformly comparable to a corresponding Sobolev norm along Σ_t. More precisely, we need to ensure the existence of a constant $C > 1$ such that on the slab $[0,T]\times\mathbb{T}^3$ of existence guaranteed by Theorem 9.10, the following estimates hold:

$$\begin{aligned}
C^{-1}\left\{\|\varphi\|^2_{H^N(\Sigma_t)} + \|\partial_t\varphi\|^2_{H^{N-1}(\Sigma_t)}\right\} &\le \sum_{0\le|\vec{I}|\le N-1} E_{\text{wave}}[\partial_{\vec{I}}\varphi](t)\\
&\le C\left\{\|\varphi\|^2_{H^N(\Sigma_t)} + \|\partial_t\varphi\|^2_{H^{N-1}(\Sigma_t)}\right\}. \qquad (9.59)
\end{aligned}$$

To see that such a constant C exists, we first use Young's inequality, (2.20), and Cauchy–Schwarz to bound the first product in braces on the last line of

RHS (9.56) as follows:

$$
\begin{aligned}
&c^2(\partial_t\varphi)u^a\partial_a\varphi\\
&\geq -\frac{1}{2}c^2\left(\sqrt{\sum_{i=1}^{3}(u^i)^2}\right)(\partial_t\varphi)^2 - \frac{1}{2}c^2\left(\sqrt{\sum_{i=1}^{3}(u^i)^2}\right)\delta^{ab}(\partial_a\varphi)\partial_b\varphi\\
&= -\frac{1}{2}c^2\left(\sqrt{(u^0)^2-1}\right)(\partial_t\varphi)^2 - \frac{1}{2}c^2\left(\sqrt{(u^0)^2-1}\right)\delta^{ab}(\partial_a\varphi)\partial_b\varphi. \qquad (9.60)
\end{aligned}
$$

Next, we recall that Theorem 9.10 guarantees that on $[0,T]\times\mathbb{T}^3$, the solution never escapes the compact subset $\mathfrak{K}$ featured in the statement of the theorem. In view of (9.60), we see that this ensures that on $[0,T]\times\mathbb{T}^3$, the product $c^2(\partial_t\varphi)u^a\partial_a\varphi$ on the last line of RHS (9.56) can be absorbed into the sum $\frac{1}{2}c^2u^0(\partial_t\varphi)^2+\frac{1}{2}c^2u^0\delta^{ab}(\partial_a\varphi)\partial_b\varphi$ from the first line of RHS (9.56), with room to spare. This implies that for solutions contained in $\mathfrak{K}$, the integrands on RHS (9.56) are in total uniformly comparable to $\sum_{\alpha=0}^{3}(\partial_\alpha\varphi)^2+\varphi^2$. This also ensures that on $[0,T]\times\mathbb{T}^3$, the volume form $\frac{\mathrm{d}\mu_{\underline{g}}}{\sqrt{|(g^{-1})^{00}|}}$ on Σ_t is uniformly comparable[38] to $\mathrm{d}x := \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$. From these observations, it readily follows that a $C>1$ exists such that (9.59) holds.

9.6.4. Proof of Proposition 9.22. Recall that the assumptions of the proposition guarantee that we have a smooth solution to the system (2.17)–(2.19) + (2.20). Consider the scalar component functions

$$h, u^\alpha, S^\alpha, \varpi^\alpha, \mathcal{C}^\alpha, \mathcal{D}, \qquad (9.61)$$

introduced in Sect. 2. According to Theorem 3.1, they satisfy the system of evolution equations given by Eqs. (3.1), (3.3), (3.7), (3.8), (3.9a), and (3.11b). Next, we recall that the hyperbolic quantities $\mathbf{H}$ and the elliptic quantities $\mathbf{E}$ were defined in Definition 9.15. To prove the proposition, we claim that it suffices to show that the following inequality holds for $t\in[0,T]$:

$$\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_t)} \leq C\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_0)} + C\int_0^t \|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)}\,\mathrm{d}\tau, \qquad (9.62)$$

where in (9.62) and in the rest of this proof, C is as in the statement of Proposition 9.22. For once we have shown (9.62), we can use Gronwall's inequality to deduce (recalling that C is allowed to depend on the initial data and can vary from line to line) that the following estimate holds for $t\in[0,T]$:

$$\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_t)} \leq C\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_0)}\exp(Ct) \leq C\exp(Ct) \leq C\exp(CT). \qquad (9.63)$$

Then from (9.39) and (9.63) we conclude, in view of Definition 9.15, the desired bound (9.44), except for the estimates for u^0, S^0, and ϖ^0. To obtain the desired estimate for these quantities, we first express u^0-1, S^0, ϖ^0, $\partial_a u^0$, $\partial_a S^0$, and $\partial_a\varpi^0$ as $\mathrm{f}(\mathbf{H},\mathbf{E})$, with f smooth and satisfying $\mathrm{f}(0)=0$ [this is

[38]To see this, it is helpful to note the following identity, which holds relative to the standard coordinates: $\frac{\mathrm{d}\mu_{\underline{g}}}{\sqrt{|(g^{-1})^{00}|}} = c^{-3}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$. This identity follows from (2.14a) and the linear algebraic identity $\det\underline{g} = (g^{-1})^{00}\det g$.

possible in view of definition (9.32e) and (9.33c)]. We then use Lemma 9.4 to deduce that $\|\mathrm{f}(\mathbf{H},\mathbf{E})\|_{H^{N-1}(\Sigma_t)} \leq C\|\mathbf{H}\|_{H^{N-1}(\Sigma_t)} + C\|\mathbf{E}\|_{H^{N-1}(\Sigma_t)}$. Finally, we use the elliptic estimate (9.39) and (9.63) to conclude that $C\|\mathbf{H}\|_{H^{N-1}(\Sigma_t)} + C\|\mathbf{E}\|_{H^{N-1}(\Sigma_t)} \leq$ RHS (9.44), which yields the desired estimates.

It remains for us to prove (9.62). We start by noting that the results described in Sects. 9.6.1–9.6.3 can be used to derive the following estimates, where we recall that $\underline{V}$ denotes the spatial components of V (i.e., the η-orthogonal projection of V onto constant-time hypersurfaces, as in Sect. 9.1.1):

$$\|h\|^2_{H^N(\Sigma_t)} + \|\partial_t h\|^2_{H^{N-1}(\Sigma_t)} \leq C\left\{\|h\|^2_{H^N(\Sigma_0)} + \|\partial_t h\|^2_{H^{N-1}(\Sigma_0)}\right\} + C\int_0^t \|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)}\,\mathrm{d}\tau, \tag{9.64}$$

$$\|\underline{u}\|^2_{H^N(\Sigma_t)} + \|\partial_t \underline{u}\|^2_{H^N(\Sigma_t)} \leq C\left\{\|\underline{u}\|^2_{H^N(\Sigma_0)} + \|\partial_t \underline{u}\|^2_{H^N(\Sigma_t)}\right\} + C\int_0^t \|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)}\,\mathrm{d}\tau, \tag{9.65}$$

$$\|\underline{S}\|^2_{H^{N-1}(\Sigma_t)} \leq C\|\underline{S}\|^2_{H^{N-1}(\Sigma_0)} + C\int_0^t \|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)}\,\mathrm{d}\tau, \tag{9.66}$$

$$\|\underline{\varpi}\|^2_{H^{N-1}(\Sigma_t)} \leq C\|\underline{\varpi}\|^2_{H^{N-1}(\Sigma_0)} + C\int_0^t \|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)}\,\mathrm{d}\tau, \tag{9.67}$$

$$\|\mathcal{D}\|^2_{H^{N-1}(\Sigma_t)} \leq C\|\mathcal{D}\|^2_{H^{N-1}(\Sigma_0)} + C\int_0^t \left\{\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)} + \|\mathbf{E}\|^2_{H^{N-1}(\Sigma_\tau)}\right\}\mathrm{d}\tau, \tag{9.68}$$

$$\|\underline{\mathcal{C}}\|^2_{H^{N-1}(\Sigma_t)} \leq C\|\underline{\mathcal{C}}\|^2_{H^{N-1}(\Sigma_0)} + C\int_0^t \left\{\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)} + \|\mathbf{E}\|^2_{H^{N-1}(\Sigma_\tau)}\right\}\mathrm{d}\tau. \tag{9.69}$$

The estimates (9.64)–(9.69) are standard and can be derived by commuting the evolution equations of Theorem 3.1 (more precisely, only the evolution equations for the spatial components of u, ϖ, S, and $\mathcal{C}$) with spatial derivative operators $\partial_{\vec{I}}$ and using the energy identities (9.52) and (9.58) (and their analogs for the $\partial_{\vec{I}}$-differentiated solution variables), the coerciveness estimate (9.59), Lemma 9.17, and the Sobolev–Moser-type estimate (9.11). We stress that RHS (9.11) is *linear* in the order r derivatives of the solution; this is the reason the integrands on RHS (9.64)–(9.69) are quadratic in $\|\mathbf{H}\|_{H^{N-1}(\Sigma_\tau)}$ and $\|\mathbf{E}\|_{H^{N-1}(\Sigma_\tau)}$ [the sup-norm factors on RHS (9.11) can be bounded by $\leq C$ since those factors are among the quantities that constants C are allowed to depend on]. The non-standard aspect of the remaining part of the proof is the appearance of the term $\|\mathbf{E}\|^2_{H^{N-1}(\Sigma_\tau)}$ on RHSs (9.68)–(9.69); we clarify that these terms are generated by the terms $\partial_a S_b$ and $\partial_a \varpi_b$ on RHSs (3.9a) and (3.11b) [see definition (9.32b)]. Next, adding (9.64)–(9.69) and appealing to Definition 9.15, we deduce that

$$\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_t)} \leq C\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_0)} + C\int_0^t \left\{\|\mathbf{H}\|^2_{H^{N-1}(\Sigma_\tau)} + \|\mathbf{E}\|^2_{H^{N-1}(\Sigma_\tau)}\right\} \mathrm{d}\tau. \tag{9.70}$$

Finally, from (9.70) and the elliptic estimate (9.39), we conclude the desired bound (9.62). □

9.7. Proof of Theorem 9.12

We now prove Theorem 9.12, which is the main result of Sect. 9. By Theorem 9.10, we need only to show that (i) under the regularity assumptions on the initial data stated in Theorem 9.12, the standard local well-posedness results (9.28a)–(9.28b) can be upgraded to (9.30a)–(9.30b) and (ii) that the solution depends continuously on the initial data, where continuity is measured in the norms corresponding to the function spaces featured in (9.30a)–(9.30b). Throughout this proof, $\mathfrak{K}$ denotes the set featured in the statement of Theorem 9.10. To proceed, we let $(\mathring{h}_{(m)}, \mathring{s}_{(m)}, \mathring{u}^i_{(m)}) \subset \left(C^\infty(\mathbb{T}^3)\right)^5$ be a sequence of smooth initial data such that as $m \to \infty$, we have

$$\left\|\mathring{h}_{(m)} - \mathring{h}\right\|_{H^N(\Sigma_0)} \to 0, \qquad \left\|\mathring{u}^i_{(m)} - \mathring{u}^i\right\|_{H^N(\Sigma_0)} \to 0, \tag{9.71}$$

$$\left\|\mathring{s}_{(m)} - \mathring{s}\right\|_{H^{N+1}(\Sigma_0)} \to 0, \qquad \left\|\mathring{\varpi}^i_{(m)} - \mathring{\varpi}^i\right\|_{H^N(\Sigma_0)} \to 0, \tag{9.72}$$

where $\mathring{\varpi}^i_{(m)}$ denotes the initial vorticity of the mth element of the sequence and $\mathring{\varpi}^i$ is as in the statement of the theorem. Let $(h_{(m)}, s_{(m)}, u^\alpha_{(m)}, S^\alpha_{(m)}, \varpi^\alpha_{(m)})$ denote the corresponding sequence of solution variables. Theorem 9.10 yields (see, for example, [31], for additional details) that for m sufficiently large, the element $(h_{(m)}, s_{(m)}, u^\alpha_{(m)})$ is a C^∞ classical solution to Eqs. (2.17)–(2.19) + (2.20) on the fixed slab $[0,T] \times \mathbb{T}^3$ with

$$(h_{(m)}(p), s_{(m)}(p), u^1_{(m)}(p), u^2_{(m)}(p), u^3_{(m)}(p)) \in \operatorname{int}\mathfrak{K}$$

for $p \in [0,T] \times \mathbb{T}^3$, and that on the same slab, $(h_{(m)}, s_{(m)}, u^\alpha_{(m)}, S^\alpha_{(m)}, \varpi^\alpha_{(m)})$ is a C^∞ solution to the equations of Theorem 3.1 [which are consequences of (2.17)–(2.19) + (2.20)]. Moreover, Theorem 9.10 also implies that the sequence converges to the solution in the following norms as $m \to \infty$:

$$\left\|h_{(m)} - h\right\|_{C\left([0,T],H^N(\mathbb{T}^3)\right)} \to 0, \tag{9.73}$$

$$\left\|u^\alpha_{(m)} - u^\alpha\right\|_{C\left([0,T],H^N(\mathbb{T}^3)\right)} \to 0, \tag{9.74}$$

$$\left\|s_{(m)} - s\right\|_{C\left([0,T],H^N(\mathbb{T}^3)\right)} \to 0, \tag{9.75}$$

$$\left\| S^{\alpha}_{(m)} - S^{\alpha} \right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} \to 0, \tag{9.76}$$

$$\left\| \varpi^{\alpha}_{(m)} - \varpi^{\alpha} \right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} \to 0. \tag{9.77}$$

Next, we use the convergence results (9.73)–(9.77), Theorem 9.10, and the a priori estimates provided by Proposition 9.22 to deduce that exist a constant $C > 0$, depending on T and on the four types of quantities listed just above (9.44), and a positive integer m_0 such that

$$\sup_{m \geq m_0} \sup_{\tau \in [0,T]} \|s_{(m)}\|_{H^{N+1}(\Sigma_\tau)} \leq C, \tag{9.78}$$

$$\sup_{m \geq m_0} \sup_{\tau \in [0,T]} \|S^{\alpha}_{(m)}\|_{H^{N}(\Sigma_\tau)} \leq C, \tag{9.79}$$

$$\sup_{m \geq m_0} \sup_{\tau \in [0,T]} \|\varpi^{\alpha}_{(m)}\|_{H^{N}(\Sigma_\tau)} \leq C. \tag{9.80}$$

Since $H^r(\mathbb{T}^3)$ is a Hilbert space for $r \in \mathbb{R}$, it follows from the norm-boundedness results (9.78)–(9.80) that for each $\tau \in [0,T]$, there exist subsequences $s_{(m_n)}$, $S^{\alpha}_{(m_n)}$, and $\varpi^{\alpha}_{(m_n)}$ that weakly converge in $H^{N+1}(\Sigma_\tau)$, $H^N(\Sigma_\tau)$, and $H^N(\Sigma_\tau)$ respectively as $n \to \infty$. Moreover, since the norm is weakly lower semicontinuous in a Hilbert space, it follows that the limits are bounded, respectively, in the norms $\|\cdot\|_{H^{N+1}(\Sigma_\tau)}$, $\|\cdot\|_{H^N(\Sigma_\tau)}$, and $\|\cdot\|_{H^N(\Sigma_\tau)}$, by $\leq C$, where C is the same constant found on RHSs (9.78)–(9.80). From (9.76) to (9.77), it follows that the limits must be s, S^α, and ϖ^α respectively. We have therefore shown that

$$\sup_{\tau \in [0,T]} \|s\|_{H^{N+1}(\Sigma_\tau)} \leq C, \tag{9.81}$$

$$\sup_{\tau \in [0,T]} \|S^{\alpha}\|_{H^{N}(\Sigma_\tau)} \leq C, \tag{9.82}$$

$$\sup_{\tau \in [0,T]} \|\varpi^{\alpha}\|_{H^{N}(\Sigma_\tau)} \leq C. \tag{9.83}$$

To complete the proof of (9.30b), we must show that for each spatial multi-index $\vec{I}$ with $|\vec{I}| = N$, the map $t \to \partial_{\vec{I}} S^\alpha(t,\cdot)$ is a continuous map from $[0,T]$ into $L^2(\mathbb{T}^3)$, and similarly for ϖ^α (the desired time-continuity results for s then follow from the relation $\partial_i s = S_i$). To keep the presentation short, we illustrate only the right-continuity of these maps at $t = 0$; the general statement can be proved by making minor modifications to the argument that we give. That is, we will show that

$$\lim_{t \downarrow 0} \|\partial_{\vec{I}} S^\alpha(t,\cdot) - \partial_{\vec{I}} \mathring{S}^\alpha(\cdot)\|_{L^2(\mathbb{T}^3)} = 0, \qquad |\vec{I}| = N, \tag{9.84a}$$

$$\lim_{t \downarrow 0} \|\partial_{\vec{I}} \varpi^\alpha(t,\cdot) - \partial_{\vec{I}} \mathring{\varpi}^\alpha(\cdot)\|_{L^2(\mathbb{T}^3)} = 0, \qquad |\vec{I}| = N, \tag{9.84b}$$

where $\mathring{S}^\alpha(\cdot) := S^\alpha(0,\cdot)$. The rest of our proof is based on Lemmas 9.6 and 9.20, but to apply the lemmas, we first have to derive some preliminary results. We will use the estimates provided by Lemma 9.4 without giving complete details each time we use them; we will refer to these estimates as the "standard

Sobolev calculus." In the rest of the proof, we will refer to the variable sets $\mathbf{H}$, $\mathbf{E}$, $\mathbf{A_H}$, $\mathbf{A_{H,E}}$, and $\mathbf{A}$ from Definition 9.15.

As a first step in proving (9.84a)–(9.84b), we will show that

$$\mathbf{H},\ \mathbf{A_H} \in C\big([0,T], H^{N-1}(\mathbb{T}^3)\big), \tag{9.85}$$

where $\mathbf{H}$ and $\mathbf{A_H}$ are defined in (9.32a) and (9.32c). Note that by (9.33a) and the standard Sobolev calculus, the desired result $\mathbf{A_H} \in C\big([0,T], H^{N-1}(\mathbb{T}^3)\big)$ would follow from $\mathbf{H} \in C\big([0,T], H^{N-1}(\mathbb{T}^3)\big)$. The latter statement is equivalent to showing that $\partial_{\vec{I}}\mathbf{H} \in C\big([0,T], L^2(\mathbb{T}^3)\big)$ for $|\vec{I}| \leq N-1$. All of these results, except in the case of the top-order (i.e., order $N-1$) derivatives of $\mathcal{C}^i$ and $\mathcal{D}$, follow from the standard local well-posedness time-continuity results (9.28a)–(9.28b), and the standard Sobolev calculus. Thus, to complete the proof of (9.85), we need only to show that for $i = 1,2,3$, we have

$$\partial_{\vec{I}}\mathcal{C}^i,\ \partial_{\vec{I}}\mathcal{D} \in C\big([0,T], L^2(\mathbb{T}^3)\big), \qquad |\vec{I}| = N-1. \tag{9.86}$$

The desired result (9.86) follows from using Eqs. (3.9a) and (3.11b) [more precisely, we need only to consider the spatial components of (3.11b)], the boundedness results (9.81)–(9.83), the standard local well-posedness time-continuity results (9.28a)–(9.28b), and the standard Sobolev calculus to deduce that $\partial_{\vec{I}}\mathcal{C}^i$ and $\partial_{\vec{I}}\mathcal{D}$ solve transport equations that satisfy the hypotheses of Lemma 9.6; put succinctly, we can apply Lemma 9.6 with $f := \partial_{\vec{I}}\mathcal{C}^i$ and $f := \partial_{\vec{I}}\mathcal{D}$. We have therefore proved (9.85). In particular, it follows from (9.85) and the definition of $\mathbf{A_H}$ that for $i = 1,2,3$, we have

$$^{(3)}\mathrm{curl}^i(\underline{\varpi}),\ ^{(3)}\mathrm{curl}^i(\underline{S}) \in C\big([0,T], H^{N-1}(\mathbb{T}^3)\big). \tag{9.87}$$

Next, we note that in view of Definition 9.15, Lemma 9.17 (in particular the relation (9.33b) for $\partial_a S^0$ and $\partial_a \varpi^0$), (9.85), and the standard Sobolev calculus, the desired results (9.84a)–(9.84b) would follow as a consequence of the following convergence result:

$$\lim_{t\downarrow 0} \|\partial_{\vec{I}}\mathbf{E}(t,\cdot) - \partial_{\vec{I}}\mathbf{E}(0,\cdot)\|_{L^2(\mathbb{T}^3)} = 0, \qquad |\vec{I}| = N-1. \tag{9.88}$$

To establish (9.88), we first use (9.85), (9.34), and the standard Sobolev calculus to deduce the following facts, where $(G^{-1})^{ij}$ is defined in Definition 9.13:

$$(G^{-1})^{ab}\partial_a\partial_{\vec{I}}S_b,\ (G^{-1})^{ab}\partial_a\partial_{\vec{I}}\varpi_b \in C\big([0,T], L^2(\mathbb{T}^3)\big), \quad |\vec{I}| = N-1. \tag{9.89}$$

In the rest of the proof, $\alpha_* > 0$ is as in the statement of Lemma 9.20 in the case $(M^{-1})^{ij}(t,x) := (G^{-1})^{ij}(t,x)$. Next, setting

$$(\mathring{G}^{-1})^{ij}(\cdot) := (G^{-1})^{ij}(0,\cdot), \tag{9.90}$$

applying Lemma 9.20 with $(M^{-1})^{ij} := (\mathring{G}^{-1})^{ij}$, and appealing to definition (9.32b), we see that in order to prove (9.88), it suffices to show the following convergence result:

$$\lim_{t\downarrow 0} \mathbb{E}_{N;\mathring{G}^{-1};\alpha_*}[(\underline{\varpi},\underline{S}) - (\mathring{\underline{\varpi}},\mathring{\underline{S}})](t) = 0, \tag{9.91}$$

where $(\mathring{\underline{\varpi}},\mathring{\underline{S}}) := (\underline{\varpi},\underline{S})|_{\Sigma_0}$.

To initiate the proof of (9.91), we let $\varphi \in H^{-N}(\mathbb{T}^3)$ be any element of the dual space of $H^N(\mathbb{T}^3)$. From the below-top-order continuity result (9.28b), the top-order boundedness results (9.82)–(9.83), and the density of C^∞ functions in $H^{-N}(\mathbb{T}^3)$, it is straightforward to deduce that the following "weak continuity" result holds for $i = 1, 2, 3$:

$$\lim_{t \downarrow 0} \int_{\mathbb{T}^3} S^i(t,x)\varphi \, \mathrm{d}x = \int_{\mathbb{T}^3} \mathring{S}^i \varphi \, \mathrm{d}x. \tag{9.92}$$

Since φ was arbitrary, we conclude that $S^i(t,\cdot)$ weakly converges to $\mathring{S}^i$ in $H^N(\mathbb{T}^3)$ as $t \downarrow 0$. Similarly, $\varpi^i(t,\cdot)$ weakly converges to $\mathring{\varpi}^i$ in $H^N(\mathbb{T}^3)$ as $t \downarrow 0$. We now let $\langle \cdot, \cdot \rangle_{\mathring{G}^{-1};\alpha_*}$ denote the inner product (9.35) on the Hilbert space $\left(H^N(\Sigma_t)\right)^3 \times \left(H^N(\Sigma_t)\right)^3$, and we let $\langle \cdot, \cdot \rangle$ denote the standard inner product on the same Hilbert space (obtained by keeping only the two sums on the last line of RHS (9.35) and replacing $N-1$ with N in the summation bounds). By Lemma 9.20, the two corresponding norms [i.e., the norms on the left- and right-hand sides of (9.37a)–(9.37b)] are equivalent. It is a basic result of functional analysis that given these two inner products with equivalent norms, a sequence weakly convergences relative to $\langle \cdot, \cdot \rangle_{\mathring{G}^{-1};\alpha_*}$ if and only if it weakly converges relative to $\langle \cdot, \cdot \rangle$. In particular, in view of the weak convergence results for $S^i(t,\cdot)$ and $\varpi^i(t,\cdot)$ proved above, we infer that $(\underline{\varpi}(t,\cdot), \underline{S}(t,\cdot))$ weakly converges to $(\underline{\mathring{\varpi}}(\cdot), \underline{\mathring{S}}(\cdot))$ relative to the inner product $\langle \cdot, \cdot \rangle_{\mathring{G}^{-1};\alpha_*}$ as $t \downarrow 0$. Moreover, it is another basic result of functional analysis that based on this weak convergence and Lemma 9.20, in order to prove the result (9.91), it suffices to show that

$$\limsup_{t \downarrow 0} \mathbb{E}_{N;\mathring{G}^{-1};\alpha_*}[(\underline{\varpi}, \underline{S})](t) \leq \mathbb{E}_{N;\mathring{G}^{-1};\alpha_*}[(\underline{\mathring{\varpi}}, \underline{\mathring{S}})]. \tag{9.93}$$

Moreover, since the standard local well-posedness time-continuity results (9.28a) and (9.9) imply that $\lim_{t \downarrow 0} \left\| (G^{-1})^{ij}(t,\cdot) - (\mathring{G}^{-1})^{ij} \right\|_{C(\mathbb{T}^3)} = 0$, it follows from definitions (9.35) and (9.36) and the top-order boundedness results (9.81)–(9.83) that in order to prove (9.93), it suffices to show that

$$\limsup_{t \downarrow 0} \mathbb{E}_{N;G^{-1};\alpha_*}[(\underline{\varpi}, \underline{S})](t) \leq \mathbb{E}_{N;\mathring{G}^{-1};\alpha_*}[(\underline{\mathring{\varpi}}, \underline{\mathring{S}})], \tag{9.94}$$

where we stress that the inverse metric G^{-1} on LHS (9.94) depends on t [which is different compared to (9.93)]. In fact, our arguments will yield a stronger statement than (9.94). More precisely, we will show the following time-continuity result:

$$\lim_{t \downarrow 0} \mathbb{E}_{N;G^{-1};\alpha_*}[(\underline{\varpi}, \underline{S})](t) = \mathbb{E}_{N;\mathring{G}^{-1};\alpha_*}[(\underline{\mathring{\varpi}}, \underline{\mathring{S}})], \tag{9.95}$$

To proceed, we use definitions (9.35) and (9.36) and the standard local well-posedness time-continuity results (9.28a)–(9.28b) to deduce that all terms in the definition of $\mathbb{E}_{N;G^{-1};\alpha_*}[(\varpi, S)](t)$ have been shown to have the desired continuous time dependence at except for the ones depending on the order N derivatives of ϖ or S [i.e., the ones corresponding to the terms on the first four lines of RHS (9.35)]. The continuous time dependence of these remaining four terms follows from (9.87), (9.89), and the fact that $(G^{-1})^{ij} \in C\big([0,T],C(\mathbb{T}^3)\big)$ [which follows from the standard local well-posedness time-continuity results (9.28a) and (9.9)]. We have therefore proved (9.95), which finishes the proof of the desired result (9.30b).

To complete our proof of Theorem 9.12, we need to show continuous dependence on the initial data. To proceed, we let $(\mathring{h}_{(m)},\mathring{u}^i_{(m)},\mathring{s}_{(m)})$ be a sequence of initial data (not necessarily C^∞ now) such that as $m\to\infty$, the convergence results (9.71)–(9.72) hold. We again let $(h_{(m)},s_{(m)},u^\alpha_{(m)},S^\alpha_{(m)},\varpi^\alpha_{(m)})$ denote the corresponding sequence of solution variables (which are not necessarily C^∞ now). We aim to show that the sequence converges to the limiting solution $(h,u^\alpha,s,S^\alpha,\varpi^\alpha)$ in the norm $\|\cdot\|_{C\left([0,T],H^N\mathbb{T}^3)\right)}$ as $m\to\infty$. To proceed, we first note that Theorem 9.10 yields that for m sufficiently large, the element $(h_{(m)},s_{(m)},u^\alpha_{(m)},S^\alpha_{(m)},\varpi^\alpha_{(m)})$ is a classical solution (not necessarily C^∞ now) to Eqs. (2.17)–(2.19) + (2.20) on the fixed slab $[0,T]\times\mathbb{T}^3$ with $(h_{(m)}(p),s_{(m)}(p),u^1_{(m)}(p),u^2_{(m)}(p),u^3_{(m)}(p))\in \operatorname{int}\mathfrak{K}$ for $p\in[0,T]\times\mathbb{T}^3$, that it also is a strong solution[39] to the equations of Theorem 3.1, that there exists an integer m_0 such that

$$\sup_{m\geq m_0}\|s_{(m)}\|_{C\left([0,T],H^{N+1}(\mathbb{T}^3)\right)}\leq C, \tag{9.96}$$

$$\sup_{m\geq m_0}\|S^\alpha_{(m)}\|_{C\left([0,T],H^{N}(\mathbb{T}^3)\right)}\leq C, \tag{9.97}$$

$$\sup_{m\geq m_0}\|\varpi^\alpha_{(m)}\|_{C\left([0,T],H^{N}(\mathbb{T}^3)\right)}\leq C, \tag{9.98}$$

and that the following convergence results (which are below top-order for S and ϖ) hold as $m\to\infty$:

$$\left\|h-h_{(m)}\right\|_{C\left([0,T],H^{N}(\mathbb{T}^3)\right)}\to 0, \tag{9.99}$$

$$\left\|u^\alpha-u^\alpha_{(m)}\right\|_{C\left([0,T],H^{N}(\mathbb{T}^3)\right)}\to 0, \tag{9.100}$$

$$\left\|s-s_{(m)}\right\|_{C\left([0,T],H^{N}(\mathbb{T}^3)\right)}\to 0, \tag{9.101}$$

$$\left\|S^\alpha-S^\alpha_{(m)}\right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)}\to 0, \tag{9.102}$$

$$\left\|\varpi^\alpha-\varpi^\alpha_{(m)}\right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)}\to 0. \tag{9.103}$$

In view of (9.99)–(9.103), we see that to complete our proof of Theorem 9.12, we need only to show continuity in the top-order norms. That is, we must show that if $|\vec{I}|=N$, then as $m\to\infty$, we have

$$\left\|\partial_{\vec{I}}S^\alpha-\partial_{\vec{I}}S^\alpha_{(m)}\right\|_{C\left([0,T],L^{2}(\mathbb{T}^3)\right)}\to 0, \tag{9.104}$$

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[39]By "strong solution," we mean in particular that at each fixed $t\in[0,T]$, the equations of Theorem 3.1 are satisfied for almost every $x\in\mathbb{T}^3$.

$$\left\| \partial_{\vec{I}} \varpi^{\alpha} - \partial_{\vec{I}} \varpi^{\alpha}_{(m)} \right\|_{C\left([0,T], L^2(\mathbb{T}^3)\right)} \to 0. \tag{9.105}$$

To proceed, we first review an approach to proving the standard estimates (9.99)–(9.103). These estimates can be proved by applying Kato's abstract framework [17–19], which is designed to handle first-order hyperbolic systems in a rather general Banach space setting. In particular, one can apply Kato's framework to the first-order system (2.17)–(2.19) + (2.20); this is described in detail, for example, in [31]. To prove (9.104)–(9.105), we will modify Kato's framework so that it applies to the hyperbolic variables $\mathbf{H}$ and the elliptic variables $\mathbf{E}$ from Definition 9.15.

To employ Kato's framework, one relies on the propagators $\mathcal{U}(t,\tau) := \mathcal{U}(t,\tau;\mathbf{H})$ for the linear homogeneous hyperbolic system corresponding to the (nonlinear) first-order hyperbolic system that $\mathbf{H}$ satisfies. To shorten the presentation, we will not explicitly state the form of this linear first-order hyperbolic system; see Remark 9.16 for further discussion of its nature. By definition, $\mathcal{U}(t,\tau;\mathbf{H})$ maps initial data given at time τ to the solution of the linear homogeneous hyperbolic system (whose principal coefficients depend on $\mathbf{H}$) at time t. Similarly, one relies on the operators $\mathcal{U}_{(m)}(t,\tau) := \mathcal{U}(t,\tau;\mathbf{H}_{(m)})$ corresponding to the homogeneous linear system whose principal coefficients depend on $\mathbf{H}_{(m)}$. By Duhamel's principle, we have

$$\mathbf{H}(t) = \mathcal{U}(t,0)\mathring{\mathbf{H}} + \int_{\tau=0}^{t} \mathcal{U}(t,\tau)\mathrm{f}\left(\mathbf{H}(\tau), \mathbf{E}(\tau)\right) \,\mathrm{d}\tau, \tag{9.106}$$

$$\mathbf{H}_{(m)}(t) = \mathcal{U}_{(m)}(t,0)\mathring{\mathbf{H}}_{(m)} + \int_{\tau=0}^{t} \mathcal{U}_{(m)}(t,\tau)\mathrm{f}\left(\mathbf{H}_{(m)}(\tau), \mathbf{E}_{(m)}(\tau)\right) \,\mathrm{d}\tau, \tag{9.107}$$

where $\mathring{\mathbf{H}}$ and $\mathring{\mathbf{H}}_{(m)}$ respectively denote the initial data of $\mathbf{H}$ and $\mathbf{H}_{(m)}$, and on RHSs (9.106)–(9.107), f denotes the inhomogeneous term in the first-order hyperbolic system satisfied by the elements of $\mathbf{H}$ and $\mathbf{H}_{(m)}$. We have not explicitly stated the form of f since its precise structure is not important for our arguments here; what matters is only the following basic facts (that can easily be checked): f is a smooth function of its arguments satisfying $\mathrm{f}(0) = 0$, and the *same* f appears on RHSs (9.106)–(9.107).

The strategy behind Kato's framework is to control the difference $\mathbf{H}(t,\cdot) - \mathbf{H}_{(m)}(t,\cdot)$ in the norm $\|\cdot\|_{H^{N-1}(\mathbb{T}^3)}$ by subtracting (9.106)–(9.107), splitting the right-hand side of the resulting equation into various pieces, and bounding each piece by exploiting some standard properties of the propagators $\mathcal{U}(t,\tau)$ and $\mathcal{U}_{(m)}(t,\tau)$. This is explained in detail in [31, Section 7.4], and most of the arguments given there for controlling $\|\mathbf{H}(t,\cdot) - \mathbf{H}_{(m)}(t,\cdot)\|_{H^{N-1}(\mathbb{T}^3)}$ go through without any substantial changes. The one part of the argument that does require substantial changes is: in order to obtain a closed inequality for $\|\mathbf{H}(t,\cdot) - \mathbf{H}_{(m)}(t,\cdot)\|_{H^{N-1}(\mathbb{T}^3)}$, one needs to show that the difference of the inhomogeneous terms on RHSs (9.106)–(9.107) satisfies the following estimate for $t \in [0,T]$:

$$\left\|\mathrm{f}(\mathbf{H},\mathbf{E}) - \mathrm{f}(\mathbf{H}_{(m)},\mathbf{E}_{(m)})\right\|_{H^{N-1}(\Sigma_t)} \le C\|\mathbf{H} - \mathbf{H}_{(m)}\|_{H^{N-1}(\Sigma_t)}, \tag{9.108}$$

where the key point is that *the quantity* $\|\mathbf{E} - \mathbf{E}_{(m)}\|_{H^{N-1}(\Sigma_t)}$ *does not appear on RHS* (9.108).

The estimate (9.108) can be obtained with the help of elliptic estimates, as we now explain. First, we note that the top-order norm-boundedness results (9.96)–(9.98) and the convergence results (9.99)–(9.103) imply that

$$\lim_{m\to\infty} \left\{ \left\|\mathbf{H} - \mathbf{H}_{(m)}\right\|_{C\left([0,T],L^2(\mathbb{T}^3)\right)} + \left\|\mathbf{E} - \mathbf{E}_{(m)}\right\|_{C\left([0,T],L^2(\mathbb{T}^3)\right)} \right\} = 0, \tag{9.109}$$

and that there exists an integer m_0 and a constant $C > 0$ such that

$$\|\mathbf{H}\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} + \|\mathbf{E}\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} \leq C, \tag{9.110}$$

$$\sup_{m\geq m_0} \left\{ \left\|\mathbf{H}_{(m)}\right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} + \left\|\mathbf{E}_{(m)}\right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} \right\} \leq C. \tag{9.111}$$

From (9.109), (9.110)–(9.111), and the Sobolev interpolation result (9.14), we deduce that if $N' < N - 1$, then

$$\lim_{m\to\infty} \left\{ \left\|\mathbf{H} - \mathbf{H}_{(m)}\right\|_{C\left([0,T],H^{N'}(\mathbb{T}^3)\right)} + \left\|\mathbf{E} - \mathbf{E}_{(m)}\right\|_{C\left([0,T],H^{N'}(\mathbb{T}^3)\right)} \right\} = 0. \tag{9.112}$$

Fixing a real number N' satisfying $3/2 < N' < 2$ and using (9.112) and the Sobolev embedding result (9.9), we deduce that

$$\lim_{m\to\infty} \left\{ \left\|\mathbf{H} - \mathbf{H}_{(m)}\right\|_{C([0,T]\times\mathbb{T}^3)} + \left\|\mathbf{E} - \mathbf{E}_{(m)}\right\|_{C([0,T]\times\mathbb{T}^3)} \right\} = 0. \tag{9.113}$$

Next, we use (9.110), (9.111), (9.113), (9.9), and (9.12) to deduce that there is a constant $C > 0$ such that if m is sufficiently large, then for $t \in [0, T]$, the following estimate holds for the function f appearing on RHSs (9.106)–(9.107):

$$\begin{aligned} \left\|\mathrm{f}(\mathbf{H}, \mathbf{E}) - \mathrm{f}(\mathbf{H}_{(m)}, \mathbf{E}_{(m)})\right\|_{H^{N-1}(\Sigma_t)} \leq C\|\mathbf{H} - \mathbf{H}_{(m)}\|_{H^{N-1}(\Sigma_t)} \\ + C\|\mathbf{E} - \mathbf{E}_{(m)}\|_{H^{N-1}(\Sigma_t)}. \end{aligned} \tag{9.114}$$

Next, we use (9.110), (9.111), (9.113), and (9.40) to deduce that if m is sufficiently large, then for $t \in [0, T]$, the last term on RHS (9.114) obeys the following bound:

$$\|\mathbf{E} - \mathbf{E}_{(m)}\|_{H^{N-1}(\Sigma_t)} \leq C\|\mathbf{H} - \mathbf{H}_{(m)}\|_{H^{N-1}(\Sigma_t)}. \tag{9.115}$$

The desired bound (9.108) follows from (9.114) and (9.115). Kato's framework (see [31, Section 7.4]) then allows one to conclude that

$$\lim_{m\to\infty} \left\|\mathbf{H} - \mathbf{H}_{(m)}\right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} = 0. \tag{9.116}$$

Moreover, (9.115) and (9.116) imply that

$$\lim_{m\to\infty} \left\|\mathbf{E} - \mathbf{E}_{(m)}\right\|_{C\left([0,T],H^{N-1}(\mathbb{T}^3)\right)} = 0. \tag{9.117}$$

Finally, in view of Definition 9.15 and the relation (9.33c), we note that the desired convergence results (9.104)–(9.105) follow from (9.116) to (9.117) and the standard Sobolev calculus [which is needed to handle the components $\alpha = 0$ in (9.104)–(9.105)]. □

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Rough sound waves in 3*D* compressible Euler flow with vorticity

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Abstract

We prove a series of intimately related results tied to the regularity and geometry of solutions to the $3D$ compressible Euler equations. The results concern "general" solutions, which can have nontrivial vorticity and entropy. Our geo-analytic framework exploits and reveals additional virtues of a recent new formulation of the equations, which decomposed the flow into a geometric "(sound) wave-part" coupled to a "transport-div-curl-part" (transport-part for short), with both parts exhibiting remarkable properties. Our main result is that the time of existence can be controlled in terms of the $H^{2^+}(\mathbb{R}^3)$-norm of the wave-part of the initial data and various Sobolev and Hölder norms of the transport-part of the initial data, the latter comprising the initial vorticity and entropy. The wave-part regularity assumptions are optimal in the scale of Sobolev spaces: Lindblad (Math Res Lett 5(5):605–622, 1998) showed that

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shock singularities can instantly form if one only assumes a bound for the $H^2(\mathbb{R}^3)$-norm of the wave-part of the initial data. Our proof relies on the assumption that the transport-part of the initial data is more regular than the wave-part, and we show that the additional regularity is propagated by the flow, even though the transport-part of the flow is deeply coupled to the rougher wave-part. To implement our approach, we derive several results of independent interest: (i) sharp estimates for the acoustic geometry, which in particular capture how the vorticity and entropy affect the Ricci curvature of the acoustical metric and therefore, via Raychaudhuri's equation, influence the evolution of the geometry of acoustic null hypersurfaces, i.e., sound cones; (ii) Strichartz estimates for quasilinear sound waves coupled to vorticity and entropy; and (iii) Schauder estimates for the transport-div-curl-part. Compared to previous works on low regularity, the main new features of the paper are that the quasilinear PDE systems under study exhibit multiple speeds of propagation and that elliptic estimates for various components of the fluid are needed, both to avoid loss of regularity and to gain space-time integrability.

Keywords Eikonal equation · Eikonal function · Low regularity · Null geometry · Raychaudhuri's equation · shocks · Schauder estimate · Strichartz estimate · Vectorfield method

Mathematics Subject Classification Primary 35Q31; Secondary 35Q35 · 35L10 · 35L67

Contents

1 Introduction and overview of the main results

In this paper, we study the compressible Euler equations in three spatial dimensions:

$$\mathbf{B}\varrho = -\varrho\,\mathrm{div}\,v, \tag{1a}$$

$$\mathbf{B}v^i = -\varrho^{-1}\delta^{ia}\partial_a p, \qquad (i = 1, 2, 3), \tag{1b}$$

$$\mathbf{B}s = 0, \tag{1c}$$

where $\varrho : \mathbb{R}^{1+3} \to [0, \infty)$, $v : \mathbb{R}^{1+3} \to \mathbb{R}^3$, and $s : \mathbb{R}^{1+3} \to \mathbb{R}$ are the fluid's density, velocity, and entropy, respectively; p is the fluid's pressure, which is a given smooth function of ϱ and s known as the *equation of state*—whose choice reflects one's assumptions about the properties of the fluid—;

$$\mathbf{B} := \partial_t + v^a\partial_a \tag{2}$$

is the material derivative vectorfield; $\mathbf{X}f := \mathbf{X}^\alpha\partial_\alpha f$ denotes the derivative of the scalar function f in the direction of the vectorfield $\mathbf{X}$; δ^{ab} is the standard Kronecker delta; and $\mathrm{div}\,v := \partial_a v^a$ is the standard (three-dimensional) Euclidean divergence of v. Equations (1) are expressed relative to Cartesian coordinates $\{x^\alpha\}_{\alpha=0,1,2,3}$ on $\mathbb{R}^{1+3}$, where here and throughout, $\{\partial_\alpha\}_{\alpha=0,1,2,3}$ denotes the corresponding partial derivative vectorfields, $x^0 := t$ denotes time, $\partial_0 := \partial_t$, $\{x^a\}_{a=1,2,3}$ are the spatial coordinates, and repeated indices are summed over their relevant ranges, with lowercase Greek indices ranging from 0 to 3 and lowercase Latin indices from from 1 to 3. We assume that $\inf_{t=0}\varrho > 0$, which allows us to avoid the well-known difficulty that the hyperbolicity of the equations can degenerate along fluid-vacuum boundaries.

Our main goal in this paper is to prove a series of intimately related results tied to the regularity and geometry of solutions. We study "general[1] solutions," which can have non-vanishing vorticity (i.e., $\mathrm{curl}\,v \neq 0$) and non-constant entropy. We allow for any[2] equation of state[3] $p = p(\varrho, s)$ with positive sound speed $c := \sqrt{\frac{\partial p(\varrho,s)}{\partial\varrho}}$. The central theme of the paper is that under low regularity assumptions on the initial data, it is possible to avoid, at least for short times, the formation of shocks, which are singularities caused by sound wave compression. These issues are fundamental for the Cauchy problem: for sufficiently rough initial data, *ill-posedness occurs* [16, 25] *due to instantaneous shock formation, which is precipitated by the degeneration of the acoustic geometry, including the intersection of the acoustic characteristics*. Shocks are of particular interest because they are the only singularities that have been shown, through constructive methods [3–5,8,26,28], to develop for open sets[4] of regular initial data. This motivates our main result: controlling the time of existence

[1] As we mentioned above, the solutions that we study have strictly positive density, i.e., we avoid studying fluid-vacuum boundaries.

[2] We assume that the equation of state is sufficiently smooth.

[3] In practice, instead of the density ϱ, we work with the logarithmic density, defined in Sect. 1.1.

[4] We also mention here the spectacular work [30] on the existence of implosion singularities in spherical symmetry under an adiabatic equation of state $p = \varrho^\gamma$ with $\gamma > 1$. These are singularities in which the density and velocity blow up at the center of symmetry in finite time. The methods of [30] suggest that the

under optimal Sobolev regularity assumptions on the data of "the part of the flow that blows up" in [5,8,26,28]. See Theorem 1.1 for a heuristic statement of the main result and Theorem 1.2 for the precise version. The proof relies on a deep analysis of the geometry of solutions that *exploits hidden structures* in the equations. We remark that, in the language of the present paper, the formation of a shock would correspond to the vanishing of the null lapse b defined in (178), more precisely to the following singular behavior: $\|b^{-1}\|_{L^1_t L^\infty_x} = \infty$. To avoid this singular scenario for short times, we prove the estimates stated in (289).

Theorem 1.1 (Control of the time of classical existence (heuristic version)) *The time of classical existence of a solution to the 3D compressible Euler equations can be controlled in terms of the $H^{2^+}(\Sigma_0)$-norm of the "wave-part" of the data (which is tied to sound waves, i.e., the part of the solution that is prone to shock formation) and additional Sobolev and Hölder norms of the "transport-part" of the data (which is tied to the transporting of vorticity and entropy), where $\Sigma_0 := \{0\} \times \mathbb{R}^3$ is the initial Cauchy hypersurface.*

We now highlight three features of our work:

- Our results are optimal in that $H^{2^+}(\Sigma_0)$ *cannot be replaced with* $H^2(\Sigma_0)$. More precisely, even in the irrotational and isentropic case (i.e., $\text{curl}v \equiv 0$ and $s \equiv \text{const}$, and thus the transport-part of the solution is trivial), the works [16,25] imply that ill-posedness occurs[5] if one assumes only an $H^2(\Sigma_0)$-bound on v and ϱ, due to the instantaneous formation of shocks.
- Our results appear to be the first of their kind for a quasilinear system featuring *multiple characteristic speeds*, i.e., sound waves coupled to the transporting of vorticity and entropy.
- In the irrotational and isentropic case, where the Euler equations reduce to a quasilinear wave equation for a potential, Theorem 1.1 recovers the low regularity well-posedness results for quasilinear wave equations proved in [41,54]. However, much like in the works [26,28] on shocks, the following theme permeates our paper: (especially) at low regularity levels, general compressible Euler solutions are *not* "perturbations of waves;" the presence of even the tiniest amount of vorticity or non-trivial entropy is a "game changer" requiring substantial new insights, particularly for controlling the acoustic geometry. This is because the vorticity and entropy are deeply and subtly coupled to the sound waves.

In proving Theorem 1.1, we derive several companion results of independent interest, including:

- *Control of the acoustic geometry* in the presence of vorticity and entropy. By "acoustic geometry," we mean an *acoustical eikonal function u*, that is, a solution

Footnote 4 continous
implosion singularities might enjoy co-dimension stability without symmetry assumptions, though perhaps not full stability for an open set of data.

[5] The Cartesian coordinate partial derivatives of the solution blow up, but in principle, it could remain smooth in different coordinates; e.g., Einstein's equations are well-posed in H^2 [24], even though they are H^2-ill-posed in wave coordinates [13].

to the acoustical eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$, where the *acoustical metric*[6] $\mathbf{g} = \mathbf{g}(\varrho, v, s)$ is a Lorentzian metric (see Definition 1.5) depending on the fluid solution. Acoustical eikonal functions are adapted to the characteristics of the "(sound) wave-part" of the solution and are fundamentally connected to shock waves. The regularity properties of u are highly (and tensorially) tied to those of the fluid, and the intersection of the level sets of u would signify the formation of a shock.

- *Strichartz estimates* for (quasilinear) sound waves coupled to vorticity and entropy.
- *Schauder estimates* for the vorticity and entropy, which solve transport-div-curl equations.

All aspects of our paper are fundamentally based on a new formulation of the compressible Euler equations as a system of wave and transport-div-curl equations, derived in [44] and stated in condensed form in Proposition 1.1. This new formulation exhibits remarkable geo-analytic properties that are crucial for our results. See also [27] for the case of a barotropic equation of state and [12] for a similar formulation of the relativistic Euler equations.

Standard proofs of local well-posedness for the compressible Euler flow are based on applying only energy estimates and Sobolev embedding to a first-order formulation of the equations, such as (1). Such proofs require $(\varrho - \bar{\varrho}, v, s) \in H^{(5/2)^+}(\Sigma_0)$, where $\bar{\varrho} > 0$ is a fixed constant background density. Compared to such standard proofs, Theorem 1.1 reduces the required Sobolev regularity of the wave-part of the data (i.e., the data of ϱ and $\mathrm{div} v$) by[7] half of a derivative, but requires additional smoothness on the transport-part of the data (i.e., of $\mathrm{curl} v$ and s); see Theorem 1.2. It is important to point out that one should not think that this additional smoothness of the transport-part of the data leads to an oversimplification of the problem. This is because, to the best of our knowledge, one cannot propagate the extra smoothness using *solely* equations (1) (or other equivalent first-order formulation), i.e., without appealing to a non-standard formulation of the equations such as the one given in Proposition 1.1 and employed here (see also [10] for another type of propagation of extra smoothness for the Euler equations that also involves reformulating the equations). Moreover, such propagation of extra regularity does not hold for general first-order symmetric hyperbolic systems, which is one of the standard frameworks used in the study of the compressible Euler equations. Furthermore, even when employing the formulation of Proposition 1.1, the propagation of extra smoothness for the transport part of the system is very delicate in that the transport- and wave-parts are coupled in a highly non-trivial way (in particular through the acoustic geometry). In this regard, a remarkable aspect of our work is:

> We propagate the regularity of the "smoother" transport-part of the compressible Euler flow, even though it is deeply coupled to the rougher wave-part.

To propagate the extra smoothness, we exploit the full nonlinear structure of the aforementioned new formulation of the equations and carry out a delicate analysis of

[6] In practice, when constructing u, we work with a rescaled version of the acoustical metric; see Sect. 9.4.

[7] Here, when discussing the regularity of v, $\mathrm{div} v$, and $\mathrm{curl} v$, we are implicitly referring to the Hodge estimate (55).

the interaction of the wave- and transport- parts of the system as well as the acoustic geometry.[8]

1.1 New formulation of the Euler equations

In Sect. 1.1.3, we provide the new formulation of the equations that we use in our analysis. We first introduce some notation and define some additional quantities that play a role in the new formulation.

Recall that we assume that the pressure p is a given smooth function of ϱ and s, and that the speed of sound c is defined by $c := \sqrt{\frac{\partial p}{\partial \varrho}|_s}$, where $\frac{\partial p}{\partial \varrho}|_s$ is the partial derivative of p with respect to ϱ at fixed s. From now on, we view p and c as smooth functions of the *logarithmic density*

$$\uprho := \ln\left(\frac{\varrho}{\bar{\varrho}}\right) : \mathbb{R}^{1+3} \to \mathbb{R}, \tag{3}$$

(as opposed to the standard density) and s, where we recall that $\bar{\varrho} > 0$ is a fixed constant background density. That is, we view $p = p(\uprho, s)$ and $c = c(\uprho, s)$. If $f = f(\uprho, s)$ is a scalar function, then we use the following notation to denote partial differentiation with respect to $\uprho$ and s: $f_{;\uprho} := \frac{\partial f}{\partial \uprho}$ and $f_{;s} := \frac{\partial f}{\partial s}$.

1.1.1 Additional fluid variables

We first recall that the fluid vorticity is the Σ_t-tangent vectorfield $\omega : \mathbb{R}^{1+3} \to \mathbb{R}^3$, where $\Sigma_t := \{(\tau, x^1, x^2, x^3) \in \mathbb{R}^{1+3} \,|\, \tau = t\}$, with the following Cartesian spatial components:

$$\omega^i := (\mathrm{curl}\, v)^i := \epsilon^{iab}\partial_a v_b, \tag{4}$$

where throughout, ϵ^{iab} denotes the fully antisymmetric symbol normalized by $\epsilon^{123} = 1$.

We will derive estimates for the *specific vorticity* and *entropy gradient*, which are vectorfields featured in the next definition. These variables solve equations with a favorable structure and thus play a key role in our analysis.

Definition 1.1 *(Specific vorticity and entropy gradient)* We define the specific vorticity $\Omega : \mathbb{R}^{1+3} \to \mathbb{R}^3$ and the entropy gradient $S : \mathbb{R}^{1+3} \to \mathbb{R}^3$ to be the Σ_t-tangent vectorfields with the following Cartesian components:

$$\Omega^i := \frac{\omega^i}{(\varrho/\bar{\varrho})} = \frac{(\mathrm{curl}\, v)^i}{\exp \uprho}, \qquad S^i := \delta^{ia}\partial_a s. \tag{5}$$

[8] Readers less familiar with Strichartz and acoustic geometry estimates can consult the arXiv version of this paper [11], wherein we provide a longer introduction with further background.

The "modified" fluid variables featured in the next definition solve equations with remarkable structures. In total, such structures allow us to prove that these variables exhibit a gain in regularity compared to standard estimates. We stress that this gain of regularity is crucial for showing that the different solution variables have enough regularity to be compatible with our approach.

Definition 1.2 *(Modified fluid variables)* We define the Cartesian components of the Σ_t-tangent vectorfield $\mathcal{C}$ and the scalar function $\mathcal{D}$ as follows:

$$\mathcal{C}^i := \exp(-\rho)(\mathrm{curl}\,\Omega)^i + \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}S^a\partial_a v^i - \exp(-3\rho)c^{-2}\frac{p_{;s}}{\bar{\varrho}}(\partial_a v^a)S^i, \tag{6a}$$

$$\mathcal{D} := \exp(-2\rho)\mathrm{div}\,S - \exp(-2\rho)S^a\partial_a\rho. \tag{6b}$$

The following definitions are primarily for notational convenience.

Definition 1.3 *(The wave variables).* We define the wave variables Ψ_ι, ($\iota = 0, 1, 2, 3, 4$), and the array $\vec{\Psi}$ of wave variables, as follows:

$$\Psi_0 := \rho, \qquad \Psi_i := v^i, \qquad (i = 1, 2, 3), \qquad \Psi_4 := s, \tag{7a}$$

$$\vec{\Psi} := (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4). \tag{7b}$$

Definition 1.4 *(Arrays of Cartesian component functions).* We define the following arrays:

$$\vec{v} := (v^1, v^2, v^3), \quad \vec{\Omega} := (\Omega^1, \Omega^2, \Omega^3), \quad \vec{S} := (S^1, S^2, S^3), \quad \vec{\mathcal{C}} := (\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3). \tag{8}$$

Throughout, we use the following notation for Cartesian partial derivative operators:

- ∂ denotes a spatial derivative with respect to the Cartesian coordinates.
- $\boldsymbol{\partial} = (\partial_t, \partial)$ denotes a spacetime derivative with respect to the Cartesian coordinates.

Moreover, $\boldsymbol{\partial}\vec{\Psi}$ denotes the array of scalar functions $\boldsymbol{\partial}\vec{\Psi} := (\partial_\alpha\Psi_\iota)_{\alpha=0,1,2,3,\iota=0,1,2,3,4}$ (recall that $\partial_0 = \partial_t$), and $\partial\vec{\Psi}$ denotes the array of scalar functions $\partial\vec{\Psi} := (\partial_a\Psi_\iota)_{a=1,2,3,\iota=0,1,2,3,4}$. Arrays such as $\boldsymbol{\partial}\vec{v}$, $\partial\vec{v}$, $\boldsymbol{\partial}\vec{\Omega}$, $\partial\vec{\Omega}$, $\boldsymbol{\partial}^2\vec{\Psi}$, etc., are defined analogously. Moreover, $\partial^{\leq 1}\vec{\Psi}$ denotes the array whose entries are those of $\vec{\Psi}$ together with those of $\partial\vec{\Psi}$, and arrays such as $\partial^{\leq 1}\vec{\Omega}$, $\partial^{\leq 1}\vec{S}$, etc., are defined analogously.

1.1.2 Acoustical metric and wave operators

Our analysis of the wave-part of the system is fundamentally tied to the acoustical metric $\mathbf{g}$ and related geometric tensors.

Definition 1.5 *(The acoustical metric and first fundamental form).* We define the *acoustical metric* $\mathbf{g} = \mathbf{g}(\rho, v, s)$ relative to the Cartesian coordinates as follows:

$$\mathbf{g} := -dt \otimes dt + c^{-2} \sum_{a=1}^{3} (dx^a - v^a dt) \otimes (dx^a - v^a dt). \tag{9}$$

We define[9] the *first fundamental form* $g = g(\rho, v, s)$ of Σ_t and the corresponding *inverse first fundamental form* $g^{-1} = g^{-1}(\rho, v, s)$ relative to the Cartesian coordinates as follows:

$$g := c^{-2} \sum_{a=1}^{3} dx^a \otimes dx^a, \qquad g^{-1} := c^2 \sum_{a=1}^{3} \partial_a \otimes \partial_a. \tag{10}$$

It is straightforward to check that relative to the Cartesian coordinates, we have

$$\mathbf{g}^{-1} = -\mathbf{B} \otimes \mathbf{B} + c^2 \sum_{a=1}^{3} \partial_a \otimes \partial_a, \qquad \det \mathbf{g} = -c^{-6}. \tag{11}$$

It is also straightforward to verify the following facts, which we will use throughout: $\mathbf{B}$ is $\mathbf{g}$-orthogonal to Σ_t and normalized by

$$\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1. \tag{12}$$

Remark 1.1 Note that $\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}(\vec{\Psi})$ and $\mathbf{B}^\alpha = \mathbf{B}^\alpha(\vec{\Psi})$. Note also that $(\mathbf{g}^{-1})^{00} = -1$. We will sometimes silently use this basic fact.

The following wave operators arise in our analysis of solutions.

Definition 1.6 *(Covariant and reduced wave operators)* $\square_{\mathbf{g}}$ denotes the covariant wave operator of $\mathbf{g}$, which acts on scalar functions φ by the coordinate invariant formula $\square_{\mathbf{g}} \varphi := \frac{1}{\sqrt{|\det \mathbf{g}|}} \partial_\alpha \left(\sqrt{|\det \mathbf{g}|} (\mathbf{g}^{-1})^{\alpha\beta} \partial_\beta \varphi \right)$. $\hat{\square}_{\mathbf{g}}$ denotes the reduced wave operator of $\mathbf{g}$, and it acts on scalar functions φ by the following formula (relative to Cartesian coordinates): $\hat{\square}_{\mathbf{g}} \varphi := (\mathbf{g}^{-1})^{\alpha\beta} \partial_\alpha \partial_\beta \varphi$.

1.1.3 Statement of the geometric wave-transport formulation of the compressible Euler equations

We now provide the geometric formulation of the compressible Euler equations that we use to study the regularity of solutions. Detailed versions of the equations were

[9] As we describe in Sect. 9.6.2, g can be extended to a Σ_t-tangent spacetime tensor. By definition, the extended version of g agrees with the original version when acting on Σ_t-tangent vectors and vanishes upon any contraction with $\mathbf{B}$. The extended g satisfies the identity $g = c^{-2} \sum_{a=1}^{3} (dx^a - v^a dt) \otimes (dx^a - v^a dt)$.

derived in [44, Theorem 1], but for our purposes here, it suffices to work with the schematic version stated in Proposition 1.1.

We will use the following schematic notation, which captures the essential structures that are relevant for our analysis. Later in the article, we will introduce additional schematic notation.

- $\mathscr{L}(A)[B]$ denotes any scalar-valued function that is linear in B with coefficients that are a (possibly nonlinear) function of A, i.e., a term of the form $\mathrm{f}(A) \cdot B$, where f denotes a generic smooth function that is free to vary from line to line.
- $\mathscr{Q}(A)[B, C]$ denotes any scalar-valued function that is quadratic in B and C with coefficients that are a (possibly nonlinear) function of A, i.e., a term of the form $\mathrm{f}(A) \cdot B \cdot C$.

Proposition 1.1 [44, The geometric wave-transport formulation of the compressible Euler equations] *Smooth solutions to the compressible Euler equations (1a)–(1c) also verify the following system of equations, where all terms on the RHSs are displayed schematically:*[10]

Wave equations: For $\Psi \in \{\rho, v^1, v^2, v^3, s\}$, we have

$$\hat{\square}_{\mathbf{g}(\vec{\Psi})} \Psi = \mathfrak{F}_{(\Psi)} := \mathscr{L}(\vec{\Psi})[\vec{\mathcal{C}}, \mathcal{D}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{13}$$

Moreover, replacing $\hat{\square}_{\mathbf{g}(\vec{\Psi})}$ on LHS (13) with the covariant wave operator $\square_{\mathbf{g}(\vec{\Psi})}$ leads to a wave equation whose RHS has the same schematic form as RHS (13).

Transport equations: The Cartesian component functions $\{\Omega^i\}_{i=1,2,3}$ and $\{S^i\}_{i=1,2,3}$ verify the following equations:

$$\mathbf{B}\Omega^i = \mathscr{L}(\vec{\Psi}, \vec{\Omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}], \qquad \mathbf{B}S^i = \mathscr{L}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{14}$$

Transport div-curl system for the specific vorticity: The scalar function $\mathrm{div}\Omega$ and the Cartesian component functions $\{\mathcal{C}^i\}_{i=1,2,3}$ verify the following equations:

$$\mathrm{div}\Omega = \mathfrak{F}_{(\mathrm{div}\Omega)} := \mathscr{L}(\vec{\Omega})[\boldsymbol{\partial}\vec{\Psi}], \tag{15a}$$

$$\mathbf{B}\mathcal{C}^i = \mathfrak{F}_{(\mathcal{C}^i)} := \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \partial\vec{\Omega}] + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \partial\vec{S}] + \mathscr{Q}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi}, \vec{\Omega}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}]. \tag{15b}$$

Transport div-curl system for the entropy gradient: The scalar function $\mathcal{D}$ and the Cartesian component functions $\{S^i\}_{i=1,2,3}$ verify the following equations:

$$\mathbf{B}\mathcal{D} = \mathfrak{F}_{(\mathcal{D})} := \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \partial\vec{S}] + \mathscr{Q}(\vec{\Psi}, \vec{S})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}] + \mathscr{L}(\vec{\Psi}, \vec{S})[\partial\vec{\Omega}], \tag{16a}$$

$$(\mathrm{curl} S)^i = 0. \tag{16b}$$

[10] The precise form of the schematic terms in Eq. (13) depends on Ψ, but the details are not important for our analysis. Similar remarks apply to the remaining equations.

Remark 1.2 We emphasize that for our main results, it is crucial that *generic* first derivatives of Ω and S do *not* appear on RHS (13); rather, only the special combinations $\vec{\mathcal{C}}$ and $\mathcal{D}$ appear.

Remark 1.3 In obtaining the form of the equations of Proposition 1.1 as a consequence of the equations presented in [44], we have used the simple relations $\Omega^i = \mathscr{L}(\vec{\Psi})[\partial\vec{\Psi}]$ and $S^i = \delta^{ic}\partial_c s = \mathscr{L}[\partial\vec{\Psi}]$.

Remark 1.4 In the equations of [44], all derivative-quadratic inhomogeneous terms are null forms. However, using Remark 1.3, we have rewritten, for example, terms of type $S \cdot S$, as $\mathscr{Q}[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]$, where $\mathscr{Q}[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]$ is not necessarily a null form. That is, the quadratic terms $\mathscr{Q}(\cdot)[\cdot, \cdot]$ in Proposition 1.1 are not necessarily null forms. While the presence of null form structures is crucial for the study of the formation of shocks, such null form structures are not important for the results of this article.

Proposition 1.1 justifies our use of the terminology "wave-parts" and "transport-parts" to refer to different parts of the system. In particular, it shows that the Cartesian velocity components v^i and ρ satisfy covariant wave equations of the form $\square_{\mathbf{g}}(v^i, \rho) = \cdots$, and we therefore refer to ϱ and v^i as the "wave-part" of the compressible Euler flow. In contrast, s, ∂s, and the specific vorticity Ω satisfy transport equations along the integral curves of the material derivative vectorfield $\mathbf{B} := \partial_t + v^a\partial_a$, and we therefore refer to these as the "transport-part" of the compressible Euler flow. Moreover, the variables $\mathcal{C}$ and $\mathcal{D}$ satisfy transport-div-curl subsystems and, therefore, we also consider these to be part of the "transport-part" of the flow.

1.2 Statement of the main result concerning control of the time of classical existence

We now precisely state the theorem on the time of classical existence. We recall that $\bar{\varrho} > 0$ is a fixed constant background density.

Theorem 1.2 (Control of the time of classical existence under low regularity assumptions on the wave-part of the data) *Consider a smooth*[11] *solution to the compressible Euler equations in 3D whose initial data obey the following three assumptions*[12] *for some real numbers*[13] $2 < N \leq 5/2$, $0 < \alpha < 1$, $0 \leq D_{N;\alpha} < \infty$, $0 < c_1 < c_2$, *and* $0 < c_3$*:*

1. $\|(\varrho - \bar{\varrho}, v, curl v)\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)} \leq D_{N;\alpha}$, *where $\bar{\varrho} > 0$ is a constant background density.*

[11] For convenience, in this paper, we will assume that the solutions are as many times differentiable as necessary. Thus, "smooth" means "as smooth as necessary for the *qualitative* arguments (such as integration by parts) to go through." However, all of our *quantitative* estimates depend only on the Sobolev and Hölder norms mentioned in Theorem 1.2.

[12] We note that since assumption 3 implies that $\varrho|_{\Sigma_0}$ is strictly positive, we have $\|\varrho - \bar{\varrho}\|_{H^N(\Sigma_0)} \approx \|\rho\|_{H^N(\Sigma_0)}$, where ρ is the logarithmic density defined in (3); this standard estimate can be proved using the product estimates of Lemma 5.3.

[13] Similar results can be proved for $N > 5/2$ using only energy estimates and Sobolev embedding.

2. *The modified fluid variables $\mathcal{C}$ and $\mathcal{D}$ from Definition 1.2 (which vanish for irrotational and isentropic solutions), verify the Hölder-norm bound $\|(\mathcal{C}, \mathcal{D})\|_{C^{0,\alpha}(\Sigma_0)} \leq D_{N;\alpha}$.*
3. *Along Σ_0, the data functions are contained in the interior of a compact subset $\mathfrak{K}$ of state-space in which $\varrho \geq c_3$ and the speed of sound is bounded from below by c_1 and above by c_2.*

Then the solution's time of classical existence T depends only on $D_{N;\alpha}$ and $\mathfrak{K}$, i.e., $T = T(D_{N;\alpha}, \mathfrak{K}) > 0$. Moreover, the Sobolev regularity of the data is propagated by the solution for $t \in [0, T]$, as is Hölder regularity.[14]

Remark 1.5 *(Regularity needed for Strichartz estimates and differences from the irrotational and isentropic case)* In Theorem 1.2, we have assumed additional Sobolev regularity on the transport-part of the flow (specifically $\text{curl}v$ and s) compared to the classical local well-posedness regime $(\varrho - \bar{\varrho}, v, s) \in H^{(5/2)^+}(\Sigma_0)$. This is because our approach to controlling $\int_0^T \|\boldsymbol{\partial}(\varrho, v, s)\|_{L^\infty(\Sigma_\tau)} \, d\tau$ (which, as we mention below (17), is crucial for the proof of Theorem 1.2) relies on deriving Strichartz estimates for the nonlinear wave equations of Proposition 1.1, which in turn requires the transport-part of the system to be more regular than the wave part. That is, at the classical local well-posedness regularity level (which is such that the transport-part does not generically enjoy any relative gain in regularity), *the approach of treating the compressible Euler equations as a coupled wave-div-curl-transport system fails*,[15] except in the irrotational and isentropic case [41,54] (where the compressible Euler equations reduce to a quasilinear wave equation for a potential function). The failure comes from the wave equation source terms[16] $\vec{\mathcal{C}}$ and $\mathcal{D}$ on RHS (13), which are the modified fluid variables from Definition 1.2. For general solutions (i.e., solutions with vorticity and non-trivial entropy), from the point of view of regularity, $\vec{\mathcal{C}}$ and $\mathcal{D}$ scale, in a naive sense, like $\partial^2 v$ and $\partial^2 s$. Therefore, at the classical local well-posedness threshold, $\vec{\mathcal{C}}$ and $\mathcal{D}$ are elements of $H^{(1/2)^+}(\Sigma_t)$. This level of source-term regularity is insufficient for using a Duhamel argument to justify the desired Strichartz estimate for the nonlinear wave equation (13); see the proof of Theorem 7.1 for details on how the source terms enter into the proof of Strichartz estimates. This is one key reason why, throughout the paper, we assume the transport-part data regularity $\|\text{curl}v\|_{H^N(\Sigma_0)} \leq D_{N;\alpha}$ and $\|s\|_{H^{N+1}(\Sigma_0)} \leq D_{N;\alpha}$ (these inequalities are automatically satisfied in the irrotational and isentropic[17] case).

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14 Proposition 5.1 allows us to propagate all of the Sobolev regularity of the initial data, while (120) allows us to propagate some Hölder regularity for $(\vec{\mathcal{C}}, \mathcal{D})$; the Hölder norm that we can control has an exponent that is controllable in terms of $N - 2$, but the exponent is possibly smaller than α. Moreover, the norms that we can control are uniformly bounded by functions of $(D_{N;\alpha}, \mathfrak{K})$ for $t \in [0, T]$.

15 At the classical local well-posedness level, one can treat the compressible Euler equations as a first-order symmetric hyperbolic system and obtain control over $\int_0^T \|\boldsymbol{\partial}(\varrho, v, s)\|_{L^\infty(\Sigma_\tau)} \, d\tau$ as a consequence of Sobolev embedding and symmetric hyperbolic energy estimates. However, symmetric hyperbolic formulations of the equations do not exhibit the intricate structures that we exploit in proving Theorem 1.2.

16 See Definition 1.4 regarding the notation "$\vec{\mathcal{C}}$."

17 Technically, s could be a non-zero constant in the isentropic case, leading to $\|s\|_{L^2(\Sigma_0)} = \infty$. However, this infinite norm would be irrelevant in that s would be constant throughout the evolution and thus trivial to control.

Given the estimates we derive in Sects. 2–8, it is known that Theorem 1.2 essentially follows from the following a priori estimate, where $\boldsymbol{\partial} f := (\partial_t f, \partial_1 f, \partial_2 f, \partial_3 f)$, Σ_τ is the standard flat hypersurface of constant time, and T is as in the statement of the theorem:

$$\int_0^T \|\boldsymbol{\partial}(\varrho, v, s)\|_{L^\infty(\Sigma_\tau)}\, d\tau \lesssim 1. \tag{17}$$

That is, we will not provide the details on how Theorem 1.2 follows from (17) via a continuity argument and persistence of regularity (see, e.g., [29, Section 2.2, Corollary 2] or [35, Lemma 9.14] for the main ideas behind the proof), but will instead focus our efforts on justifying the a priori estimate[18] (17) for $T > 0$ sufficiently small (where the required smallness depends only the norms of the data and the set $\mathfrak{K}$ mentioned in Theorem 1.2). More precisely, our approach requires us to prove a stronger result, namely Theorem 7.1, whose proof in turn is coupled to all of the other ingredients mentioned above. We also remark that, as we explain in Sects. 7–11, most of the arguments needed for the proof of Theorem 7.1, including a series of technical-but-known reductions, are supplied by other papers cited in Sects. 7–11. In this paper, our main focus will be showing how to control the vorticity and entropy in norms that allow to use the machinery from these other papers. Our proof relies on norms of the vorticity and entropy on constant-time hypersurfaces and sound cones, and the main novelties of our work are: (i) we can propagate substantial smoothness for the vorticity, entropy, and modified fluid variables $\mathcal{C}$ and $\mathcal{D}$ from Definition 1.2, even though these variables are intimately coupled to the rougher wave part of the solution; (ii) we can obtain suitable estimates for the acoustic geometry by exploiting the precise structure of the new formulation of compressible Euler flow provided by Proposition 1.1, which allows us to show that the main top-order vorticity/entropy-dependent terms driving the evolution of the acoustic geometry are[19] $\mathcal{C}$ and $\mathcal{D}$—as opposed to generic first-order derivatives of Ω and S.

Remark 1.6 *(Remarks on local well-posedness).* Theorem 1.2 provides the main ingredient, namely a priori estimates for smooth solutions, needed for a full proof of local well-posedness, including existence in the regularity spaces featured in the theorem and uniqueness in related spaces.

We anticipate that the remaining aspects of a full proof of local well-posedness could be shown by deriving, using the ideas that we use to prove Theorem 1.2, uniform estimates for sequences of smooth solutions and their differences. For ideas on how to proceed, readers can consult [41], in which existence and uniqueness were proved at low regularity levels for quasilinear wave equations.

We now further describe some ingredients of independent interest that we use in the proof of Theorem 1.2.

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[18] Actually, under our framework, the bound $\int_0^T \|\boldsymbol{\partial} s\|_{L^\infty(\Sigma_\tau)}\, d\tau \lesssim 1$ will be trivial to justify since we will prove the stronger result $s \in L^\infty\left([0,T], H^{N+1}(\mathbb{R}^3)\right)$.

[19] See, for example, the first terms on RHSs (228a) and (228b).

(I) Control of the acoustic geometry. For quasilinear wave systems with a single wave operator, there has been remarkable progress on obtaining control of the acoustic geometry and applications to low regularity local well-posedness, see [18–24,41,54]. A fundamental new aspect of the present work is that *the vorticity and entropy appear as source terms in the acoustic geometry estimates*, signifying a coupling between the geometry of sound cones and transport phenomena. The coupling enters in particular through the Ricci curvature of the acoustical metric $\mathbf{g}$ (see Definition 1.5), which, by virtue of the compressible Euler equations, can be expressed in terms of quantities involving the vorticity and entropy; see Lemma 9.6. We also exploit some remarkable consequences of the compressible Euler formulation provided by Proposition 1.1, namely, through careful geometric decompositions we show that high order derivatives of vorticity and entropy *occur only the special combinations $\mathcal{C}$ and $\mathcal{D}$*; see Proposition 9.7. The point is that the modified fluid variables $\mathcal{C}$ and $\mathcal{D}$—as opposed to generic first-order derivatives of Ω and S—*enjoy good estimates up to top-order along sound cones*, and such estimates turn out to be crucial for obtaining control of the acoustic geometry. This unexpected-but-critical structure should not be taken for granted since *generic* high order derivatives of the vorticity and entropy can be controlled *only along constant-time hypersurfaces*.

(II) Strichartz estimates for the wave-part of solutions. As in the works cited in **I**, our derivation of Strichartz estimates is fundamentally based on having suitable quantitative control of the acoustic geometry; see Sect. 11. Therefore, in view of the discussion in **I**, we see that the Strichartz estimates are tied to the delicate regularity properties of the vorticity and entropy along sound cones.

(III) New Schauder estimates for the transport-div-curl equations appearing in the compressible Euler formulation; see Sect. 8. These provide us with mixed spacetime estimates for the transport-part that complement the Strichartz estimates, allowing us to control the new (compared to the previously treated case of irrotational and isentropic solutions) kinds of derivative-quadratic terms that we encounter in the energy and elliptic estimates.

1.3 Some general remarks and connections with prior work

Much of the remarkable progress that has been obtained for quasilinear hyperbolic PDEs over the last two decades stems from studying specific systems of geometric or physical interest (as opposed to "general systems"), where very delicate structural features of the equations can be exploited in combination with a precise understanding of the regularity of the system's characteristics. Moreover, a common theme in these developments is that the special structural and/or regularity features of the system become visible only after one rewrites the equations in some novel way, which might involve a coordinate system adapted to the problem in question and/or a new formulation of the equations of motion in the spirit of the equations of Proposition 1.1.

A primary example is Einstein's equations, where the following notable results were obtained in recent years: the formation of trapped surfaces [6], the stability of the Kerr Cauchy horizon [9], stable curvature blowup [36,37,43], instability of anti de

Sitter space [31,32], and the proof [24] of the bounded L^2 curvature conjecture. For the compressible Euler equations, we can cite Christodoulou's breakthrough works [5,8] on the formation of shocks in the irrotational and isentropic case, and, more recently, the works [3,4,26,28] on the formation of shocks for solutions with vorticity and entropy.

Regarding the problem of low regularity, in the case of an irrotational and isentropic flow, where the compressible Euler equations can be written as a system of quasilinear wave equations with a single wave speed, our result follows directly from the optimal low regularity local well-posedness by Smith and Tataru [41] or also from the more recent physical-space approach to the problem by Wang [54]. This highlights, once more, that the main novelty of our work is to obtain control of the fluid flow under optimal regularity assumptions on the wave-part of the system *in the presence of vorticity and entropy.*

In order to highlight the difference between our result and what can be obtained using solely techniques from quasilinear wave equations, we now discuss an approach that one could take for controlling the wave-part of the system at sub-$H^{(5/2)^+}(\Sigma_0)$ regularity levels,[20] one that is simpler than the approach that we use here, but less powerful in that it would *not* allow one to reach the $H^{2^+}(\Sigma_0)$ regularity threshold for the wave-part. Specifically, one could control the wave-part of the system at a regularity level below $H^{(5/2)^+}(\Sigma_0)$ by invoking the technology of Strichartz estimates for *linear* wave equations with rough coefficients, based on Fourier integral parametrix representations, developed in a series of works by Tataru [45–47], which improved the foundational work [1] of Bahouri–Chemin; see also the related work [39]. By "linear," we mean in particular that the proofs do not exploit any information about the principal coefficients of the wave operator besides their pre-specified regularity. In particular, when combined with the bootstrap-type arguments given in Sects. 3–8, the methods of [45,47] (see in particular [45, Theorem 6] and [47, Theorem 5.1]) would allow one to prove local well-posedness assuming that $(\varrho - \bar{\varrho}, v) \in H^{(13/6)^+}(\Sigma_0)$ and that the transport-part of the data enjoys the same relative gain in regularity that we assume for our results (e.g., $s \in H^{(19/6)^+}(\Sigma_0)$ and $\partial^2 s \in C^{0,0^+}(\Sigma_0)$); see Sect. 2.1.3 for further discussion. The work [40] shows that without further information about the principal coefficients of the wave equation, Tataru's linear Strichartz estimates are optimal. Thus, since our results further lower the Sobolev regularity threshold by $1/6$, our analysis *necessarily exploits* the specific nonlinear structure of the equations of Proposition 1.1. We also refer to the works (some of which we mentioned earlier) [18,21,24,41,54] for further low regularity results in which the nonlinear structure of the PDE plays a fundamental role.

1.4 Paper outline

The remainder of the paper is organized as follows:

[20] Recall that $H^{(5/2)^+}(\Sigma_0)$ is what is required for standard local well-posedness based on energy estimates and Sobolev embedding.

- In Sect. 2, we outline the main ideas of our analysis through the study of a model problem.
- In Sect. 3, we recall some standard constructions from Littlewood–Paley theory, define the norms that we use until Sect. 9, define the parameters that play a role in our analysis, state our assumptions on the data, and formulate bootstrap assumptions. The two key bootstrap assumptions are Strichartz estimates for the wave-part of the solution and complementary mixed spacetime estimates for the transport-part.
- In Sect. 4, we use the bootstrap assumptions to derive preliminary below-top-order energy and elliptic estimates, which are useful for controlling simple error terms.
- In Sect. 5, we use the bootstrap assumptions and the results of Sect. 4 to derive top-order energy and elliptic estimates along constant-time hypersurfaces.
- In Sect. 6, we derive energy estimates along acoustic null hypersurfaces, which complement the estimates from Sect. 5. We need these estimates along null hypersurfaces in Sect. 10, when we control the acoustic geometry. Compared to prior works, the main contribution of Sect. 6 is the estimate (102), which shows that the modified fluid variables $(\vec{\mathcal{C}}, \mathcal{D})$ can be controlled in L^2 up to top-order along acoustic null hypersurfaces, i.e., sound cones; as we described in Sect. 1.2, *such control along sound cones is not available for generic top-order derivatives of the vorticity and entropy*.
- In Sect. 7, we prove Theorem 7.1, which yields Strichartz estimates for the wave-part of the solution, thereby improving the first key bootstrap assumption and justifying the estimate (17). The proof of Theorem 7.1 is conditional on Theorem 7.2, whose proof in turn relies on the estimates for the acoustic geometry that we derive in Sect. 10.
- In Sect. 8, we use Schauder estimates to derive mixed spacetime estimates for the transport-part of the solution, thereby improving the second key bootstrap assumption. At this point in the paper, to close the bootstrap argument and complete the proof of Theorem 1.2, it only remains for us to prove Theorem 7.2.
- In Sect. 9, in service of proving Theorem 7.2, we construct the acoustic geometry on spacetime slabs corresponding to a partition of the bootstrap time interval; see Sect. 7.2 for the construction of the partition. The acoustic geometry is centered around an acoustical eikonal function. We also define corresponding geometric norms.
- In Sect. 10, we derive estimates for the acoustic geometry. The main result is Proposition 10.1.
- In Sect. 11, we review some results derived in [54], which in total show that the results of Sect. 10 imply Theorem 7.2. This closes the bootstrap argument, justifies the estimate (17), and completes the proof of Theorem 1.2.

Note added. After the completion of this manuscript, the work [55] became available, in which the author considers the compressible Euler equations under a barotropic equation of state $p = p(\varrho)$ (and thus the variable s is absent from the analysis). In this case, the author was able to lower the regularity of $\mathrm{curl} v|_{\Sigma_0}$ compared to Theorem 1.2 by eliminating the Hölder-norm bound assumption and showing that it suffices to assume $\mathrm{curl} v \in H^{N'}(\Sigma_0)$, where $2 < N' < \left(\frac{N-2}{5}\right)^2$. Moreover, in the wake of [55], there also appeared [56], where a $2D$ local well-posedness result is established in the barotropic case such that the density, velocity, and specific vorticity are in H^2, and [57], which provides an alternative proof of the results of [55].

2 A model problem

In this section, we discuss a model problem that serves as a blueprint for the rest of the paper. The purpose of this section is to provide insight into the analysis and is entirely independent of the rest of the paper. Readers not interested in a schematic guide to the main ideas of the paper can skip this section.

2.1 Overview of the analysis via a model problem

In this subsection, we exhibit some of the main ideas behind our analysis by discussing a model problem.

2.1.1 Statement of the model system

We will study the following schematically depicted model system in the scalar unknown Ψ and the Σ_t-tangent unknown vectorfield W on $\mathbb{R}^{1+3}$:

$$\hat{\square}_{\mathbf{g}(\Psi)}\Psi = \operatorname{curl} W + \boldsymbol{\partial}\Psi \cdot \boldsymbol{\partial}\Psi, \tag{18a}$$

$$\operatorname{div} W = \partial\Psi, \tag{18b}$$

$$\{\partial_t + \Psi\partial_1\} \operatorname{curl} W = \partial\Psi \cdot \partial W. \tag{18c}$$

We intend for the system (18a)–(18c) to be a caricature of the equations of Proposition 1.1. Above, $\mathbf{g}_{\alpha\beta}(\Psi)$ are given Cartesian component functions (assumed to depend smoothly on Ψ) of the Lorentzian metric $\mathbf{g}$, and $\hat{\square}_{\mathbf{g}(\Psi)} := (\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta$. Ψ may be thought of as a model for the wave-part of the compressible Euler equations, while W may be thought of as a model for the transport-part (e.g., the vorticity and entropy gradient), with $\partial_t + \Psi\partial_1$ a model quasilinear transport operator (the fact that it involves only ∂_t and ∂_1 is not important). That is, from the point of view of regularity, we can think that $\Psi \sim (\rho, v)$ and $W \sim (\operatorname{curl} v, \partial s)$. We intend for the reader to interpret the inhomogeneous terms schematically (especially, since, for example, LHS (18a) is a scalar while the first term on RHS (18a) appears to be a vector).

We will outline how to control the time of existence for solutions to the model system (18a)–(18c) assuming the data-bound

$$\|(\Psi, \partial_t\Psi)\|_{H^N(\Sigma_0)\times H^{N-1}(\Sigma_0)} + \|\partial W\|_{H^{N-1}(\Sigma_0)} < \infty,$$

where $2 < N \leq 5/2$ is a fixed real number. In Sect. 2.1.4, we will find that we need to make the further Hölder regularity assumption $\|\operatorname{curl} W\|_{C^{0,\alpha}(\Sigma_0)} < \infty$ for some $\alpha > 0$, much like we did in Theorem 1.2. In the rest of Sect. 2.1, "data" schematically denotes any quantity depending on $\|(\Psi, \partial_t\Psi)\|_{H^N(\Sigma_0)\times H^{N-1}(\Sigma_0)} + \|\partial W\|_{H^{N-1}(\Sigma_0)}$.

2.1.2 A priori energy and elliptic estimates along $\boldsymbol{\Sigma}_t$ for the model system

The most fundamental step in controlling the time of existence is to derive a priori energy and elliptic estimates along Σ_t. In the context of the compressible Euler equations, we provide the analog of this step in Proposition 5.1 below. To obtain the desired a priori estimate for the model system, we first note that equation (18b) and the standard elliptic Hodge estimate

$$\|\partial W\|_{H^{N-1}(\Sigma_t)} \lesssim \|\mathrm{div}\, W\|_{H^{N-1}(\Sigma_t)} + \|\mathrm{curl}\, W\|_{H^{N-1}(\Sigma_t)} \tag{19}$$

together imply the following bound:

$$\|\partial W\|_{H^{N-1}(\Sigma_t)} \lesssim \|\partial \Psi\|_{H^{N-1}(\Sigma_t)} + \|\mathrm{curl}\, W\|_{H^{N-1}(\Sigma_t)}. \tag{20}$$

Next, by combining standard estimates for the wave equation (18a), based on energy estimates and the Littlewood–Paley calculus, we deduce (where we ignore all numerical constants "C") that

$$\begin{aligned} &\|(\Psi, \partial_t \Psi)\|^2_{H^N(\Sigma_t)\times H^{N-1}(\Sigma_t)} \\ &\leq \mathrm{data} + \int_0^t \left\{1 + \|\boldsymbol{\partial} \Psi\|_{L^\infty_x(\Sigma_\tau)}\right\} \left\{\|(\Psi, \partial_t \Psi)\|^2_{H^N(\Sigma_\tau)\times H^{N-1}(\Sigma_\tau)} + \|\mathrm{curl}\, W\|^2_{H^{N-1}(\Sigma_\tau)}\right\} d\tau. \end{aligned} \tag{21}$$

Similarly, with the help of the Littlewood–Paley calculus, we can derive energy estimates for the transport equation (18c) and use (20) to control the top-order derivatives of the factor ∂W on RHS (18c), thereby obtaining the following bound:

$$\begin{aligned} &\|\mathrm{curl}\, W\|^2_{H^{N-1}(\Sigma_t)} \\ &\leq \mathrm{data} + \int_0^t \left\{1 + \|\boldsymbol{\partial} \Psi\|_{L^\infty(\Sigma_\tau)} + \|\partial W\|_{L^\infty(\Sigma_\tau)}\right\} \\ &\qquad \times \left\{\|(\Psi, \partial_t \Psi)\|^2_{H^N(\Sigma_\tau)\times H^{N-1}(\Sigma_\tau)} + \|\mathrm{curl}\, W\|^2_{H^{N-1}(\Sigma_\tau)}\right\} d\tau. \end{aligned} \tag{22}$$

Adding (21) and (22), applying Grönwall's inequality, and finally again using (20), we obtain (again ignoring all numerical constants "C") the following estimate:

$$\begin{aligned} &\|(\Psi, \partial_t \Psi)\|_{H^N(\Sigma_t)\times H^{N-1}(\Sigma_t)} + \|\partial W\|_{H^{N-1}(\Sigma_t)} \\ &\leq \mathrm{data} \times \exp\left(1 + \|\boldsymbol{\partial} \Psi\|_{L^1([0,t])L^\infty_x} + \|\partial W\|_{L^1([0,t])L^\infty_x}\right). \end{aligned} \tag{23}$$

Thus, (23) would immediately imply the desired a priori estimate if we were able to simultaneously show that for $T > 0$ sufficiently small, we have the following key bounds for some $\delta > 0$ and $\delta_1 > 0$ with $0 < \delta_1 \leq \alpha$:

$$\|\boldsymbol{\partial} \Psi\|_{L^2([0,T])L^\infty_x}, \; \|\partial W\|_{L^2([0,T])L^\infty_x} \lesssim T^\delta \mathrm{data} + T^\delta \|\mathrm{curl}\, W\|_{C^{0,\delta_1}(\Sigma_0)}. \tag{24}$$

The rest of the discussion in Sect. 2.1 concerns the proof of (24).

2.1.3 Strichartz estimates and acoustic geometry for the model system

We now discuss how to establish (24) for the term $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x}$ using Strichartz estimates. In practice, this can be accomplished by first making a bootstrap assumption that is weaker than (24), then combining it with (23) to deduce the energy bound

$$\|(\Psi, \partial_t \Psi)\|_{H^N(\Sigma_t)\times H^{N-1}(\Sigma_t)} + \|\partial W\|_{H^{N-1}(\Sigma_t)} \leq \text{data},$$

and then finally proving Strichartz estimates that imply the "improved" estimate

$$\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x} + \|\partial W\|_{L^2([0,T])L^\infty_x} \lesssim T^{\delta}\text{data} + T^{\delta}\|\text{curl}\, W\|_{C^{0,\delta_1}(\Sigma_0)}.$$

Thus, to illustrate the main ideas, we will assume the energy bound and sketch how to prove $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x} \lesssim 1$, where, for convenience, we will ignore the small power of T^{δ} (which in reality is important for gaining smallness in various estimates) and also ignore term "data" by considering it to be $\lesssim 1$. At this point in our discussion of the model system, we will also ignore the following important technical point: to close some estimates, one must achieve control of not only $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x}$, but also $\sum_{\nu\geq 2}\nu^{2\delta_1}\|P_\nu\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T])L^\infty_x}$ and $\|\boldsymbol{\partial}\vec{\Psi}\|_{L^2([0,T])C^{0,\delta_1}_x}$, where P_ν are standard dyadic Littlewood–Paley projections and $\delta_1 > 0$ is a small Hölder exponent; see Theorem 7.1 and Corollary 7.1 for the details. We will elaborate on the importance of controlling $\|\boldsymbol{\partial}\vec{\Psi}\|_{L^2([0,T])C^{0,\delta_1}_x}$ in Sect. 2.1.4, when we explain how to control $\|\partial W\|_{L^2([0,T])L^\infty_x}$. As we describe starting two paragraphs below, our approach to deriving the Strichartz estimates is fundamentally connected to the geometry of $\mathbf{g}$-null hypersurfaces, i.e., hypersurfaces whose normals V verify $\mathbf{g}(V, V) = 0$, and in order to control the geometry of null hypersurfaces, we use arguments that rely on having a bound for $\sum_{\nu\geq 2}\nu^{2\delta_1}\|P_\nu\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T])L^\infty_x}$.

The basic idea behind obtaining the desired bound for $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x}$ is to establish an appropriate Strichartz estimate for the wave equation (18a). The analog estimate in the context of the standard flat linear wave equation $-\partial_t^2\varphi + \Delta\varphi = 0$ on $\mathbb{R}^{1+3}$ is the well-known Strichartz estimate $\|\boldsymbol{\partial}\varphi\|_{L^2_t([0,1])L^\infty_x} \lesssim \|\boldsymbol{\partial}\varphi\|_{H^{1+\varepsilon}(\Sigma_0)}$, valid for any $\varepsilon > 0$. As we mentioned in Sect. 1.3, the important work of Tataru [45,47] (see in particular [45, Theorem 6] and [47, Theorem 5.1]), which provided Strichartz estimates for linear wave equations with *rough* coefficients, would in fact yield the desired bound $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x} \lesssim 1$ *under the stronger assumption* $N > 13/6$, *provided one can simultaneously bound* $\overline{\textit{RHS (18a) in}}$ $\|\cdot\|_{L^\infty([0,T])H^{N-1}_x}$, i.e, provided one can control $\|\text{curl}\, W + \boldsymbol{\partial}\Psi\cdot\boldsymbol{\partial}\Psi\|_{L^\infty([0,T])H^{N-1}_x}$. For the model system, there is no difficulty in extending the estimate (23) to the case $N > 13/6$. Thus, assuming that one can also control the term $\|\partial W\|_{L^2([0,t])L^\infty_x}$ on RHS (23), we obtain (using Tataru's framework) the desired bound $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x} \lesssim 1$ under this stronger assumption $N > 13/6$. We stress that in the case of the compressible Euler equations, controlling the analog of the term $\|\text{curl}\, W\|_{L^\infty([0,T])H^{N-1}_x}$ *is* possible (see Proposition 5.1), but only by exploiting the special structures of the equations of Proposition 1.1. Moreover, it is not possible to achieve such control at the classical local well-posedness level $(\varrho - \bar{\varrho}, v, s) \in H^{(5/2)^+}(\Sigma_0)$; see Remark 1.5.

It is known [40] that without further information about the principal coefficients $(\mathbf{g}^{-1})^{\alpha\beta}$ of the wave operator $\hat{\square}_{\mathbf{g}}$, Tataru's linear Strichartz estimates are optimal. Thus, to achieve the goal of lowering the Sobolev regularity threshold to $N > 2$, we must exploit the specific structure of the system (18a)–(18c). Over the last two decades, a robust framework for achieving this goal for quasilinear wave systems with a *single wave speed*[21] has emerged, starting with [18], progressing through the results [19–23,41,54], and, in the case of the Einstein-vacuum equations, culminating in the proof [24] of the bounded L^2 curvature conjecture. As we will further explain below, the most significant difference between the case of single-speed quasilinear wave systems and the model system (18a)–(18c) is the presence of the terms on RHSs (18a)–(18c) that depend on one derivative of W. Despite the presence of these terms, our approach here allows us to initiate the derivation of Strichartz estimates for the model system starting from the same crucial ingredient found in the works cited above on single-speed quasilinear wave systems: an outgoing acoustical eikonal function u, which is a solution to the following eikonal equation (Footnote 6 also applies here, i.e., as we describe in Sect. 9.4, when constructing u, we work with a rescaled version of the acoustical metric):

$$(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0 \tag{25}$$

such that $\partial_t u > 0$.

A glaring point is that *the regularity properties of u are tied to those of the solution of* (18a)–(18c) *through the dependence of the coefficients* $(\mathbf{g}^{-1})^{\alpha\beta}$ *of the eikonal equation* (25) *on* Ψ. Thus, if one studies solutions of (18a)–(18c) using arguments that rely on estimates for u and its derivatives, one must carefully confirm that the regularity of u needed for the arguments is compatible with that of Ψ. This serious technical issue, which we further discuss below, was first handled by Christodoulou–Klainerman [7] in their proof of the stability of Minkowski spacetime as a solution to the Einstein–vacuum equations. In our study of compressible Euler flow, we dedicate the entirety of Sect. 9 towards the construction of an appropriate u (where the role of $\mathbf{g}$ is played by the acoustical metric of Definition 1.5) and related geometric quantities, while in Sect. 10, we derive the difficult, tensorial regularity properties of these quantities.

The level sets of u, denoted by $\mathcal{C}_u$, are $\mathbf{g}$-null hypersurfaces, and in this paper, we will construct u so that the $\mathcal{C}_u$ are outgoing sound cones; see Fig. 2. Through a long series of reductions, originating in [46,47] and with further insights provided by [17,18,41,54], it is known that the desired Strichartz estimate $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x} \lesssim 1$ for solutions to equation (18a) can be proved for $N > 2$, thanks in part to the availability of the bound (23), *provided one can derive complementary, highly tensorial, Sobolev estimates for the derivatives of u up to top-order, both along Σ_t and along null hypersurfaces $\mathcal{C}_u$*. We refer to this task as "controlling the acoustic geometry," and our above remarks make clear that the regularity of the acoustic geometry depends on that of Ψ and W; see the discussion surrounding equation (26) for further clarification of this point. In Sect. 11, we review the main ideas behind deriving the Strichartz estimate as a

[21] By this, we mean wave equation systems featuring only one Lorentzian metric.

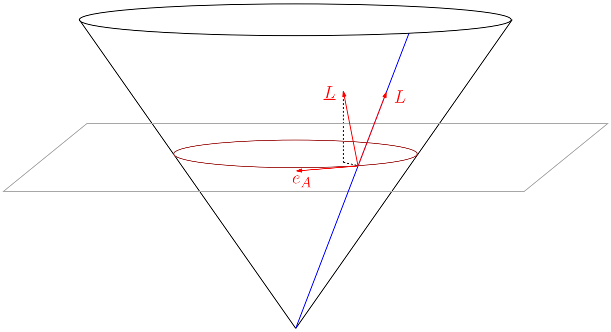

Fig. 1 The null frame

consequence of control of the acoustic geometry. The basic chain of logic[22] is: control over the acoustic geometry $\implies$ an estimate for an L^2-type (weighted) conformal energy for solutions to $\square_{\mathbf{g}}\varphi = 0$ $\implies$ dispersive decay estimates for φ $\implies$ (via a TT^* argument) linear Strichartz estimates $\implies$ (by Duhamel's principle, the energy estimates, and the Schauder estimates for the transport-part of the system discussed in Sect. 2.1.4) Strichartz estimates for the quasilinear wave equation (13).

The task of controlling the acoustic geometry is quite involved and occupies the second half of the paper; see Proposition 10.1 for a lengthy list of estimates that we use to control the acoustic geometry. In the case of quasilinear wave equations, many of the ideas for how to control u originated in [7,18–23,54]. For the model system, the main new difficulty is the presence of the term curl W on the right-hand side of the wave equation (18a), whose regularity properties strongly influence those of u; below we will elaborate on this issue. In this subsubsection, we cannot hope to discuss all of the technical difficulties that arise when controlling u, so we will mainly highlight a few key points that are new compared to earlier works. Readers can consult the introduction to [54] for an overview of many of the technical difficulties that arise in the case of quasilinear wave equations and for how they can be overcome. At the end of this subsubsection, we will mention some of these difficulties since they occur in the present work as well.

As is standard in the theory of wave equations, our analysis relies on a $\mathbf{g}$-null frame $\{L, \underline{L}, e_1, e_2\}$ adapted to u, where the vectorfield L is rescaled version of the gradient vectorfield of u, normalized by $Lt = 1$; see (184). Thus, by (25), L is null (i.e., $\mathbf{g}(L, L) = 0$), tangent to $\mathcal{C}_u$, and orthogonal to the spheres $S_{t,u} := \mathcal{C}_u \cap \Sigma_t$. Moreover, $\underline{L}$ is null, transversal to $\mathcal{C}_u$, orthogonal to $S_{t,u}$, and normalized by $\underline{L}t = 1$, and $\{e_A\}_{A=1,2}$ are a $\mathbf{g}$-orthonormal frame tangent to $S_{t,u}$; see Fig. 1, and see Sect. 9 for details on the construction of the objects depicted in the figure.

Controlling the acoustic geometry means, essentially, deriving estimates for various connection coefficients[23] of the null frame and their derivatives. There are many

[22] In our detailed proof, we partition $[0, T]$ into appropriate subintervals and derive estimates on each subinterval; see Sect. 7.2. This strategy is part of the series of reductions mentioned above. Here we are ignoring this technical aspect of the proof.

[23] These are, roughly, first $\mathbf{g}$-covariant derivatives of the frame in the directions of the frame.

quantities that we need to estimate, but for brevity, in our discussion of the model problem, we will discuss only one of them. Specifically, of primary importance for applications to Strichartz estimates is the null mean curvature of the level sets of u (i.e., of sound cones in the context of compressible Euler flow), denoted by $\mathrm{tr}_{\mathscr{g}}\chi$ and defined by $\mathrm{tr}_{\mathscr{g}}\chi = \sum_{A=1}^{2} \mathbf{g}(\mathbf{D}_{e_A} L, e_A)$, with $\mathbf{D}$ the Levi-Civita connection of $\mathbf{g}$. Analytically, $\mathrm{tr}_{\mathscr{g}}\chi$ corresponds to a special combination of up-to-second-order derivatives of u with coefficients that depend, relative to Cartesian coordinates, on the up-to-first-order derivatives of $\mathbf{g}$. To bound $\mathrm{tr}_{\mathscr{g}}\chi$, one exploits that it verifies *Raychaudhuri's equation* (see (212c) and (228a)), which is an evolution equation with source terms depending on the Ricci curvature of $\mathbf{g}$. A careful decomposition of the Ricci curvature (see Lemma 9.6) allows one to express Raychaudhuri's equation in the form[24]

$$L(\mathrm{tr}_{\mathscr{g}}\chi + \mathbf{\Gamma}_L) = \frac{1}{2} L^\alpha L^\beta \hat{\square}_{\mathbf{g}}(\mathbf{g}_{\alpha\beta}(\Psi)) + \cdots, \tag{26}$$

where $\mathbf{\Gamma}_L := L^\alpha \mathbf{\Gamma}_\alpha$, and $\mathbf{\Gamma}^\alpha \sim (\mathbf{g}^{-1})^2 \cdot \boldsymbol{\partial}\mathbf{g}$ is a contracted Cartesian Christoffel symbol of $\mathbf{g}$. Here we emphasize that the regularity properties of $\mathrm{tr}_{\mathscr{g}}\chi + \mathbf{\Gamma}_L$ are tied to those of the source terms in the wave equation (18a), since the first term on RHS (26) can be expressed via (18a) and the chain rule. It turns out that in order to obtain enough control of the acoustic geometry to prove the Strichartz estimates, one needs to control, among other terms, the $\mathcal{C}_u$-tangential derivatives, namely L and $\nabla\!\!\!/$, of $\mathrm{tr}_{\mathscr{g}}\chi$ in various norms along $\mathcal{C}_u$, where $\nabla\!\!\!/$ is the Levi-Civita connection of the Riemannian metric $\mathscr{g}$ induced on the spheres $S_{t,u}$ by $\mathbf{g}$; see, for example, the estimate (288d). This suggests, in view of Eq. (18a) and the presence of the product $\frac{1}{2} L^\alpha L^\beta \hat{\square}_{\mathbf{g}}(\mathbf{g}_{\alpha\beta}(\Psi))$ on RHS (26), that we in particular have to control $\|\nabla\!\!\!/ \mathrm{curl} W\|_{L^2(\mathcal{C}_u)}$. In fact, one needs control of a slightly higher Lebesgue exponent than 2 in the angular variables to close the proof, though we will downplay this technical issue in our simplified discussion here. For the compressible Euler equations, see Proposition 6.1 for the precise estimates that we need for the fluid variables along null hypersurfaces. We emphasize that in reality, the needed control of $\nabla\!\!\!/ \mathrm{curl} W$ is at the top-order level (i.e., it relies on the assumption $\|\partial W\|_{H^{N-1}(\Sigma_0)} < \infty$). To achieve the desired control, we use two crucial structural features of the equations.

1. $\mathrm{curl} W$ satisfies the transport equation (18c). Therefore, using standard energy estimates for transport equations and the energy estimate (23) along Σ_t (which can be used to obtain spacetime control of the source terms in the transport equation), one can control, roughly,[25] $\mathrm{curl} W$ in $\|\cdot\|_{H^{N-1}(\mathcal{H})}$ along *any hypersurface* $\mathcal{H}$ that is transversal to the transport operator $\partial_t + \Psi\partial_1$ on LHS (18c). Note that *the needed estimate along* $\mathcal{H}$ *would not be available if, instead of* $\mathrm{curl} W$ *on RHS* (18a), *we had a generic spatial derivative* ∂W; we can control *generic* top-order spatial derivatives of W in L^2 *only along the hypersurfaces* Σ_t, since elliptic Hodge estimates of type (19) hold only along such hypersurfaces. In the compressible

24 Some of the terms denoted by "$\cdots$" on RHS (26) are important from the point of view of their L^∞-size; we are ignoring those terms in the present discussion because we are focusing on issues tied to regularity.

25 The precise norm that we need to control along null hypersurfaces is the one on LHS (102), which involves Littlewood–Paley projections adapted to Σ_t.

Euler equations, this miraculous structural feature is manifested by the fact that the principal transport terms on RHS (13) are precisely $\vec{\mathcal{C}}$ and $\mathcal{D}$, which satisfy the transport equations (15b) and (16a). We again refer to Proposition 6.1 for the precise estimates that we derive for $\vec{\mathcal{C}}$ and $\mathcal{D}$ along null hypersurfaces.

2. To control the acoustic geometry, one must consider the case $\mathcal{H} := \mathcal{C}_u$, and thus one needs to know that $\partial_t + \Psi\partial_1$ is transversal to the sound cones $\mathcal{C}_u$. For the model system, the transversality could be guaranteed only by making assumptions on the structure of the component functions $(\mathbf{g}^{-1})^{\alpha\beta}(\Psi)$. However, for the compressible Euler equations, the needed transversality is guaranteed by a crucial geometric fact: the relevant transport vectorfield operator is $\mathbf{B}$, and it enjoys the timelike property $\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1$ (see (12)), thus ensuring that $\mathbf{B}$ is transversal to *any* $\mathbf{g}$-null hypersurface.

We close this subsubsection by highlighting a few key technical issues that were also present in [54] and related works.

- To close our bootstrap argument, we find it convenient to partition the bootstrap interval and to work with a rescaled version of the solution adapted to the partition. We define the partitioning in Sect. 7.2 and the rescaling in Sect. 9.1. Moreover, for each partition and corresponding rescaled solution, we construct an eikonal function adapted to that specific partition; we will ignore this technical issue for the rest of this subsubsection.
- It turns out that the connection coefficients of the null frame do not satisfy PDEs that allow us to derive the desired estimates. Thus, one must instead work with a collection of "modified" connection coefficients that satisfy better PDEs, for which we can derive the desired estimates. This is already apparent from equation (26), which suggests that $\mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L$ is the "correct" quantity to study from the point of view of PDE analysis. We define these modified quantities in Sect. 9.7.
- To close the proof, we need to control $\|\mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L\|_{L_t^\infty L_x^\infty}$ via the transport equation (26); see, for example, the estimate[26] (288a). However, given the low regularity, it is not automatic that we have quantitative control of the "data-term" $\|\mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L\|_{L^\infty(\Sigma_0)}$, as such control depends on the initial condition for u (which we are free to choose). In Proposition 9.8, we recall a result of [54], which shows that there exists a foliation of Σ_0 that can be used to define an initial condition for u with many good properties, leading in particular to the desired quantitative control of $\|\mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L\|_{L^\infty(\Sigma_0)}$.
- In the proof of the conformal energy estimate from [54] (the results of which we quote in our proof of the Strichartz estimate), there is a technical part of the argument in which one needs to work with a conformally rescaled metric $e^{2\sigma}\mathbf{g}$, constructed such that its null second fundamental form has a trace equal to the quantity $\mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L$ highlighted above; we refer readers to [54, Section 1.4.1] for further discussion on this issue. In Sect. 9.7.1, we construct the conformally rescaled metric. To close the conformal energy estimate, we must derive estimates for various geometric derivatives of σ up to second order; see Proposition 10.1.

[26] The actual estimates that we need involve $\tilde{r}$ weights, where $\tilde{r}$ is defined in (176). We also note that in the bulk of the article, we denote $\mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L$ by $\mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}$; see (204b).

2.1.4 Mixed spacetime estimates for the transport variable

We now discuss how to establish (24) for the term $\|\partial W\|_{L^2([0,T])L^\infty_x}$ on the left-hand side. As in Sect. 2.1.3, we will assume the energy bound

$$\|(\Psi, \partial_t\Psi)\|_{H^N(\Sigma_t)\times H^{N-1}(\Sigma_t)} + \|\partial W\|_{H^{N-1}(\Sigma_t)} \lesssim 1,$$

we will ignore the small power of T^δ, and, imagining that we are carrying out a bootstrap argument, we will assume the results of that subsubsection, i.e., we assume the bound $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x} \lesssim 1$. The main idea of controlling $\|\partial W\|_{L^2([0,T])L^\infty_x}$ is to in fact control, for some small constant $\delta_1 > 0$, the stronger norm[27] $\|\partial W\|_{L^2([0,T])C^{0,\delta_1}_x}$ by combining estimates for the transport-div-curl system (18b)–(18c) with the following standard elliptic Schauder-type estimate (see Lemma 8.2):

$$\|\partial W\|_{C^{0,\delta_1}(\mathbb{R}^3)} \lesssim \|\mathrm{div} W\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|\mathrm{curl} W\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|\partial W\|_{L^2(\mathbb{R}^3)}. \tag{27}$$

It is well-known that (27) is false when the space $C^{0,\delta_1}(\mathbb{R}^3)$ is replaced (on both sides) with $L^\infty(\mathbb{R}^3)$; this explains our reliance on Hölder norms. To control RHS (27), we will use the following important fact, mentioned already in the first paragraph of Sect. 2.1.3: *the Strichartz estimate $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])L^\infty_x} \lesssim 1$ can be slightly strengthened, under the scope of our approach, to $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])C^{0,\delta_1}_x} \lesssim 1$*; see Corollary 7.1. To proceed, we take the norm $\|\cdot\|_{C^{0,\delta_1}(\Sigma_t)}$ of the transport equation (18c) and integrate in time, use (27) to bound the source term factor ∂W on RHS (18c), use (18b) to substitute for the first term on RHS (27), and use the strengthened Strichartz estimate $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])C^{0,\delta_1}_x} \lesssim 1$ (which in particular, as the arguments of Lemma 8.3 show, yields control of the integral curves of the transport operator $\partial_t + \Psi\partial_1$ on LHS (18c)) to obtain the following estimate (see Sect. 8.5 for the details):

$$\|\mathrm{curl} W\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|\mathrm{curl} W\|_{C^{0,\delta_1}(\Sigma_0)} + \mathrm{data} + \int_0^t \|\boldsymbol{\partial}\Psi\|_{C^{0,\delta_1}(\Sigma_\tau)}\|\mathrm{curl} W\|_{C^{0,\delta_1}(\Sigma_\tau)}\, d\tau. \tag{28}$$

To control the first term on RHS (28), we need to assume that $\|\partial W\|_{C^{0,\alpha}(\Sigma_0)} < \infty$, for some $\alpha > 0$ (and then $\delta_1 > 0$ is chosen to be $\le \alpha$). There seems to be no way to avoid this assumption by the method we are using since transport equation solutions do not gain regularity or satisfy Strichartz estimates (which are tied to dispersion). From (28), Grönwall's inequality, and the bound $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])C^{0,\delta_1}_x} \lesssim 1$, we find that

$$\|\mathrm{curl} W\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|\mathrm{curl} W\|_{C^{0,\delta_1}(\Sigma_0)} + \mathrm{data}.$$

[27] One might be tempted to avoid using the Hölder-based norms $\|\cdot\|_{L^2([0,T])C^{0,\delta_1}_x}$ and to instead use elliptic theory to obtain control of $\|\partial W\|_{L^2([0,T])BMO_x}$. The difficulty is that control of $\|\partial W\|_{L^2([0,T])BMO_x}$ is insufficient for controlling the nonlinear term $\partial\Psi \cdot \partial W$ on RHS (18c) in the norm $\|\cdot\|_{L^2_t([0,T])H^N_x}$, which in turn would obstruct closure of the energy estimates.

From this bound, (27), Eq. (18b) (which we again use to substitute for the first term on RHS (27)), and the assumed energy bound, we find that

$$\|\partial W\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|\mathrm{curl}\, W\|_{C^{0,\delta_1}(\Sigma_0)} + \text{data} + \|\boldsymbol{\partial}\Psi\|_{C^{0,\delta_1}(\Sigma_t)}.$$

Finally, squaring this estimate, integrating in time, and again using the bound $\|\boldsymbol{\partial}\Psi\|_{L^2([0,T])C_x^{0,\delta_1}} \lesssim 1$, we obtain the desired bound $\|\partial W\|_{L^2([0,T])C_x^{0,\delta_1}} \lesssim \|\mathrm{curl}\, W\|_{C^{0,\delta_1}(\Sigma_0)} + \text{data}$.

We have therefore sketched how to establish (24) which, in view of (23), justifies (for t sufficiently small) the fundamental estimate $\|(\Psi, \partial_t\Psi)\|_{H^N(\Sigma_t)\times H^{N-1}(\Sigma_t)} + \|\partial W\|_{H^{N-1}(\Sigma_t)} \leq \text{data}$.

3 Littlewood–Paley projections, standard norms, parameters, assumptions on the initial data, bootstrap assumptions, and notation regarding constants

In this section, we define the standard Littlewood–Paley projections, define various norms and parameters that we use in our analysis, state our assumption on the initial data, formulate the bootstrap assumptions that we use in proving Theorem 1.2, and state our conventions for constants C.

3.1 Littlewood–Paley projections

We fix a smooth function $\eta : \mathbb{R}^3 \to [0, 1]$ supported on the frequency-space annulus $\{\xi \in \mathbb{R}^3 \mid 1/2 \leq |\xi| \leq 2\}$ such that for $\xi \neq 0$, we have $\sum_{k\in\mathbb{Z}} \eta(2^k\xi) = 1$. For dyadic frequencies $\lambda = 2^k$ with $k \in \mathbb{Z}$, we define the standard Littlewood–Paley projection P_λ, which acts on scalar functions $F : \mathbb{R}^3 \to \mathbb{C}$, as follows:

$$P_\lambda F(x) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\eta(\lambda^{-1}\xi)\hat{F}(\xi)\,d\xi, \tag{29}$$

where $\hat{F}(\xi) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi}F(x)\,dx$ (with $dx := dx^1dx^2dx^3$) is the Fourier transform of F. If F is an array-valued function, then $P_\lambda F$ denotes the array of projections of its components. If $I \subset 2^{\mathbb{Z}}$ is an interval of dyadic frequencies, then $P_I F := \sum_{\nu\in I} P_\nu F$, and $P_{\leq\lambda}F := P_{(-\infty,\lambda]}F$.

If F is a function on Σ_t, then $P_\lambda F(t, x) := P_\lambda G(x)$, where $G(x) := F(t, x)$, and similarly for $P_I F(t, x)$ and $P_{\leq\lambda}F(t, x)$.

3.2 Norms and seminorms

In this subsection, we define some standard norms and seminorms that we will use in the first part of the paper, before we control the acoustic geometry. To control the acoustic geometry, we will use additional norms, defined in Sect. 9.10.

For scalar- or array-valued functions F and $1 \leq q < \infty$, $\|F\|_{L^q(\Sigma_t)} := \left\{\int_{\Sigma_t} |F(t,x)|^q \, dx\right\}^{1/q}$ and $\|F\|_{L^\infty(\Sigma_t)} := \text{ess sup}_{x\in\mathbb{R}^3}|F(t,x)|$ are standard Lebesgue norms of F, where we recall that Σ_t is the standard constant-time slice. Lebesgue norms on subsets $D \subset \Sigma_t$ are defined in an analogous fashion, e.g., if $D = \{t\} \times D'$, then $\|F\|_{L^q(D)} := \left\{\int_{D'} |F(t,x)|^q \, dx\right\}^{1/q}$. Similarly, if $\{A_\lambda\}_{\lambda\in 2^{\mathbb{N}}}$ is a dyadic-indexed sequence of real numbers and $1 \leq q < \infty$, then $\|A_\nu\|_{\ell^q_\nu} := \left\{\sum_{\nu\geq 1} A_\nu^q\right\}^{1/q}$.

We will rely on the following family of seminorms, parameterized by real numbers M (where we will have $M > 0$ in our applications below):

$$\|\Lambda^M F\|_{L^2(\Sigma_t)} := \sqrt{\sum_{\nu\geq 2} \nu^{2M} \|P_\nu F\|^2_{L^2(\Sigma_t)}}, \tag{30}$$

where on RHS (30) and throughout, sums involving Littlewood–Paley projections are understood to be dyadic sums.

For real numbers $M \geq 0$, we define the following standard Sobolev norm for functions F on Σ_t:

$$\|F\|_{H^M(\Sigma_t)} := \left\{\|P_{\leq 1} F\|^2_{L^2(\Sigma_t)} + \|\Lambda^M F\|^2_{L^2(\Sigma_t)}\right\}^{1/2}. \tag{31}$$

Throughout, we will rely on the standard fact that when M is an integer, the norm defined in (31) is equivalent to $\sum_{|\vec{I}|\leq M} \|\partial_{\vec{I}} F\|_{L^2(\Sigma_t)}$, where $\vec{I}$ are spatial derivative multi-indices.

If F is a function defined on a subset $D \subset \mathbb{R}^3$ and $\beta \geq 0$, then we define the Hölder norm $\|\cdot\|_{C^{0,\beta}(D)}$ of F as follows:

$$\|F\|_{C^{0,\beta}(D)} := \|F\|_{L^\infty(D)} + \sup_{x,y\in D, 0<|x-y|} \frac{|F(x)-F(y)|}{|x-y|^\beta}. \tag{32}$$

Similarly, if F is a function defined on a subset $D \subset \Sigma_t$ of the form $D = \{t\} \times D'$, then $\|F\|_{C^{0,\beta}(D)} := \|G\|_{C^{0,\beta}(D')}$, where $G(x) := F(t,x)$.

We will also use the following mixed norms for functions F defined on $\mathbb{R}^{1+3}$, where $1 \leq q_1 < \infty$, $1 \leq q_2 \leq \infty$, and I is an interval of time:

$$\|F\|_{L^{q_1}(I)L^{q_2}_x} := \left\{\int_I \|F\|^{q_1}_{L^{q_2}(\Sigma_\tau)} \, d\tau\right\}^{1/q_1}, \qquad \|F\|_{L^\infty(I)L^{q_2}_x} := \text{ess sup}_{\tau\in I}\|F\|_{L^{q_2}(\Sigma_\tau)}, \tag{33a}$$

$$\|F\|_{L^{q_1}(I)C^{0,\beta}_x} := \left\{\int_I \|F\|^{q_1}_{C^{0,\beta}(\Sigma_\tau)} \, d\tau\right\}^{1/q_1}, \quad \|F\|_{L^\infty(I)C^{0,\beta}_x} := \text{ess sup}_{\tau\in I}\|F\|_{C^{0,\beta}(\Sigma_\tau)}. \tag{33b}$$

Similarly, if $\{F_\lambda\}_{\lambda\in 2^{\mathbb{N}}}$ is a dyadic-indexed sequence of functions F_λ on Σ_t, then

$$\|F_\nu\|_{\ell^2_\nu L^2(\Sigma_t)} := \left\{\sum_{\nu\geq 1}\|F_\nu\|^2_{L^2(\Sigma_t)}\right\}^{1/2}. \tag{34}$$

3.3 Choice of parameters

In this subsection, we introduce the parameters that will play a role in our analysis. We recall that $2 < N \leq 5/2$ and $0 < \alpha < 1$ denote given real numbers corresponding, respectively, to the assumed Sobolev regularity of the data and the assumed Hölder regularity of the transport part of the data; see (38a)–(38b). We then choose positive numbers q, ϵ_0, δ_0, δ, and δ_1 that satisfy the following conditions:

$$2 < q < \infty, \tag{35a}$$

$$0 < \epsilon_0 := \frac{N-2}{10} < \frac{1}{10}, \tag{35b}$$

$$\delta_0 := \min\left\{\epsilon_0^2, \frac{\alpha}{10}\right\}, \tag{35c}$$

$$0 < \delta := \frac{1}{2} - \frac{1}{q} < \epsilon_0, \tag{35d}$$

$$\delta_1 := \min\{N - 2 - 4\epsilon_0 - \delta(1 - 8\epsilon_0), \alpha\} > 8\delta_0 > 0. \tag{35e}$$

More precisely, we consider N, α, ϵ_0, and δ_0 to be fixed throughout the paper, while in some of our arguments below, we will treat q, δ, and δ_1 as parameters, where $q > 2$ will need to be chosen to be sufficiently close to 2 (i.e., $\delta > 0$ will need to be chosen to be sufficiently small).

3.4 Assumptions on the initial data

The following definition captures the subset of solution space in which the compressible Euler equations are hyperbolic in a non-degenerate sense.

Definition 3.1 *(Regime of hyperbolicity)*. We define $\mathcal{K}$ as follows, where $c(\rho, s)$ is the speed of sound:

$$\mathcal{K} := \left\{(\rho, s, \vec{v}, \vec{\Omega}, \vec{S}) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 \mid 0 < c(\rho, s) < \infty\right\}. \tag{36}$$

We set

$$(\mathring{\rho}, \mathring{s}, \mathring{\vec{v}}, \mathring{\vec{\Omega}}, \mathring{\vec{S}}, \mathring{\vec{\mathcal{C}}}, \mathring{\mathcal{D}}) := (\rho, s, \vec{v}, \vec{\Omega}, \vec{S}, \vec{\mathcal{C}}, \mathcal{D})|_{\Sigma_0}. \tag{37}$$

With N and α as in Sect. 3.3, we assume that

$$\|\mathring{\rho}\|_{H^N(\Sigma_0)} + \|\mathring{\vec{v}}\|_{H^N(\Sigma_0)} < \infty, \tag{38a}$$

$$\|\mathring{\vec{\Omega}}\|_{H^N(\Sigma_0)} + \|\mathring{s}\|_{H^{N+1}(\Sigma_0)} + \|\mathring{\vec{\mathcal{C}}}\|_{C^{0,\alpha}(\Sigma_0)} + \|\mathring{\mathcal{D}}\|_{C^{0,\alpha}(\Sigma_0)} < \infty. \tag{38b}$$

(38a) corresponds to "rough" regularity assumptions on the wave-part of the data, while (38b) corresponds to regularity assumptions on the transport-part of the data.

Let $\text{int}U$ denote the interior of the set U. We assume that there are compact subsets $\mathring{\mathfrak{K}}$ and $\mathfrak{K}$ of $\text{int}\mathcal{K}$ such that

$$(\mathring{\rho}, \mathring{s}, \mathring{\vec{v}}, \mathring{\vec{\Omega}}, \mathring{\vec{S}})(\mathbb{R}^3) \subset \text{int}\mathring{\mathfrak{K}} \subset \mathring{\mathfrak{K}} \subset \text{int}\mathfrak{K} \subset \mathfrak{K} \subset \text{int}\mathcal{K}. \tag{39}$$

3.5 Bootstrap assumptions

For the rest of the article, $0 < T_* \ll 1$ denotes a "bootstrap time" that we will choose to be sufficiently small in a manner that depends only on the quantities introduced in Sect. 3.4. We assume that $(\rho, s, \vec{v}, \vec{\Omega}, \vec{S})$ is a smooth (see Footnote 11) solution to the equations of Proposition 1.1 on the "bootstrap slab" $[0, T_*] \times \mathbb{R}^3$.

3.5.1 Bootstrap assumptions tied to $\mathcal{K}$.

Let $\mathfrak{K}$ be the subset from Sect. 3.4. We assume that

$$(\rho, s, \vec{v}, \vec{\Omega}, \vec{S})([0, T_*] \times \mathbb{R}^3) \subset \mathfrak{K}. \tag{40}$$

In Corollary 8.1, we derive a strict improvement of (40).

Remark 3.1 *(Uniform $L^\infty(\Sigma_t)$ bounds).* Note that the bootstrap assumption (40) implies, in particular, uniform $L^\infty(\Sigma_t)$ bounds, depending on $\mathfrak{K}$, for ρ, s, $\vec{v}$, $\vec{\Omega}$, and $\vec{S} \sim \partial s$. Throughout the article, we will often use these simple $L^\infty(\Sigma_t)$ bounds without explicitly mentioning that we are doing so.

3.5.2 Mixed spacetime norm bootstrap assumptions

We assume that the following estimates hold:

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t([0,T_*])L^\infty_x} + \sum_{\nu \geq 2} \nu^{2\delta_0} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2_t([0,T_*])L^\infty_x} \leq 1, \tag{41a}$$

$$\|\partial(\vec{\Omega}, \vec{S})\|^2_{L^2_t([0,T_*])L^\infty_x} + \sum_{\nu \geq 2} \nu^{2\delta_0} \|P_\nu \partial(\vec{\Omega}, \vec{S})\|^2_{L^2_t([0,T_*])L^\infty_x} \leq 1. \tag{41b}$$

In Theorem 7.1, we derive a strict improvement of (41a). In Theorem 8.1, we derive a strict improvement of (41b).

Remark 3.2 When deriving the energy estimates, we will only use the bounds for $\|\boldsymbol{\partial}\vec{\Psi}\|_{L^2_t([0,T_*])L^\infty_x}$ and $\|\partial(\vec{\Omega},\vec{S})\|_{L^2_t([0,T_*])L^\infty_x}$. We use the bounds for the two sums in (41a)–(41b) to obtain control over the acoustic geometry, that is, for proving Proposition 10.1. In turn, such control over the acoustic geometry will allow us to prove a frequency-localized Strichartz estimate (Theorem 7.2), and then to improve the Strichartz-type assumption for the wave variables (i.e., to prove Theorem 7.1). For more details about this strategy, we refer to Sect. 2.1.3.

3.6 Notation regarding constants

In the rest of the paper, $C > 0$ denotes a constant that is free to vary from line to line. C is allowed to depend on N, α, the parameters from Sect. 3.3, the norms of the data from Sect. 3.4, and the set $\mathfrak{K}$ from Sect. 3.4. We often bound explicit functions of t by $\leq C$ since $t \leq T_* \ll 1$. For given quantities $A, B \geq 0$, write $A \lesssim B$ to mean that there exists a $C > 0$ such that $A \leq CB$. We write $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$.

4 Preliminary energy and elliptic estimates

Our main goal in this section is to prove preliminary energy and elliptic estimates that yield $H^2(\Sigma_t)$-control of the velocity, density, and specific vorticity, and $H^3(\Sigma_t)$-control of the entropy. The main result is provided by Proposition 4.1. These preliminary below-top-order estimates are useful, in the context of controlling the solution's top-order derivatives, for handling all but the most difficult error terms. The proof of Proposition 4.1 is located in Sect. 4.4. Before proving the proposition, we first provide two standard ingredients: the geometric energy method for wave equations and transport equations, and estimates in $L^2(\Sigma_t)$-based spaces for div-curl systems.

Proposition 4.1 (Preliminary energy and elliptic estimates) *There exists a continuous strictly increasing function $F : [0,\infty) \to [0,\infty)$ such that under the initial data and bootstrap assumptions of Sect. 3, smooth solutions to the compressible Euler equations satisfy the following estimates for $t \in [0, T_*]$:*

$$\sum_{k=0}^{2} \|\partial_t^k(\rho,\vec{v},\vec{\Omega})\|_{H^{2-k}(\Sigma_t)} + \sum_{k=0}^{2} \|\partial_t^k s\|_{H^{3-k}(\Sigma_t)} + \sum_{k=0}^{1} \|\partial_t^k(\vec{\mathcal{C}},\mathcal{D})\|_{H^{1-k}(\Sigma_t)} \leq F\left(\|(\rho,\vec{v},\vec{\Omega})\|_{H^2(\Sigma_0)} + \|s\|_{H^3(\Sigma_0)}\right). \tag{42}$$

Moreover, for any a and b with $0 \leq a \leq b \leq T_*$, solutions φ to the inhomogeneous wave equation

$$\square_{g(\vec{\Psi})}\varphi = \mathfrak{F} \tag{43}$$

satisfy the following estimate:

$$\|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_b)} \lesssim \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_a)} + \|\mathfrak{F}\|_{L^1([a,b])L^2_x}. \tag{44}$$

4.1 The geometric energy method for wave equations

To derive energy estimates for solutions to the wave equations in (13), we will use the well-known vectorfield multiplier method. In this subsection, we set up this geometric energy method. Throughout this subsection, we lower and raise Greek indices with the acoustical metric $\mathbf{g} = \mathbf{g}(\vec{\Psi})$ from Definition 1.5 and its inverse. Moreover, we recall that $\mathbf{D}$ denotes the Levi-Civita connection of $\mathbf{g}$ and $\square_{\mathbf{g}} := (\mathbf{g}^{-1})^{\alpha\beta}\mathbf{D}_\alpha\mathbf{D}_\beta$ denotes the corresponding covariant wave operator.

4.1.1 Energy-momentum tensor, energy current, and deformation tensor

We define the energy-momentum tensor associated to a scalar function φ to be the following symmetric type $\binom{0}{2}$ tensorfield:

$$\mathbf{Q}_{\alpha\beta}[\varphi] := \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}\mathbf{g}_{\alpha\beta}(\mathbf{g}^{-1})^{\kappa\lambda}\partial_\kappa\varphi\partial_\lambda\varphi. \tag{45}$$

Given φ and any "multiplier" vectorfield $\mathbf{X}$, we define the corresponding energy current ${}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi]$ vectorfield:

$${}^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi] := \mathbf{Q}^{\alpha\beta}[\varphi]\mathbf{X}_\beta. \tag{46}$$

We define the deformation tensor of $\mathbf{X}$ to be the following symmetric type $\binom{0}{2}$ tensorfield:

$${}^{(\mathbf{X})}\boldsymbol{\pi}_{\alpha\beta} := \mathbf{D}_\alpha\mathbf{X}_\beta + \mathbf{D}_\beta\mathbf{X}_\alpha. \tag{47}$$

A straightforward computation yields the following identity, which will form the starting point for our energy estimates for the wave equations:

$$\mathbf{D}_\kappa{}^{(\mathbf{X})}\mathbf{J}^\kappa[\varphi] = (\square_{\mathbf{g}}\varphi)\mathbf{X}\varphi + \frac{1}{2}\mathbf{Q}^{\kappa\lambda}{}^{(\mathbf{X})}\boldsymbol{\pi}_{\kappa\lambda}. \tag{48}$$

4.1.2 The basic energy along Σ_t

To derive energy estimates for solutions φ to wave equations $\square_{\mathbf{g}}\varphi = \mathfrak{F}$, we will rely on the following energy $\mathbb{E}[\varphi](t)$, where $\mathbf{B} = \partial_t + v^a\partial_a$ is the material derivative vectorfield:

$$\mathbb{E}[\varphi](t) := \int_{\Sigma_t}\left\{{}^{(\mathbf{B})}\mathbf{J}^\kappa[\varphi]\mathbf{B}_\kappa + \varphi^2\right\}d\varpi_g = \int_{\Sigma_t}\left\{\mathbf{Q}^{00}[\varphi] + \varphi^2\right\}d\varpi_g. \tag{49}$$

In (49) and throughout, $d\varpi_g$ is the volume form induced on Σ_t by the first fundamental form g of $\mathbf{g}$. A straightforward computation yields that relative to the Cartesian coordinates, we have

$$d\varpi_g = \sqrt{\det g}\, dx^1 dx^2 dx^3 = c^{-3} dx^1 dx^2 dx^3. \tag{50}$$

Also, (12) implies that $\mathbf{B}$ is timelike with respect to $\mathbf{g}$. This leads to the coercivity of $\mathbb{E}[\varphi](t)$, as we show in the next lemma.

Lemma 4.2 (Coerciveness of $\mathbb{E}[\varphi](t)$). *Under the bootstrap assumptions of Sect. 3, the following estimate holds for $t \in [0, T_*]$:*

$$\mathbb{E}[\varphi](t) \approx \|(\varphi, \partial_t\varphi)\|^2_{H^1(\Sigma_t)\times L^2(\Sigma_t)}. \tag{51}$$

Proof Since the bootstrap assumption (40) guarantees that the solution is contained in $\mathfrak{K}$, we have $c \approx 1$ and thus, by (50), $d\varpi_g = c^{-3} dx^1 dx^2 dx^3 \approx dx^1 dx^2 dx^3$. Next, using (11), (12), (45), and (46), we compute that ${}^{(\mathbf{B})}\mathbf{J}^\kappa[\varphi]\mathbf{B}_\kappa = \frac{1}{2}(\mathbf{B}\varphi)^2 + \frac{1}{2}c^2\delta^{ab}\partial_a\varphi\partial_b\varphi$. Using that $\mathbf{B}\varphi = \partial_t\varphi + v^a\partial_a\varphi$, that $|\vec{v}|$ is uniformly bounded for solutions contained in $\mathfrak{K}$, and that $c \approx 1$, and applying Young's inequality to the cross term $2(\partial_t\varphi)(v^a\partial_a\varphi)$ in $(\mathbf{B}\varphi)^2$, we deduce ${}^{(\mathbf{B})}\mathbf{J}^\kappa[\varphi]\mathbf{B}_\kappa \approx |\boldsymbol{\partial}\varphi|^2$. From these estimates and definition (49), the desired estimate (51) easily follows. □

In the next lemma, we provide the basic energy inequality that we will use when deriving energy estimates for solutions to the wave equations.

Lemma 4.3 (Basic energy inequality for the wave equations). *Let φ be smooth on $[0, T_*] \times \mathbb{R}^3$. Under the bootstrap assumptions of Sect. 3, the following inequality holds for $t \in [0, T_*]$:*

$$\begin{aligned} \|(\varphi, \partial_t\varphi)\|^2_{H^1(\Sigma_t)\times L^2(\Sigma_t)} \lesssim{}& \|(\varphi, \partial_t\varphi)\|^2_{H^1(\Sigma_0)\times L^2(\Sigma_0)} \\ &+ \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)} \|(\varphi, \partial_t\varphi)\|^2_{H^1(\Sigma_\tau)\times L^2(\Sigma_\tau)}\, d\tau \\ &+ \int_0^t \|\hat{\square}_{\mathbf{g}}\varphi\|_{L^2(\Sigma_\tau)} \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_\tau)}\, d\tau. \end{aligned} \tag{52}$$

Proof Let ${}^{(\mathbf{B})}\widetilde{\mathbf{J}}^\alpha[\varphi] := {}^{(\mathbf{B})}\mathbf{J}^\alpha[\varphi] - \varphi^2\mathbf{B}^\alpha$, where ${}^{(\mathbf{B})}\mathbf{J}^\alpha[\varphi]$ is defined by (46). Note that (12) implies ${}^{(\mathbf{B})}\widetilde{\mathbf{J}}^\kappa[\varphi]\mathbf{B}_\kappa = {}^{(\mathbf{B})}\mathbf{J}^\kappa[\varphi]\mathbf{B}_\kappa + \varphi^2$ and thus ${}^{(\mathbf{B})}\widetilde{\mathbf{J}}^\kappa[\varphi]\mathbf{B}_\kappa$ is equal to the integrand in the middle term in (49). Next, taking into account definition (47), we compute that $\mathbf{D}_\kappa {}^{(\mathbf{B})}\widetilde{\mathbf{J}}^\kappa[\varphi] = \mathbf{D}_\kappa {}^{(\mathbf{B})}\mathbf{J}^\kappa[\varphi] - 2\varphi\mathbf{B}\varphi - \frac{1}{2}\varphi^2(\mathbf{g}^{-1})^{\kappa\lambda}{}^{(\mathbf{B})}\boldsymbol{\pi}_{\kappa\lambda}$. Applying the divergence theorem on the spacetime region $[0, t] \times \mathbb{R}^3$ relative to the volume form $d\varpi_{\mathbf{g}} = \sqrt{|\det \mathbf{g}|}\, dx^1 dx^2 dx^3 d\tau = d\varpi_g d\tau$ (where the last equality follows from (10)–(11) and (50)), recalling that $\mathbf{B}$ is the future-directed $\mathbf{g}$-unit normal to Σ_t, appealing to definition (49), and using Eq.Q(48) with $\mathbf{X} := \mathbf{B}$, we deduce

$$\mathbb{E}[\varphi](t) = \mathbb{E}[\varphi](0) - \int_0^t \int_{\Sigma_\tau} (\square_{\mathbf{g}}\varphi)\mathbf{B}\varphi \, d\varpi_g \, d\tau + 2\int_0^t \int_{\Sigma_\tau} \varphi\mathbf{B}\varphi \, d\varpi_g \, d\tau + \frac{1}{2}\int_0^t \int_{\Sigma_\tau} (\mathbf{g}^{-1})^{\kappa\lambda} {}^{(\mathbf{B})}\boldsymbol{\pi}_{\kappa\lambda}\varphi^2 \, d\varpi_g \, d\tau - \frac{1}{2}\int_0^t \int_{\Sigma_\tau} \mathbf{Q}^{\kappa\lambda}[\varphi] {}^{(\mathbf{B})}\boldsymbol{\pi}_{\kappa\lambda} \, d\varpi_g \, d\tau. \tag{53}$$

Next, we note that since the bootstrap assumption (40) guarantees that the compressible Euler solution is contained in $\mathfrak{K}$, we have the following estimates for $\alpha, \beta = 0, 1, 2, 3$: $|\mathbf{B}^\alpha| \lesssim 1$, $|\mathbf{g}_{\alpha\beta}| \lesssim 1$, $|(\mathbf{g}^{-1})^{\alpha\beta}| \lesssim 1$, and $|\boldsymbol{\partial}\mathbf{g}_{\alpha\beta}| \lesssim |\boldsymbol{\partial}\vec{\Psi}|$. It follows that $\square_{\mathbf{g}}\varphi = \hat{\square}_{\mathbf{g}}\varphi + \mathcal{O}(|\boldsymbol{\partial}\vec{\Psi}|)|\boldsymbol{\partial}\varphi|$, $|\mathbf{B}\varphi| \lesssim |\boldsymbol{\partial}\varphi|$, $\mathbf{Q}[\varphi] \lesssim |\boldsymbol{\partial}\varphi|^2$, and $|{}^{(\mathbf{B})}\boldsymbol{\pi}_{\kappa\lambda}| \lesssim |\boldsymbol{\partial}\vec{\Psi}|$. From these estimates, the identity (53), the coercivity estimate (51), and the Cauchy–Schwarz inequality along Σ_τ, we conclude (52). □

4.2 The energy method for transport equations

In this subsection, we provide a simple lemma that yields a basic energy inequality for solutions to transport equations.

Lemma 4.4 (Energy estimates for transport equations). *Let φ be smooth on $[0, T_*] \times \mathbb{R}^3$. Under the bootstrap assumptions of Sect. 3, the following inequality holds for $t \in [0, T_*]$:*

$$\|\varphi\|^2_{L^2(\Sigma_t)} \lesssim \|\varphi\|^2_{L^2(\Sigma_0)} + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\|\varphi\|^2_{L^2(\Sigma_\tau)} \, d\tau + \int_0^t \|\varphi\|_{L^2(\Sigma_\tau)}\|\mathbf{B}\varphi\|_{L^2(\Sigma_\tau)} \, d\tau. \tag{54}$$

Proof Let $\mathbf{J}^\alpha := \varphi^2\mathbf{B}^\alpha$. Then $\partial_\alpha\mathbf{J}^\alpha = 2\varphi\mathbf{B}\varphi + (\partial_a v^a)\varphi^2$. Thus, we have $|\partial_\alpha\mathbf{J}^\alpha| \lesssim |\varphi||\mathbf{B}\varphi| + |\boldsymbol{\partial}\vec{\Psi}|\varphi^2$. From this estimate, a routine application of the divergence theorem on the spacetime region $[0, t] \times \mathbb{R}^3$ relative to the Cartesian coordinates that exploits the positivity of $\mathbf{J}^0 = \varphi^2$, and the Cauchy–Schwarz inequality along Σ_τ, we conclude the desired estimate (54). □

4.3 The standard elliptic div-curl identity in L^2 spaces

To control the top-order spatial derivatives of the specific vorticity and entropy, we will rely on the following standard elliptic identity.

Lemma 4.5 (Elliptic div-curl identity in L^2 spaces). *For vectorfields $V \in H^1(\mathbb{R}^3; \mathbb{R}^3)$, the following identity holds:*

$$\sum_{a,b=1}^{3} \|\partial_a V^b\|^2_{L^2(\mathbb{R}^3)} = \|divV\|^2_{L^2(\mathbb{R}^3)} + \|curlV\|^2_{L^2(\mathbb{R}^3)}. \tag{55}$$

Proof It suffices to prove the desired identity for smooth, compactly supported vectorfields, since these are dense in $H^1(\mathbb{R}^3;\mathbb{R}^3)$. For smooth, compactly supported vectorfields, the desired identity follows from integrating the divergence identity $\sum_{a,b=1}^3(\partial_a V_b)^2 = (\mathrm{div}V)^2 + |\mathrm{curl}V|^2 + \partial_a\left\{V^b\partial_b V^a\right\} - \partial_a\{V^a\mathrm{div}V\}$ over $\mathbb{R}^3$ with respect to volume form of the standard Euclidean metric on $\mathbb{R}^3$. □

4.4 Proof of Proposition 4.1

We first note that the estimates for the terms

$$\|\partial_t^2(\rho,\vec{v})\|_{L^2(\Sigma_t)},\ \sum_{k=1}^2\|\partial_t^k\vec{\Omega}\|_{H^{2-k}(\Sigma_t)},\ \sum_{k=1}^2\|\partial_t^k s\|_{H^{3-k}(\Sigma_t)},\ \text{and } \|\partial_t(\vec{\mathcal{C}},\mathcal{D})\|_{L^2(\Sigma_t)}$$

on LHS (42) follow once we have obtained the desired estimates for the remaining terms on LHS (42). The reason is that these time-derivative-involving terms can be bounded by $\lesssim$ the sum of products of the other terms on LHS (42) by using the equations of Proposition 1.1 to solve for the relevant time derivatives in terms of spatial derivatives and then using standard product estimates as well as our bootstrap assumption that the compressible Euler solution is contained in $\mathfrak{K}$ (i.e., (40)); we omit these straightforward details. Thus, it suffices for us to bound the remaining terms on LHS (42).

To proceed, we commute the equations of Proposition 1.1 with up to one spatial derivative, appeal to Definition 1.2, consider Remark 1.3, and use the bootstrap assumption (40), thereby deducing that for $\Psi\in\{\rho,v^1,v^2,v^3,s\}$, we have the following pointwise estimates:

$$|\hat{\square}_{\mathbf{g}}\partial^{\le 1}\Psi| \lesssim |\partial(\vec{\mathcal{C}},\mathcal{D})| + \left\{|\boldsymbol{\partial}\vec{\Psi}|+1\right\}|\partial\boldsymbol{\partial}\vec{\Psi}| + \sum_{P=1}^3|\boldsymbol{\partial}\vec{\Psi}|^P, \tag{56}$$

$$|\mathbf{B}\partial^{\le 1}(\vec{\Omega},\vec{S})| \lesssim |\partial\boldsymbol{\partial}\vec{\Psi}| + \left\{|(\boldsymbol{\partial}\vec{\Psi},\partial\vec{\Omega},\partial\vec{S})|+1\right\}|\boldsymbol{\partial}\vec{\Psi}|, \tag{57}$$

$$|\partial(\mathrm{div}\Omega,\mathrm{curl}S)| \lesssim |\partial\boldsymbol{\partial}\vec{\Psi}| + |\partial\vec{\Omega}||\boldsymbol{\partial}\vec{\Psi}|, \tag{58}$$

$$|\partial(\mathrm{curl}\Omega,\mathrm{div}S)| \lesssim |\partial(\vec{\mathcal{C}},\mathcal{D})| + |(\boldsymbol{\partial}\vec{\Psi},\partial\vec{\Omega},\partial\vec{S})| + \left\{|(\boldsymbol{\partial}\vec{\Psi},\partial\vec{\Omega},\partial\vec{S})|+1\right\}|\boldsymbol{\partial}\vec{\Psi}|, \tag{59}$$

$$|\mathbf{B}\partial(\vec{\mathcal{C}},\mathcal{D})| \lesssim \left\{|\boldsymbol{\partial}\vec{\Psi}|+1\right\}|\partial^2(\vec{\Omega},\vec{S})| + \left\{|(\boldsymbol{\partial}\vec{\Psi},\partial\vec{\Omega},\partial\vec{S})|+1\right\}|\partial\boldsymbol{\partial}\vec{\Psi}| + |\boldsymbol{\partial}\vec{\Psi}|^2|(\partial\vec{\Omega},\partial\vec{S})| + \sum_{P=1}^3|\boldsymbol{\partial}\vec{\Psi}|^P. \tag{60}$$

We clarify that in deriving (59), we used Definition 1.2 to algebraically solve for $\mathrm{curl}\Omega$ and $\mathrm{div}S$.

Using the estimates (56)–(60), we will derive estimates for the "controlling quantity" $Q_2(t)$ defined by

$$Q_2(t) := \|(\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi})\|^2_{H^2(\Sigma_t)\times H^1(\Sigma_t)} + \|\partial(\vec{\mathcal{C}}, \mathcal{D})\|^2_{L^2(\Sigma_t)} + \|(\vec{\Omega}, \vec{S})\|^2_{H^1(\Sigma_t)}. \quad (61)$$

We will prove the following two estimates:

$$Q_2(t) \lesssim Q_2(0) + \int_0^t \left\{\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^\infty(\Sigma_\tau)} + \|\partial(\vec{\Omega}, \vec{S})\|_{L^\infty(\Sigma_\tau)} + 1\right\} Q_2(\tau)\, d\tau, \quad (62)$$

$$\|\partial^2(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} \lesssim Q_2(t) + Q_2^3(t). \quad (63)$$

Then from the bootstrap assumptions (41a)–(41b), (62), and Grönwall's inequality, we deduce that for $t \in [0, T_*]$, we have $Q_2(t) \lesssim Q_2(0)$. From this estimate, (63), and the remarks made at the beginning the proof, we arrive at the desired estimate (42).

It remains for us to prove (62) and (63). We start with the elliptic estimates needed to control $\partial^2\vec{\Omega}$ and $\partial^2\vec{S}$ in $\|\cdot\|_{L^2(\Sigma_t)}$. From (55) with $\partial\Omega$ and ∂S in the role of V, (58), and (59), we find that

$$\|\partial^2(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} \lesssim \|\partial(\vec{\mathcal{C}}, \mathcal{D})\|^2_{L^2(\Sigma_t)} + \|\partial\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2(\Sigma_t)} + \left\{\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^\infty(\Sigma_t)} + 1\right\}\left\{\|\partial(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2(\Sigma_t)}\right\}, \quad (64)$$

which, in view of definition (61), implies that

$$\|\partial^2(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} \lesssim \left\{\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^\infty(\Sigma_t)} + 1\right\} Q_2(t). \quad (65)$$

Moreover, through an argument similar to the one we used to derive (65), based on (55), (58), and (59), but modified in that we now use the interpolation-product estimate[28]

$$\|G_1 \cdot G_2\|_{L^2(\Sigma_t)} \lesssim \|G_1\|^{1/2}_{L^2(\Sigma_t)}\|G_1\|^{1/2}_{H^1(\Sigma_t)}\|G_2\|_{H^1(\Sigma_t)},$$

to derive the bound

$$\||(\boldsymbol{\partial}\vec{\Psi}, \partial\vec{\Omega}, \partial\vec{S})||\boldsymbol{\partial}\vec{\Psi}|\|^2_{L^2(\Sigma_t)} \lesssim \|\partial(\vec{\Omega}, \vec{S})\|_{L^2(\Sigma_t)}\|\partial(\vec{\Omega}, \vec{S})\|_{H^1(\Sigma_t)}\|\boldsymbol{\partial}\vec{\Psi}\|^2_{H^1(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|^4_{H^1(\Sigma_t)},$$

we deduce that

$$\begin{aligned}\|\partial^2(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} &\lesssim \|\partial^2(\vec{\Omega}, \vec{S})\|_{L^2(\Sigma_t)} Q_2^{3/2}(t) + Q_2(t) + Q_2^2(t)\\ &\le \frac{1}{2}\|\partial^2(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} + C Q_2(t) + C Q_2^3(t),\end{aligned}$$

[28] This standard estimate can be obtained by using Hölder's inequality, Sobolev embedding, and interpolation estimates. For a more detailed proof, we refer to the proof of (79b).

from which the desired bound (63) readily follows.

We now derive energy estimates for the evolution equations. From (52) with $\partial^{\leq 1}\vec{\Psi}$ in the role of φ, (56), the Cauchy–Schwarz inequality along Σ_τ, Young's inequality, and definition (61), we deduce (occasionally using the non-optimal bound $|\boldsymbol{\partial}\vec{\Psi}| \lesssim |\boldsymbol{\partial}\vec{\Psi}|^2 + 1$) that

$$\|(\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi})\|^2_{H^2(\Sigma_t)\times H^1(\Sigma_t)} \lesssim \mathcal{Q}_2(0) + \int_0^t \left\{\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^\infty(\Sigma_\tau)} + 1\right\} \mathcal{Q}_2(\tau)\, d\tau. \tag{66}$$

Using a similar argument based on (54) with $\partial^{\leq 1}\vec{\Omega}$ and $\partial^{\leq 1}\vec{S}$ in the role of φ and Eq. (57), we deduce

$$\|(\vec{\Omega}, \vec{S})\|^2_{H^1(\Sigma_t)} \lesssim \mathcal{Q}_2(0) + \int_0^t \left\{\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^\infty(\Sigma_\tau)} + 1\right\} \mathcal{Q}_2(\tau)\, d\tau. \tag{67}$$

Using a similar argument based on (54) with $\partial\vec{\mathcal{C}}$ and $\partial\mathcal{D}$ in the role of φ and Eq. (60), and using the elliptic estimate (65) to control the norm $\|\cdot\|_{L^2(\Sigma_t)}$ of the (linear) factor of $\partial^2(\vec{\Omega}, \vec{S})$ on RHS (60), we deduce

$$\|\partial(\vec{\mathcal{C}}, \mathcal{D})\|^2_{L^2(\Sigma_t)} \lesssim \mathcal{Q}_2(0) + \int_0^t \left\{\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^\infty(\Sigma_\tau)} + \|\partial(\vec{\Omega}, \vec{S})\|_{L^\infty(\Sigma_\tau)} + 1\right\} \mathcal{Q}_2(\tau)\, d\tau. \tag{68}$$

Adding (66), (67), and (68), we conclude, in view of definition (61), the desired bound (62). □

5 Energy and elliptic estimates along constant-time hypersurfaces up to top-order

Our main goal in this section is to use the bootstrap assumptions to prove energy and elliptic estimates along Σ_t up to top-order. The main result is Proposition 5.1, which we prove in Sect. 5.3 after providing some preliminary technical estimates.

Proposition 5.1 (Energy and elliptic estimates up to top-order). *There exists a continuous strictly increasing function $F : [0, \infty) \to [0, \infty)$ such that under the initial data and bootstrap assumptions of Sect. 3, the following estimate holds for $t \in [0, T_*]$:*

$$\begin{aligned}\sum_{k=0}^{2} \|\partial_t^k(\rho, \vec{v}, \vec{\Omega})\|_{H^{N-k}(\Sigma_t)} + \sum_{k=0}^{2} \|\partial_t^k s\|_{H^{N+1-k}(\Sigma_t)} + \sum_{k=0}^{1} \|\partial_t^k(\vec{\mathcal{C}}, \mathcal{D})\|_{H^{N-1-k}(\Sigma_t)} \\ \leq F\left(\|(\rho, \vec{v}, \vec{\Omega})\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)}\right).\end{aligned} \tag{69}$$

5.1 Equations satisfied by the frequency-projected solution variables

In proving Proposition 5.1, we will derive energy and elliptic estimates for projections of the solution variables onto dyadic frequencies $\nu \in 2^{\mathbb{N}}$. In the next lemma, as a preliminary step in deriving these estimates, we derive the equations satisfied by the frequency-projected solution variables.

Lemma 5.2 (Equations satisfied by the frequency-projected solution variables). *Let $\nu \in 2^{\mathbb{N}}$. For solutions to the equations of Proposition 1.1, the following equations hold, where $\mathbf{g} = \mathbf{g}(\vec{\Psi})$, $\Psi \in \{\rho, v^1, v^2, v^3, s\}$, and the terms $\mathfrak{F}_{(\Psi)}, \cdots, \mathfrak{F}_{(\mathcal{D})}$ on RHSs (73a)–(75) are defined in Proposition 1.1:*

$$\hat{\square}_{\mathbf{g}} P_\nu \Psi = \hat{\mathfrak{R}}_{(\Psi);\nu}, \tag{70a}$$
$$\square_{\mathbf{g}} P_\nu \Psi = \mathfrak{R}_{(\Psi);\nu}, \tag{70b}$$

$$\mathit{div} P_\nu \Omega = \mathfrak{R}_{(\mathit{div}\Omega);\nu}, \tag{71a}$$
$$\mathbf{B} P_\nu \mathcal{C}^i = \mathfrak{R}_{(\mathcal{C}^i);\nu}, \tag{71b}$$
$$\mathbf{B} P_\nu \mathcal{D} = \mathfrak{R}_{(\mathcal{D});\nu}, \tag{72a}$$
$$(\mathit{curl} P_\nu S)^i = 0, \tag{72b}$$

where the inhomogeneous terms take the following form:

$$\hat{\mathfrak{R}}_{(\Psi);\nu} = P_\nu \mathfrak{F}_{(\Psi)} + \sum_{(\alpha,\beta)\neq(0,0)} \left\{(\mathbf{g}^{-1})^{\alpha\beta} - P_{\leq\nu}(\mathbf{g}^{-1})^{\alpha\beta}\right\} P_\nu \partial_\alpha \partial_\beta \Psi$$
$$+ \sum_{(\alpha,\beta)\neq(0,0)} \left\{\left(P_{\leq\nu}(\mathbf{g}^{-1})^{\alpha\beta}\right) P_\nu \partial_\alpha \partial_\beta \Psi - P_\nu\left[(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta\Psi\right]\right\}, \tag{73a}$$
$$\mathfrak{R}_{(\Psi);\nu} = \hat{\mathfrak{R}}_{(\Psi);\nu} - \mathbf{\Gamma}^\alpha P_\nu \partial_\alpha \Psi, \tag{73b}$$

$\mathbf{\Gamma}^\alpha = (\mathbf{g}^{-1})^{\alpha\beta}(\mathbf{g}^{-1})^{\gamma\delta}\partial_\gamma \mathbf{g}_{\beta\delta} - \frac{1}{2}(\mathbf{g}^{-1})^{\alpha\beta}(\mathbf{g}^{-1})^{\gamma\delta}\partial_\beta \mathbf{g}_{\gamma\delta} = \mathscr{L}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}]$ are the contracted Cartesian Christoffel symbols of $\mathbf{g}(\vec{\Psi})$, and

$$\mathfrak{R}_{(\mathit{div}\Omega);\nu} = P_\nu \mathfrak{F}_{(\mathit{div}\Omega)}, \tag{74a}$$
$$\mathfrak{R}_{(\mathcal{C}^i);\nu} = P_\nu \mathfrak{F}_{(\mathcal{C}^i)} + \left\{v^a - P_{\leq\nu} v^a\right\} P_\nu \partial_a \mathcal{C}^i + \left\{(P_{\leq\nu} v^a) P_\nu \partial_a \mathcal{C}^i - P_\nu[v^a \partial_a \mathcal{C}^i]\right\}, \tag{74b}$$

$$\mathfrak{R}_{(\mathcal{D});\nu} = P_\nu \mathfrak{F}_{(\mathcal{D})} + \left\{v^a - P_{\leq\nu} v^a\right\} P_\nu \partial_a \mathcal{D} + \left\{(P_{\leq\nu} v^a) P_\nu \partial_a \mathcal{D} - P_\nu[v^a \partial_a \mathcal{D}]\right\}. \tag{75}$$

Moreover,

$$\square_{\mathbf{g}} P_{\vee} \boldsymbol{\partial} \Psi = \mathfrak{R}_{(\boldsymbol{\partial}\Psi);\vee}, \quad (76a)$$

$$\mathbf{B} P_{\vee} \boldsymbol{\partial} \mathcal{C}^i = \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^i);\vee}, \quad (76b)$$

$$\mathbf{B} P_{\vee} \boldsymbol{\partial} \mathcal{D} = \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\vee}, \quad (76c)$$

where

$$\begin{aligned} \mathfrak{R}_{(\boldsymbol{\partial}\Psi);\vee} = & P_{\vee} \boldsymbol{\partial} \mathfrak{F}_{(\Psi)} - \sum_{(\alpha,\beta)\neq(0,0)} P_{\vee} \left\{ \left(\boldsymbol{\partial} (\mathbf{g}^{-1})^{\alpha\beta} \right) \partial_\alpha \partial_\beta \Psi \right\} - \boldsymbol{\Gamma}^\alpha P_{\vee} \partial_\alpha \boldsymbol{\partial} \Psi \\ & + \sum_{(\alpha,\beta)\neq(0,0)} \left\{ (\mathbf{g}^{-1})^{\alpha\beta} - P_{\leq \vee} (\mathbf{g}^{-1})^{\alpha\beta} \right\} P_{\vee} \partial_\alpha \partial_\beta \boldsymbol{\partial} \Psi \\ & + \sum_{(\alpha,\beta)\neq(0,0)} \left\{ \left(P_{\leq \vee} (\mathbf{g}^{-1})^{\alpha\beta} \right) P_{\vee} \partial_\alpha \partial_\beta \boldsymbol{\partial} \Psi - P_{\vee} \left[(\mathbf{g}^{-1})^{\alpha\beta} \partial_\alpha \partial_\beta \boldsymbol{\partial} \Psi \right] \right\}, \end{aligned} \quad (77a)$$

$$\begin{aligned} \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^i);\vee} = & P_{\vee} \boldsymbol{\partial} \mathfrak{F}_{(\mathcal{C}^i)} - P_{\vee} [(\boldsymbol{\partial} v^a) \partial_a \mathcal{C}^i] \\ & + \left\{ v^a - P_{\leq \vee} v^a \right\} P_{\vee} \partial_a \boldsymbol{\partial} \mathcal{C}^i + \left\{ (P_{\leq \vee} v^a) P_{\vee} \partial_a \boldsymbol{\partial} \mathcal{C}^i - P_{\vee} [v^a \partial_a \boldsymbol{\partial} \mathcal{C}^i] \right\}, \end{aligned} \quad (77b)$$

$$\begin{aligned} \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\vee} = & P_{\vee} \boldsymbol{\partial} \mathfrak{F}_{(\mathcal{D})} - P_{\vee} [(\boldsymbol{\partial} v^a) \partial_a \mathcal{D}] \\ & + \left\{ v^a - P_{\leq \vee} v^a \right\} P_{\vee} \partial_a \boldsymbol{\partial} \mathcal{D} + \left\{ (P_{\leq \vee} v^a) P_{\vee} \partial_a \boldsymbol{\partial} \mathcal{D} - P_{\vee} [v^a \partial_a \boldsymbol{\partial} \mathcal{D}] \right\}. \end{aligned} \quad (77c)$$

Proof The lemma follows from straightforward computations and the fact that $\square_{\mathbf{g}} \varphi = \hat{\square}_{\mathbf{g}} \varphi - \boldsymbol{\Gamma}^\alpha \partial_\alpha \varphi$ for scalar functions φ. We therefore omit the details. □

5.2 Product and commutator estimates

In this subsection, we derive estimates for various norms of the inhomogeneous terms $\hat{\mathfrak{R}}_{(\Psi);\vee}, \cdots, \mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\vee}$ on RHSs (73a)–(75). We provide the main result in Lemma 5.4.

5.2.1 Preliminary product and commutator estimates

In the next lemma, we provide some standard product and commutator estimates that are based on the Littlewood–Paley calculus.

Lemma 5.3 (Preliminary product and commutator estimates). *The following estimates hold, where we assume that F, G_i, and φ are (possibly array-valued) functions on Σ_t, that f is a smooth function of its arguments, and that f' denotes the derivative of f with respect to its arguments.*
Product estimates: *For any ε such that $0 < \varepsilon < 1$ (in our forthcoming applications, we will set $\varepsilon := N - 2$), the following product estimates hold, where the implicit*

constants are allowed to depend on ε, $\|\mathrm{f}\circ\varphi\|_{L^\infty(\Sigma_t)}$, *and* $\|\mathrm{f}'\circ\varphi\|_{L^\infty(\Sigma_t)}$, *and the projection operators* P_ν *on the RHSs of the estimates are allowed to correspond to a slightly different projection operator, localized at the same frequency, than the ones on the LHSs:*

$$\|\Lambda^{1+\varepsilon}F\|_{L^2(\Sigma_t)} \approx \|\Lambda^\varepsilon \partial F\|_{L^2(\Sigma_t)}, \tag{78}$$

$$\|G_1\cdot G_2\|_{L^2(\Sigma_t)} \lesssim \|G_1\|_{H^1(\Sigma_t)}\|G_2\|_{H^1(\Sigma_t)}, \tag{79a}$$

$$\|G_1\cdot G_2\|_{L^2(\Sigma_t)} \lesssim \|G_1\|^{1/2}_{L^2(\Sigma_t)}\|G_1\|^{1/2}_{H^1(\Sigma_t)}\|G_2\|_{H^1(\Sigma_t)}, \tag{79b}$$

$$\|G_1\cdot G_2\cdot G_3\|_{L^2(\Sigma_t)} \lesssim \|G_1\|_{H^1(\Sigma_t)}\|G_2\|_{H^1(\Sigma_t)}\|G_3\|_{H^1(\Sigma_t)}. \tag{79c}$$

In addition, for dyadic frequencies $\nu \geq 1$, *we have:*

$$\|P_\nu(\mathrm{f}\circ\varphi\cdot G)\|_{L^\infty(\Sigma_t)} \lesssim \nu^{-1/2}\|\partial\varphi\|_{L^\infty(\Sigma_t)}\|G\|_{H^1(\Sigma_t)} + \|P_\nu G\|_{L^\infty(\Sigma_t)}. \tag{80}$$

Moreover,

$$\|\Lambda^\varepsilon(\mathrm{f}\circ\varphi\cdot G)\|_{L^2(\Sigma_t)} \lesssim \|\Lambda^\varepsilon G\|_{L^2(\Sigma_t)} + \|\partial\varphi\|_{H^1(\Sigma_t)}\|G\|_{H^\varepsilon(\Sigma_t)}, \tag{81a}$$

$$\|\Lambda^\varepsilon(F\cdot G)\|_{L^2(\Sigma_t)} \lesssim \|F\|_{H^{1/2+\varepsilon}(\Sigma_t)}\|G\|_{H^1(\Sigma_t)} + \|G\|_{H^{1/2+\varepsilon}(\Sigma_t)}\|F\|_{H^1(\Sigma_t)}, \tag{81b}$$

$$\|\Lambda^\varepsilon(F\cdot\partial G)\|_{L^2(\Sigma_t)} \lesssim \|F\|_{L^\infty(\Sigma_t)}\|\partial G\|_{H^\varepsilon(\Sigma_t)} + \|G\|_{L^\infty(\Sigma_t)}\|\partial F\|_{H^\varepsilon(\Sigma_t)}, \tag{81c}$$

$$\|\Lambda^\varepsilon(G_1\cdot G_2\cdot G_3)\|_{L^2(\Sigma_t)} \lesssim \sum_{j=1}^{3}\|G_j\|_{H^{1+\varepsilon}(\Sigma_t)}\prod_{k\neq j}\|G_k\|_{H^1(\Sigma_t)}. \tag{81d}$$

<u>Commutator estimates</u>*: The following commutator estimates hold for dyadic frequencies* $\nu \geq 1$*:*

$$\|[\mathrm{f}\circ\varphi - P_{\leq\nu}(\mathrm{f}\circ\varphi)]\cdot P_\nu G\|_{L^2(\Sigma_t)} \lesssim \nu^{-1}\|\partial\varphi\|_{L^\infty(\Sigma_t)}\|P_\nu G\|_{L^2(\Sigma_t)}, \tag{82a}$$

$$\begin{aligned}\|P_\nu[\mathrm{f}\circ\varphi\cdot\partial G] - P_{\leq\nu}(\mathrm{f}\circ\varphi)\cdot P_\nu\partial G\|_{L^2(\Sigma_t)} &\lesssim \|\partial\varphi\|_{L^\infty(\Sigma_t)}\|P_\nu G\|_{L^2(\Sigma_t)} \\ &\quad + \|G\|_{L^\infty(\Sigma_t)}\|P_\nu[\mathrm{f}'\circ\varphi\cdot\partial\varphi]\|_{L^2(\Sigma_t)} \\ &\quad + \sum_{\lambda>\nu}\lambda^{-1}\|\partial\varphi\|_{L^\infty(\Sigma_t)}\|P_\lambda\partial G\|_{L^2(\Sigma_t)}.\end{aligned} \tag{82b}$$

Convolution-type estimate for dyadic-indexed sums: If $\{A_\lambda\}_{\lambda \in 2^{\mathbb{N}}}$ is a dyadic-indexed sequence of real numbers, then

$$\left\| \nu^{1+\varepsilon} \sum_{\lambda > \nu} \lambda^{-1} A_\lambda \right\|_{\ell^2_\nu} \lesssim \| \nu^\varepsilon A_\nu \|_{\ell^2_\nu} . \tag{83}$$

Proof (78) is a basic result in harmonic analysis; see, e.g., [2, Chapter 2]. (81b) is proved in [53, Lemma 17]. (81c) follows from the proof of [53, Lemma 19], which yielded a similar estimate, differing only in the following minor fashion: the terms $\|\partial G\|_{H^\varepsilon(\Sigma_t)}$ and $\|\partial F\|_{H^\varepsilon(\Sigma_t)}$ on the right-hand side were replaced, respectively, with $\|G\|_{H^{1+\varepsilon}(\Sigma_t)}$ and $\|F\|_{H^{1+\varepsilon}(\Sigma_t)}$. (81d) is proved as [53, Lemma 18]. (80) follows from the proof of [54, Equation (8.2)] and the standard Sobolev embedding estimate $\|G\|_{L^6(\Sigma_t)} \lesssim \|G\|_{H^1(\Sigma_t)}$. (79c) follows from the Hölder estimate $\|G_1 \cdot G_2 \cdot G_3\|_{L^2(\Sigma_t)} \le \|G_1\|_{L^6(\Sigma_t)} \|G_2\|_{L^6(\Sigma_t)} \|G_3\|_{L^6(\Sigma_t)}$ and the Sobolev embedding estimate $\|G_i\|_{L^6(\Sigma_t)} \lesssim \|G_i\|_{H^1(\Sigma_t)}$, while (79a) follows from the Hölder estimate $\|G_1 \cdot G_2\|_{L^2(\Sigma_t)} \le \|G_1\|_{L^4(\Sigma_t)} \|G_2\|_{L^4(\Sigma_t)}$ and the Sobolev embedding estimate $\|G_i\|_{L^4(\Sigma_t)} \lesssim \|G_i\|_{H^1(\Sigma_t)}$. Similarly (79b), follows from the Hölder estimate $\|G_1 \cdot G_2\|_{L^2(\Sigma_t)} \le \|G_1\|_{L^3(\Sigma_t)} \|G_2\|_{L^6(\Sigma_t)}$, the Sobolev embedding estimate $\|G_2\|_{L^6(\Sigma_t)} \lesssim \|G_2\|_{H^1(\Sigma_t)}$, and the Sobolev interpolation estimate $\|G_1\|_{L^3(\Sigma_t)} \lesssim \|G_1\|_{L^2(\Sigma_t)}^{1/2} \|G_1\|_{H^1(\Sigma_t)}^{1/2}$. With the help of the Sobolev embedding result $\|\partial\varphi\|_{L^6(\Sigma_t)} \lesssim \|\partial\varphi\|_{H^1(\Sigma_t)}$, the estimate (81a) follows from a straightforward adaptation of the proof of [54, Equation (8.1)], which provided a similar estimate in the case $0 < \varepsilon < 1/2$. The estimates (82a) and (82b) follow from the proof of [54, Lemma 2.4]. To obtain (83), we first observe that $\nu^{1+\varepsilon} \sum_{\lambda > \nu} \lambda^{-1} A_\lambda = \sum_{\lambda > \nu} \left(\frac{\lambda}{\nu}\right)^{-(1+\varepsilon)} \lambda^\varepsilon A_\lambda = (\widetilde{A} * B)_\nu$, where $\widetilde{A}$ denotes the dyadic sequence $\widetilde{A}_\lambda := \lambda^\varepsilon A_\lambda$, B denotes the dyadic sequence $B_\lambda := \mathbf{1}_{[1,\infty)}(\lambda)\lambda^{-(1+\varepsilon)}$, $\mathbf{1}_{[1,\infty)}(\lambda)$ denotes the characteristic function of the dyadic interval $[1, \infty)$, and $(\widetilde{A} * B)_\nu$ denotes the convolution of $\widetilde{A}$ and B, viewed as a function of ν. Thus, from Young's $L^2 * L^1 \to L^2$ convolution inequality and the bound $\|B_\lambda\|_{\ell^1_\lambda} \lesssim 1$, we deduce that $\|\widetilde{A} * B\|_{\ell^2_\nu} \lesssim \|\widetilde{A}_\nu\|_{\ell^2_\nu}$, which is the desired bound. □

5.2.2 Product and commutator estimates estimates for the compressible Euler equations

In the next lemma, we derive bounds that are sufficient for controlling the error terms in the top-order energy-elliptic estimates of Proposition 5.1 and the top-order energy estimates along null hypersurfaces of Proposition 6.1.

Lemma 5.4 (Product and commutator estimates estimates for the compressible Euler equations) *Under the bootstrap assumptions of Sect. 3 and the $H^2(\Sigma_t)$ energy estimates of Proposition 4.1, for solutions to the equations of Proposition 1.1, the inhomogeneous terms from the equations of Lemma 5.2 verify the following estimates for $t \in [0, T_*]$, where the implicit constants are allowed to depend in a continuous increasing fashion on the data norms $\|(\rho, \vec{v}, \vec{\Omega})\|_{H^N(\Sigma_0)} + \|s\|_{H^{N+1}(\Sigma_0)}$.*

Frequency-summed control of the inhomogeneous terms: *The following estimates hold, where in* $\|\cdot\|_{\ell^2_\nu L^2(\Sigma_t)}$, *the* ℓ^2_ν*-seminorm is taken over dyadic frequencies* $\nu \geq 1$*:*

$$
\begin{aligned}
&\|\nu^{N-1}\hat{\mathfrak{R}}_{(\Psi);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}, \ \|\nu^{N-1}\mathfrak{R}_{(\Psi);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)},\\
&\|\nu^{N-2}\partial\hat{\mathfrak{R}}_{(\Psi);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}, \ \|\nu^{N-2}\partial\mathfrak{R}_{(\Psi);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}, \ \|\nu^{N-2}\mathfrak{R}_{(\boldsymbol{\partial}\Psi);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}\\
&\lesssim \|\partial(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_t)} + \left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+1\right\}\|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+1,
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
&\|\nu^{N-1}\mathfrak{R}_{(\mathcal{C}^i);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}, \ \|\nu^{N-1}\mathfrak{R}_{(\mathcal{D});\nu}\|_{\ell^2_\nu L^2(\Sigma_t)},\\
&\|\nu^{N-2}\partial\mathfrak{R}_{(\mathcal{C}^i);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}, \ \|\nu^{N-2}\partial\mathfrak{R}_{(\mathcal{D});\nu}\|_{\ell^2_\nu L^2(\Sigma_t)},\\
&\|\nu^{N-2}\mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^i);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}, \ \|\nu^{N-2}\mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}\\
&\lesssim \left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+1\right\}\|\partial(\vec{\Omega},\vec{S})\|_{H^{N-1}(\Sigma_t)}\\
&\quad + \left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)} + \|\partial(\vec{\Omega},\vec{S})\|_{L^\infty(\Sigma_t)}+1\right\}\|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}\\
&\quad + \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+1,
\end{aligned} \tag{85}
$$

$$
\|\nu^{N-1}\mathfrak{R}_{(div\Omega);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}, \ \|\nu^{N-2}\partial\mathfrak{R}_{(div\Omega);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}+1. \tag{86}
$$

Control of $curl\Omega$ ***and*** $divS$ ***in terms of the modified fluid variables***: *The following estimates hold, where the modified fluid variables* $\mathcal{C}$ *and* $\mathcal{D}$ *are as in Definition* 1.2*:*

$$
\|\Lambda^{N-1}curl\Omega\|_{L^2(\Sigma_t)} \lesssim \|\partial\vec{\mathcal{C}}\|_{H^{N-2}(\Sigma_t)} + \|\partial\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}+1, \tag{87a}
$$

$$
\|\Lambda^{N-1}divS\|_{L^2(\Sigma_t)} \lesssim \|\partial\mathcal{D}\|_{H^{N-2}(\Sigma_t)} + \|\partial\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}+1. \tag{87b}
$$

Proof All of these estimates are standard consequences of Lemma 5.3 and we therefore prove only one representative estimate; we refer to [54, Lemmas 2.2, 2.3, 2.4, and 2.7] for the proof of very similar estimates. Specifically, we will prove (84). Throughout the proof, we use the convention for implicit constants stated in the lemma. We will silently use our bootstrap assumption that the compressible Euler solution is contained in $\mathfrak{K}$ (i.e., (40)). We will also silently use the estimate (78), the estimates of Proposition 4.1, and simple estimates of the type $\|\vec{\Psi}\|_{H^N(\Sigma_t)} \lesssim \|\vec{\Psi}\|_{H^2(\Sigma_t)} + \|\partial\vec{\Psi}\|_{H^{N-1}(\Sigma_t)} \lesssim 1 + \|\partial\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}$, the point being that by Proposition 4.1, we have already shown that $\|\vec{\Psi}\|_{H^2(\Sigma_t)} \lesssim 1$ (and similarly for the variables $\vec{\Omega}$ and $\vec{S}$).

In proving (84), we will show only how to obtain the desired bound for the term $\|\nu^{N-1}\mathfrak{R}_{(\Psi);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}$; the remaining terms on LHS (84) can be bounded using nearly identical arguments. To proceed, we start by bounding the first term $P_\nu\mathfrak{F}_{(\Psi)}$ on RHS (73a). That is, we must bound $\|\nu^{N-1}P_\nu \text{RHS } (13)\|_{\ell^2_\nu L^2(\Sigma_t)}$. We begin by bounding the first product on RHS (13), which is of the form $\mathrm{f}(\vec{\Psi})(\vec{\mathcal{C}},\mathcal{D})$. Repeatedly using the product estimates of Lemma 5.3 and appealing to Definition 1.2, we deduce (where throughout, we allow f to vary from line to line, in particular denoting the derivatives of f also by f), with $\mathscr{P}$ a polynomial with bounded coefficients that is allowed to vary from line to line, that

$$\begin{aligned}
&\|\Lambda^{N-1}[\mathrm{f}(\vec{\Psi})(\vec{\mathcal{C}},\mathcal{D})]\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \|\Lambda^{N-2}\partial[\mathrm{f}(\vec{\Psi})(\vec{\mathcal{C}},\mathcal{D})]\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \|\Lambda^{N-2}[\mathrm{f}(\vec{\Psi})\partial(\vec{\mathcal{C}},\mathcal{D})]\|_{L^2(\Sigma_t)} + \|\Lambda^{N-2}[\mathrm{f}(\vec{\Psi})\partial\vec{\Psi}\cdot(\vec{\mathcal{C}},\mathcal{D})]\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \left\{\|\partial(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_t)}+1\right\}\mathscr{P}\left(\|\partial\vec{\Psi}\|_{H^1(\Sigma_t)},\|(\vec{\mathcal{C}},\mathcal{D})\|_{H^1(\Sigma_t)}\right) \\
&\quad\lesssim \|\partial(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_t)}+1
\end{aligned} \tag{88}$$

as desired. The second product on RHS (13) is of the form $\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}$. Thus, using the product estimates of Lemma 5.3 and the bound $\|\vec{\Psi}\|_{H^2(\Sigma_t)}\lesssim 1$, we deduce that

$$\begin{aligned}
&\|\Lambda^{N-1}[\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}]\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \|\Lambda^{N-2}\partial[\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}]\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \|\Lambda^{N-2}[\mathrm{f}(\vec{\Psi})\boldsymbol{\partial}\vec{\Psi}\cdot\partial\boldsymbol{\partial}\vec{\Psi}]\|_{L^2(\Sigma_t)} + \|\Lambda^{N-2}[\mathrm{f}(\vec{\Psi})\partial\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}]\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \|\mathrm{f}(\vec{\Psi})\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}\|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}\|\partial[\mathrm{f}(\vec{\Psi})\boldsymbol{\partial}\vec{\Psi}]\|_{H^{N-2}(\Sigma_t)} \\
&\qquad + \|\mathrm{f}(\vec{\Psi})\partial\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}\|\boldsymbol{\partial}\vec{\Psi}\|^2_{H^1(\Sigma_t)} \\
&\qquad + \|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}\|\mathrm{f}(\vec{\Psi})\partial\vec{\Psi}\|_{H^1(\Sigma_t)}\|\boldsymbol{\partial}\vec{\Psi}\|_{H^1(\Sigma_t)} \\
&\quad\lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}\left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+\mathscr{P}\left(\|\boldsymbol{\partial}\vec{\Psi}\|_{H^1(\Sigma_t)}\right)\right\} \\
&\qquad + \left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+1\right\}\mathscr{P}\left(\|\boldsymbol{\partial}\vec{\Psi}\|_{H^1(\Sigma_t)}\right) \\
&\quad\lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}\left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+1\right\} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}+1
\end{aligned} \tag{89}$$

as desired. It remains for us to bound the two sums on RHS (73a) in the norm $\|\nu^{N-1}\cdot\|_{\ell^2_\nu L^2(\Sigma_t)}$. To handle the first sum, we use (82a) with $\vec{\Psi}$ in the role of φ and $\partial\boldsymbol{\partial}\vec{\Psi}$ in the role of G to deduce

$$\begin{aligned}
&\sum_{(\alpha,\beta)\neq(0,0)}\|\nu^{N-1}\left\{(\mathbf{g}^{-1})^{\alpha\beta}-P_{\leq\nu}(\mathbf{g}^{-1})^{\alpha\beta}\right\}P_\nu\partial_\alpha\partial_\beta\Psi\|_{\ell^2_\nu L^2(\Sigma_t)} \\
&\quad\lesssim \|\partial\vec{\Psi}\|_{L^\infty(\Sigma_t)}\|\Lambda^{N-2}\partial\boldsymbol{\partial}\vec{\Psi}\|_{L^2(\Sigma_t)} \\
&\quad\lesssim \|\partial\vec{\Psi}\|_{L^\infty(\Sigma_t)}\|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}
\end{aligned} \tag{90}$$

as desired. To bound the last sum on RHS (73a) in the norm $\|\nu^{N-1}\cdot\|_{\ell^2_\nu L^2(\Sigma_t)}$, we use (82b) with $\vec{\Psi}$ in the role of φ and $\boldsymbol{\partial}\vec{\Psi}$ in the role of G, the bound $\|\Lambda^{N-1}[\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}]\|_{L^2(\Sigma_t)}\lesssim\|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)}+1$ (which follows from the product estimates of Lemma 5.3 and the bound $\|\vec{\Psi}\|_{H^2(\Sigma_t)}\lesssim 1$), and the convolution estimate (83) with $\|P_\lambda\partial\boldsymbol{\partial}\vec{\Psi}\|_{L^2(\Sigma_t)}$ in the role of A_λ to deduce

$$\begin{aligned}
&\sum_{(\alpha,\beta)\neq(0,0)} \|\nu^{N-1}\left\{\left(P_{\leq \nu}(\mathbf{g}^{-1})^{\alpha\beta}\right) P_\nu \partial_\alpha\partial_\beta \Psi - P_\nu\left[(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta\Psi\right]\right\}\|_{\ell^2_\nu L^2(\Sigma_t)} \\
&\quad\lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}\|\Lambda^{N-1}[\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}]\|_{L^2(\Sigma_t)} \\
&\qquad + \|\partial\vec{\Psi}\|_{L^\infty(\Sigma_t)}\left\|\nu^{N-1}\sum\nolimits_{\lambda>\nu}\lambda^{-1}\|P_\lambda\partial\boldsymbol{\partial}\vec{\Psi}\|_{L^2(\Sigma_t)}\right\|_{\ell^2_\nu} \\
&\quad\lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}\|\boldsymbol{\partial}\vec{\Psi}\|_{H^{N-1}(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_t)}
\end{aligned} \tag{91}$$

as desired.

The remaining estimates in the lemma can be proved using similar arguments, and we omit the details. We clarify that **i)** to derive some of the estimates in their stated form, one must use Definition 1.2 to express $\vec{\mathcal{C}}$ and $\mathcal{D}$ in terms of the other solution variables and **ii)** in order to bound the term $\|\nu^{N-2}\mathfrak{R}_{(\boldsymbol{\partial}\Psi);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}$ on LHS (84) and the terms $\|\nu^{N-2}\mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^i);\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}$ and $\|\nu^{N-2}\mathfrak{R}_{(\boldsymbol{\partial}\mathcal{D});\nu}\|_{\ell^2_\nu L^2(\Sigma_t)}$ on LHS (85) using arguments of the type given above, one must derive Sobolev estimates for products featuring the time-derivative-involving terms $\partial_t^2\vec{\Psi}$, $\partial_t\vec{\mathcal{C}}$, $\partial_t\mathcal{D}$, $\partial_t\vec{\Omega}$, and $\partial_t\vec{S}$. These time-derivative-involving terms can be handled by first using the equations of Proposition 1.1 to solve for the relevant time derivatives in terms of spatial derivatives and then using the estimates of Lemma 5.3, as we did above. □

5.3 Proof of Proposition 5.1

Throughout the proof, we rely on the remarks made in the first paragraph of the proof of Lemma 5.4. In particular, we silently use the already proven below-top-order estimates (42). Moreover, we use the convention that our implicit constants are allowed to depend on functions F of the norms of the data of the type stated on RHS (69); in particular, we consider such functions of the norms of the data to be bounded by $\lesssim 1$. Finally, whenever convenient, we consider factors of t to be bounded by $\lesssim 1$.

We first note that, for the same reasons stated at the beginning of the proof of Proposition 4.1, the estimates for the terms $\|\partial_t^2(\rho,\vec{v})\|_{H^{N-2}(\Sigma_t)}$, $\sum_{k=1}^2\|\partial_t^k\vec{\Omega}\|_{H^{N-k}(\Sigma_t)}$, $\sum_{k=1}^2\|\partial_t^k s\|_{H^{N+1-k}(\Sigma_t)}$, and $\|\partial_t(\vec{\mathcal{C}},\mathcal{D})\|_{H^{N-2}(\Sigma_t)}$ on LHS (69) follow from straightforward arguments once we have obtained the desired estimates for the remaining terms on LHS (69); we therefore omit the details for bounding these terms.

To prove the desired estimates for the remaining terms on LHS (69), we will derive energy and elliptic estimates for the solution variables at fixed frequency, which satisfy the equations of Lemma 5.2. After summing over dyadic frequencies, this will allow us to obtain estimates for the "controlling quantity" $Q_N(t)$ defined by

$$Q_N(t) := \|\boldsymbol{\partial}\vec{\Psi}\|^2_{H^{N-1}(\Sigma_t)} + \|\partial(\vec{\mathcal{C}},\mathcal{D})\|^2_{H^{N-2}(\Sigma_t)}. \tag{92}$$

Our assumptions on the initial data imply that $Q_N(0)\lesssim 1$, and we will use this fact throughout the proof.

The main steps in deriving a bound for $Q_N(t)$ are proving the following two bounds:

$$Q_N(t) \lesssim 1 + \int_0^t \left\{ \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)} + \|\partial(\vec{\Omega}, \vec{S})\|_{L^\infty(\Sigma_\tau)} + 1 \right\} Q_N(\tau)\, d\tau, \tag{93}$$

$$\|\partial(\vec{\Omega}, \vec{S})\|^2_{H^{N-1}(\Sigma_t)} \lesssim Q_N(t) + 1. \tag{94}$$

Then from the bootstrap assumptions (41a)–(41b), (93), and Grönwall's inequality, we deduce that for $t \in [0, T_*]$, we have $Q_N(t) \lesssim 1$. From this estimate, (94), and the below-top-order energy estimates (42), we conclude, in view of the remarks made above, the desired bound (69).

It remains for us to prove (93) and (94). To prove (94), we first use the elliptic identity (55) with $P_\nu\vec{\Omega}$ and $P_\nu\vec{S}$ in the role of V and equations (71a) and (72b) to deduce, after multiplying by $\nu^{2(N-1)}$ and summing over $\nu \geq 1$, that

$$\|\Lambda^{N-1}\partial(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} = \|\nu^{N-1}\mathfrak{R}_{(\mathrm{div}\Omega);\nu}\|^2_{\ell^2_\nu L^2(\Sigma_t)} + \|\Lambda^{N-1}(\mathrm{curl}\Omega, \mathrm{div}S)\|^2_{L^2(\Sigma_t)}. \tag{95}$$

Using (86), (87a), and (87b), and appealing to definition (92), we find that RHS (95) $\lesssim$ RHS (94). Also using Proposition 4.1 to deduce that $\|P_{\leq 1}\partial(\vec{\Omega}, \vec{S})\|^2_{L^2(\Sigma_t)} \lesssim 1$, we conclude the desired estimate (94).

We now derive energy estimates for the evolution equations. To proceed, we first use equation (70a) and (52) with $P_\nu\vec{\Psi}$ in the role of φ to deduce that

$$\begin{aligned}\|(P_\nu\vec{\Psi}, P_\nu\partial_t\vec{\Psi})\|^2_{H^1(\Sigma_t)\times L^2(\Sigma_t)} &\lesssim \|(P_\nu\vec{\Psi}, P_\nu\partial_t\vec{\Psi})\|^2_{H^1(\Sigma_0)\times L^2(\Sigma_0)} \\ &\quad + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\|(P_\nu\vec{\Psi}, P_\nu\partial_t\vec{\Psi})\|^2_{H^1(\Sigma_\tau)\times L^2(\Sigma_\tau)}\, d\tau \\ &\quad + \sum_{\iota=0}^{4}\int_0^t \|\hat{\mathfrak{R}}_{(\Psi_\iota);\nu}\|_{L^2(\Sigma_\tau)}\|\boldsymbol{\partial} P_\nu\vec{\Psi}\|_{L^2(\Sigma_\tau)}\, d\tau.\end{aligned} \tag{96}$$

Multiplying (96) by $\nu^{2(N-1)}$, summing over dyadic frequencies $\nu \geq 1$, using the Cauchy–Schwarz inequality for ℓ^2_ν, using (84), and using Young's inequality, we deduce, in view of definition (92), that

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{H^{N-1}(\Sigma_t)} \lesssim 1 + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\, d\tau + \int_0^t \left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)} + 1\right\} Q_N(\tau)\, d\tau. \tag{97}$$

Similarly, using Eqs. (71b), (72a), and (54) with $P_\nu\vec{\mathcal{C}}$ and $P_\nu\mathcal{D}$ in the role of φ, we deduce that

$$\begin{aligned}\|(P_\nu\vec{\mathcal{C}}, P_\nu\mathcal{D})\|^2_{L^2(\Sigma_t)} \lesssim \|(P_\nu\vec{\mathcal{C}}, P_\nu\mathcal{D})\|^2_{L^2(\Sigma_0)} \\ + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\|(P_\nu\vec{\mathcal{C}}, P_\nu\mathcal{D})\|^2_{L^2(\Sigma_\tau)}\, d\tau \\ + \sum_{i=1}^3 \int_0^t \|P_\nu\mathcal{C}^i\|_{L^2(\Sigma_\tau)}\|\mathfrak{R}_{(\mathcal{C}^i);\nu}\|_{L^2(\Sigma_\tau)}\, d\tau \\ + \int_0^t \|P_\nu\mathcal{D}\|_{L^2(\Sigma_\tau)}\|\mathfrak{R}_{(\mathcal{D});\nu}\|_{L^2(\Sigma_\tau)}\, d\tau.\end{aligned} \tag{98}$$

Multiplying (98) by $\nu^{2(N-1)}$, summing over dyadic frequencies $\nu \geq 1$, using the Cauchy–Schwarz inequality for ℓ^2_ν, using (85), using (94) to bound the factor $\|\partial(\vec{\Omega}, \vec{S})\|_{H^{N-1}(\Sigma_t)}$ on RHS (85), and using Young's inequality, we deduce, in view of definition (92), that

$$\begin{aligned}\|\partial(\vec{\mathcal{C}}, \mathcal{D})\|^2_{H^{N-2}(\Sigma_t)} \lesssim 1 + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\, d\tau \\ + \int_0^t \left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)} + \|\partial(\vec{\Omega}, \vec{S})\|_{L^\infty(\Sigma_\tau)} + 1\right\} Q_N(\tau)\, d\tau.\end{aligned} \tag{99}$$

Finally, adding (97) and (99), and controlling the second term on RHS (99) by using the bootstrap assumption (41a) to infer that $\int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\, d\tau \lesssim 1$, we conclude (93). We have therefore proved the proposition. □

6 Energy estimates along acoustic null hypersurfaces

Our main goal in this section is to derive energy estimates for the fluid variables along acoustic null hypersurfaces (which we sometimes refer to as "**g**-null hypersurfaces" to clarify their tie to the acoustical metric, or simply "null hypersurfaces" for short). We will use these estimates in Sect. 10, when we derive quantitative control of the acoustic geometry (for example, in the proof of Proposition 10.4). Compared to prior works, the main contribution of the present section is the estimate (102), which shows that the modified fluid variables $(\vec{\mathcal{C}}, \mathcal{D})$ can be controlled in L^2 up to top-order along acoustic null hypersurfaces; as we described in point **I** of Sect. 1.2, *such control along acoustic null hypersurfaces is not available for generic top-order derivatives of the vorticity and entropy*.

6.1 Geometric ingredients

We assume that in some subset of $[0, T_*] \times \mathbb{R}^3$ equal to the closure of an open set, U is an acoustical eikonal function. More precisely, we assume that U is a solution to the eikonal equation $(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha U \partial_\beta U = 0$ such that $\partial_t U > 0$ and such that U is smooth and non-degenerate (i.e. $|\boldsymbol{\partial} U| \neq 0$) away from the integral curve of $\mathbf{B}$ emanating from a point $\mathbf{z} \in \Sigma_T$ for some $T \in [0, T_*]$; see Sects. 9.4 and 9.4.1 for discussion of our choice of $\mathbf{z}$ and the integral curve.

In Sect. 9, we will construct a related eikonal function, one that is equivalent to the eikonal functions considered here, differing only in that we work with rescaled solution variables starting in Sect. 9 (see Sect. 9.1 for their definition). We let $l := \frac{-1}{(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha U \partial_\beta t} > 0$ denote the null lapse,[29] and we define $V^\alpha := -l(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta U$. Thus, $\mathbf{g}(V, V) = 0$ and $Vt = 1$. We assume that the hypersurface $\mathcal{N}$ is equal to some portion of a level set of U. Note that V is normal to $\mathcal{N}$ and thus $\mathcal{N}$ is a $\mathbf{g}$-null hypersurface. We define the two-dimensional spacelike surfaces $\mathcal{S}_t := \Sigma_t \cap \mathcal{N}$. We let $g\!\!\!/$ denote the Riemannian metric induced by $\mathbf{g}$ on $\mathcal{S}_t$, we let $\nabla\!\!\!\!/$ denote the corresponding Levi-Civita connection, and we let $d\varpi_{g\!\!\!/}$ denote the volume form on $\mathcal{S}_t$ induced by $g\!\!\!/$.

We now define acoustic null fluxes along $\mathcal{N}$.

Definition 6.1 *(Acoustic null fluxes).* For scalar functions φ defined on $\mathcal{N}$, we define the acoustic null fluxes $\mathbb{F}_{(Wave)}[\varphi; \mathcal{N}]$ and $\mathbb{F}_{(Transport)}[\varphi; \mathcal{N}]$ as follows, where relative to arbitrary coordinates on $\mathcal{S}_t$, $|\nabla\!\!\!\!/\varphi|^2_{g\!\!\!/} := (g\!\!\!/^{-1})^{AB}\nabla\!\!\!\!/_A\varphi\nabla\!\!\!\!/_B\varphi$:

$$\mathbb{F}_{(Wave)}[\varphi; \mathcal{N}] := \int_{\mathcal{N}} \left\{(V\varphi)^2 + |\nabla\!\!\!\!/\varphi|^2_{g\!\!\!/}\right\} d\varpi_{g\!\!\!/}dt, \quad \mathbb{F}_{(Transport)}[\varphi; \mathcal{N}] := \int_{\mathcal{N}} \varphi^2 \, d\varpi_{g\!\!\!/}dt. \tag{100}$$

6.2 Energy estimates along acoustic null hypersurfaces

In this subsection, we establish the main energy estimate for the fluid solution variables along null hypersurfaces. As we mentioned at the start of Sect. 6, the main new ingredient of interest is (102), whose proof relies on the special structure of the equations of Proposition 1.1. In Sect. 10, we will apply Proposition 6.1 along a family of null hypersurfaces that are equal to the level sets of an acoustical eikonal function that we construct in Sect. 9.4 (we denote the acoustical eikonal function by "u" starting in Sect. 9).

Proposition 6.1 (Energy estimates along acoustic null hypersurfaces). *Let $\mathcal{N}$ be any of the null hypersurface portions from Sect. 6.1. Assume that for some pair of times $0 \leq t_I < t_F \leq T_*$, $\mathcal{N}$ and some subsets of Σ_{t_I} and Σ_{t_F} collectively form the boundary a compact subset of $[0, T_*] \times \mathbb{R}^3$. Then under the initial data and bootstrap assumptions of Sect. 3 and the conclusions of Proposition 5.1, the following estimates hold for*

[29] We use the symbol "b" to denote the null lapse of the eikonal function constructed in Sect. 9. Moreover, starting in Sect. 9, we use the symbol "L" to denote the analog of the vectorfield denoted by "V" in the present subsection.

$\Psi \in \{\rho, v^1, v^2, v^3, s\}$*:*

$$\mathbb{F}_{(Wave)}[\boldsymbol{\partial}\Psi; \mathcal{N}] + \sum_{\nu > 1} \nu^{2(N-2)} \mathbb{F}_{(Wave)}[P_\nu \boldsymbol{\partial}\Psi; \mathcal{N}] \lesssim 1. \tag{101}$$

Moreover,

$$\mathbb{F}_{(Transport)}[\boldsymbol{\partial}(\vec{\mathcal{C}}, \mathcal{D}); \mathcal{N}] + \sum_{\nu > 1} \nu^{2(N-2)} \mathbb{F}_{(Transport)}[P_\nu \boldsymbol{\partial}(\vec{\mathcal{C}}, \mathcal{D}); \mathcal{N}] \lesssim 1. \tag{102}$$

Proof We first prove (102) for $\boldsymbol{\partial}\mathcal{C}^i$. We set $\mathbf{J}^\alpha := |\boldsymbol{\partial}\mathcal{C}^i|^2 \mathbf{B}^\alpha$ and compute, relative to the Cartesian coordinates, that $\mathbf{D}_\alpha \mathbf{J}^\alpha = 2(\boldsymbol{\partial}\mathcal{C}^i) \cdot \mathbf{B}\boldsymbol{\partial}\mathcal{C}^i + (\partial_a v^a)|\boldsymbol{\partial}\mathcal{C}^i|^2 + \mathbf{\Gamma}_{\alpha\ \beta}^{\ \alpha} \mathbf{B}^\beta |\boldsymbol{\partial}\mathcal{C}^i|^2$, where $\mathbf{\Gamma}_{\alpha\ \beta}^{\ \gamma} = \frac{1}{2}(\mathbf{g}^{-1})^{\gamma\sigma} \left\{\partial_\alpha \mathbf{g}_{\sigma\beta} + \partial_\beta \mathbf{g}_{\alpha\sigma} - \partial_\sigma \mathbf{g}_{\alpha\beta}\right\}$ are the Cartesian Christoffel symbols of $\mathbf{g}$. From the constructions carried out Sect. 6.1, we find that $\mathbf{g}(\mathbf{B}, V) = -Vt = -1$ and thus $\mathbf{g}(\mathbf{J}, V) = -|\boldsymbol{\partial}\mathcal{C}^i|^2$. Note also that since $\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1$, we have $\mathbf{g}(\mathbf{J}, \mathbf{B}) = -|\boldsymbol{\partial}\mathcal{C}^i|^2$. We now apply the divergence theorem (where the Riemannian volume forms are induced by $\mathbf{g}$) using the vectorfield $\mathbf{J}^\alpha$ on the compact spacetime region bounded by Σ_{t_I}, Σ_{t_F}, and $\mathcal{N}$. Considering also the fact that $\mathbf{\Gamma}_{\alpha\ \beta}^{\ \alpha} = \mathrm{f}(\vec{\Psi})\boldsymbol{\partial}\vec{\Psi}$, we arrive at the following inequality for $\boldsymbol{\partial}\mathcal{C}^i$:

$$\begin{aligned}
\int_{\mathcal{N}} |\boldsymbol{\partial}\mathcal{C}^i|^2 \, d\varpi_g dt &= -\int_{\mathcal{N}} \mathbf{g}(\mathbf{J}, V) \, d\varpi_g dt \\
&\lesssim \int_{\Sigma_{t_I}} |\mathbf{g}(\mathbf{J}, \mathbf{B})| \, d\varpi_g + \int_{\Sigma_{t_F}} |\mathbf{g}(\mathbf{J}, \mathbf{B})| \, d\varpi_g \\
&\quad + \int_{t_I}^{t_F} \int_{\Sigma_\tau} |\boldsymbol{\partial}\mathcal{C}^i||\mathbf{B}\boldsymbol{\partial}\mathcal{C}^i| \, d\varpi_g \, d\tau + \int_{t_I}^{t_F} \int_{\Sigma_\tau} \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)} |\boldsymbol{\partial}\mathcal{C}^i|^2 \, d\varpi_g \, d\tau,
\end{aligned} \tag{103}$$

where $d\varpi_g$ is the volume form induced on constant-time hypersurfaces by their first fundamental form g. Here we clarify that the normalization condition $Vt = 1$ has the following virtue: it guarantees that the volume element on $\mathcal{N}$ appearing in the divergence theorem is precisely $d\varpi_g dt$. From the energy estimates of Proposition 5.1, we deduce that the two integrals $\int_{\Sigma_{t_I}} \cdots$ and $\int_{\Sigma_{t_F}} \cdots$ on RHS (103) are $\lesssim 1$. Next, commuting the evolution equation (15b) with $\boldsymbol{\partial}$, using the resulting expression to substitute for the factor $\mathbf{B}\boldsymbol{\partial}\mathcal{C}^i$ on RHS (103), using the bootstrap assumptions and the energy estimates of Proposition 5.1, and using the Cauchy–Schwarz and Young's inequalities, we deduce that the two integrals $\int_{\Sigma_\tau} \cdots$ on RHS (103) are $\lesssim 1 + \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}^2 + \|\partial(\vec{\Omega}, \vec{S})\|_{L^\infty(\Sigma_\tau)}^2$. Also using the bootstrap assumptions (41a)–(41b), we see that RHS (103) $\lesssim 1$, which, in view of definition (100), yields the desired bound $\mathbb{F}_{(Transport)}[\boldsymbol{\partial}\vec{\mathcal{C}}; \mathcal{N}] \lesssim 1$.

To obtain the desired bound for the sum on LHS (102) involving the terms $P_\nu \boldsymbol{\partial}\mathcal{C}$, we repeat the above argument with $P_\nu \boldsymbol{\partial}\mathcal{C}^i$ in the role of $\boldsymbol{\partial}\mathcal{C}^i$. Considering also the evolution equation (76b), we obtain the following bound:

$$\begin{aligned}
&\mathbb{F}_{(Transport)}[P_\nu \boldsymbol{\partial}\mathcal{C}^i;\mathcal{N}] \\
&\quad\lesssim \int_{\Sigma_{t_I}} |P_\nu \boldsymbol{\partial}\mathcal{C}^i|^2\, d\varpi_g + \int_{\Sigma_{t_F}} |P_\nu \boldsymbol{\partial}\mathcal{C}^i|^2\, d\varpi_g \\
&\qquad + \int_{t_I}^{t_F}\int_{\Sigma_\tau} |P_\nu \boldsymbol{\partial}\mathcal{C}^i||\mathfrak{R}_{(\boldsymbol{\partial}\mathcal{C}^i);\nu}|\, d\varpi_g\, d\tau + \int_{t_I}^{t_F}\int_{\Sigma_\tau} \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}|P_\nu \boldsymbol{\partial}\mathcal{C}^i|^2\, d\varpi_g\, d\tau.
\end{aligned} \tag{104}$$

Multiplying (104) by $\nu^{2(N-2)}$, summing over $\nu > 1$, using the estimate (85) and the Cauchy–Schwarz inequality for $L^2(\Sigma_\tau)$ and ℓ^2_ν, and using the energy estimates of Proposition 5.1 and the bootstrap assumptions (41a)–(41b), we conclude that RHS (104) $\lesssim 1$ as desired.

The estimate (102) for the terms involving $\mathcal{D}$ can be obtained in a similar fashion with the help of the evolution equations (16a) and (76c), and we omit the details.

The estimate (101) can be obtained using similar arguments, with a few minor adjustments that we now describe. To bound the first term on LHS (101), we apply the divergence theorem with the vectorfield ${}^{(\mathbf{B})}\mathbf{J}^\alpha[\boldsymbol{\partial}\Psi]$ defined by (46). The integrand appearing on the analog of LHS (103) is $\mathbf{g}({}^{(\mathbf{B})}\mathbf{J}, V)$, which through standard arguments (for example, using a null frame as in Sect. 9.6.2) can be shown to be equal to $\frac{1}{2}\left\{|V\boldsymbol{\partial}\Psi|^2 + |\nabla\!\!\!/\boldsymbol{\partial}\Psi|^2_{g\!\!\!/}\right\}$, that is, equal to the integrand in the definition (100) of $\mathbb{F}_{(Wave)}[\boldsymbol{\partial}\Psi;\mathcal{N}]$ (aside from the factor of $1/2$). The spacetime error integrals appearing on the analog of RHS (103) have integrands equal to RHS (48) (with $\mathbf{B}$ in the role of $\mathbf{X}$), where one commutes the wave equation (13) with $\boldsymbol{\partial}$ to obtain algebraic expressions for $\square_{\mathbf{g}}\boldsymbol{\partial}\Psi$. One can then argue as we did above to show that the error integrals are $\lesssim 1$ as desired. To bound the sum on LHS (101), we can use a similar argument based on the wave equation (76a) and the estimate (84). □

7 Strichartz estimates for the wave equation and control of Hölder norms of the wave variables

The main results of this section are Theorem 7.1, which yields a strict improvement of the Strichartz-type bootstrap assumption (41a) for the wave variables, and Corollary 7.1. Our proof of Theorem 7.1 relies on a frequency-localized Strichartz estimate provided by Theorem 7.2. We outline the proof of Theorem 7.2 in Sect. 11; given the estimates for the acoustic geometry that we derive in Sect. 10, the proof of Theorem 7.2 is essentially the same as the proof of an analogous frequency-localized Strichartz estimate featured in [54].

Remark 7.1 *(Reminder concerning the various parameters).* Our analysis in this section extensively refers to the collection of parameters from Sect. 3.3.

7.1 Statement of Theorem 7.1 and proof of Corollary 7.1

We now provide the main results of Sect. 7, starting with Theorem 7.1. The proof of the theorem is located in Sect. 7.4.

Theorem 7.1 (Improvement of the Strichartz-type bootstrap assumption for the wave variables). *If $\delta > 0$ is sufficiently small, then under the initial data and bootstrap assumptions of Sect. 3, the following estimate for the wave variables $\vec{\Psi} = (\rho, v^1, v^2, v^3, s)$ holds, where δ_1 is defined by (35e):*

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])L^\infty_x} + \sum_{\nu \geq 2} \nu^{2\delta_1} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])L^\infty_x} \lesssim T_*^{2\delta}. \tag{105}$$

The second main result of this section is the following corollary, which is a simple consequence of Theorem 7.1. It plays a fundamental role in Sect. 8, when we derive Schauder estimates for Ω and S.

Corollary 7.1 (Strichartz-type estimate with a Hölder spatial norm for the wave variables). *Under the assumptions and conclusions of Theorem 7.1, the following estimate holds for the wave variable array $\vec{\Psi} = (\rho, v^1, v^2, v^3, s)$:*

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])C^{0,\delta_1}_x} + \sum_{\nu \geq 2} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])C^{0,\delta_1}_x} \lesssim T_*^{2\delta}. \tag{106}$$

Proof (Discussion of proof) Given Theorem 7.1, Corollary 7.1 follows from standard results in harmonic analysis; see, for example, [48, Equation (A.1.5)] and the discussion surrounding it. □

7.2 Partitioning of the bootstrap time interval

In proving Theorem 7.1, we will follow the strategy of [54] by constructing an appropriate partition of the bootstrap time interval $[0, T_*]$. The partition refers to a parameter Λ_0, where in the rest of the paper, $\Lambda_0 \gg 1$ denotes a dyadic frequency that is chosen to be sufficiently large (we adjust the largeness of Λ_0 as needed throughout the course of the analysis). In view of the bootstrap assumptions (41a)–(41b), it is straightforward to see that for $\lambda \geq \Lambda_0$, we can partition $[0, T_*]$ into intervals $[t_k, t_{k+1}]$ of length $|t_{k+1} - t_k| \leq \lambda^{-8\epsilon_0} T_*$ such that the total number of intervals is $\approx \lambda^{8\epsilon_0}$ and such that

$$\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([t_k,t_{k+1}])L^\infty_x} + \sum_{\nu \geq 2} \nu^{2\delta_0} \|P_\nu \boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([t_k,t_{k+1}])L^\infty_x} \leq \lambda^{-8\epsilon_0}, \tag{107a}$$

$$\|(\partial\vec{\Omega}, \partial\vec{S})\|^2_{L^2([t_k,t_{k+1}])L^\infty_x} + \sum_{\nu \geq 2} \nu^{2\delta_0} \|(P_\nu \partial\vec{\Omega}, P_\nu \partial\vec{S})\|^2_{L^2([t_k,t_{k+1}])L^\infty_x} \leq \lambda^{-8\epsilon_0}. \tag{107b}$$

We refer readers to [18, Remark 1.3] for more details on the construction of a partition of $[0, T_*]$ such that (107a)–(107b) hold.

7.3 Frequency-localized Strichartz estimate

The main step in the proof of Theorem 7.1 is proving a frequency-localized version, specifically Theorem 7.2; see Sect. 11 for an outline of its proof, which relies on estimates for the acoustic geometry that we derive in Sect. 10.

Theorem 7.2 (Frequency-localized Strichartz estimate). *Fix $\lambda \geq \Lambda_0$, and let φ be a solution to the following covariant linear wave equation on the slab $[t_k, t_{k+1}] \times \mathbb{R}^3$, where $\{[t_k, t_{k+1}]\}_{k=1,\ldots}$ denotes the finite collection of time intervals constructed in Sect. 7.2:*

$$\square_{\mathbf{g}(\vec{\Psi})}\varphi = 0. \tag{108}$$

Under the initial data and bootstrap assumptions of Sect. 3, if Λ_0 is sufficiently large, then for any $q > 2$ sufficiently close to 2 and any $\tau \in [t_k, t_{k+1}]$, we have the following estimate:

$$\|P_\lambda \boldsymbol{\partial}\varphi\|_{L^q([\tau,t_{k+1}])L^\infty_x} \lesssim \lambda^{\frac{3}{2}-\frac{1}{q}} \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_\tau)}. \tag{109}$$

7.4 Proof of Theorem 7.1 given Theorem 7.2

In this proof, we often suppress the x-dependence of functions, and we use the remarks made in the first paragraph of Sect. 5.3. Let $W(t,\tau)[f, f_0]$ be the solution at time t to the covariant linear wave equation $\square_{\mathbf{g}(\vec{\Psi})}(W(t,\tau)[f, f_0]) = 0$ whose data at time τ are $W(\tau,\tau)[f, f_0] := f$ and $\partial_t W(\tau,\tau)[f, f_0] := f_0$. We assume that $\lambda \geq \Lambda_0$, as in Theorem 7.2. Let $\widetilde{P}_\lambda := \sum_{1/2 \leq \frac{\mu}{\lambda} \leq 2} P_\mu$, so that in particular, $P_\lambda = \widetilde{P}_\lambda P_\lambda$. Then from Eq. (70b) and Duhamel's principle, for $\Psi \in \{\rho, v^1, v^2, v^3, s\}$ and $t \in [t_k, t_{k+1}]$, we have

$$P_\lambda \Psi(t) = W(t, t_k)[P_\lambda \Psi(t_k), P_\lambda \partial_t \Psi(t_k)] + \int_{t_k}^{t} W(t,\tau)[0, \mathfrak{R}_{(\Psi);\lambda}(\tau)]\, d\tau. \tag{110}$$

Differentiating (110) with $\boldsymbol{\partial}$ and applying $\widetilde{P}_\lambda$, and letting $\mathbf{1}_{[t_k,t]}(\cdot)$ denote the characteristic function of the interval $[t_k, t]$, we find that

$$\begin{aligned} P_\lambda \boldsymbol{\partial}\Psi(t) &= \widetilde{P}_\lambda \{\boldsymbol{\partial} W(t, t_k)[P_\lambda \Psi(t_k), P_\lambda \partial_t \Psi(t_k)]\} \\ &\quad + \int_{t_k}^{t_{k+1}} \mathbf{1}_{[t_k,t]}(\tau) \widetilde{P}_\lambda \boldsymbol{\partial} W(t,\tau)[0, \mathfrak{R}_{(\Psi);\lambda}(\tau)]\, d\tau \\ &:= I_\lambda(t) + II_\lambda(t). \end{aligned} \tag{111}$$

We now recall that $\delta = \frac{1}{2} - \frac{1}{q} > 0$ (see (35d)), where $q > 2$ is any number for which Theorem 7.2 holds. Then from (109) with $\widetilde{P}_\lambda$ in the role of P_λ, Hölder's inequality, the covariant wave equation (70b) satisfied by $P_\lambda \Psi$, and the energy estimate (44), we find that

$$\begin{aligned} \| I_\lambda \|_{L^2([t_k,t_{k+1}])L^\infty_x} &\lesssim |t_{k+1}-t_k|^\delta \| I_\lambda \|_{L^q([t_k,t_{k+1}])L^\infty_x} \\ &\lesssim |t_{k+1}-t_k|^\delta \lambda^{\frac{3}{2}-\frac{1}{q}} \| \pmb{\partial} P_\lambda \Psi \|_{L^2(\Sigma_{t_k})} \\ &\lesssim |t_{k+1}-t_k|^\delta \lambda^{1+\delta} \left\{ \| \pmb{\partial} P_\lambda \Psi \|_{L^2(\Sigma_0)} + \| \mathfrak{R}_{(\Psi);\lambda} \|_{L^1([0,T_*])L^2_x} \right\}. \end{aligned} \tag{112}$$

Similarly, using (109) (again with $\widetilde{P}_\lambda$ in the role of P_λ) and Minkowski's inequality for integrals, we find that

$$\begin{aligned} \| II_\lambda \|_{L^2([t_k,t_{k+1}])L^\infty_x} &\lesssim \int_{t_k}^{t_{k+1}} \| \mathbf{1}_{[t_k,t]}(\tau) P_\lambda \pmb{\partial} W(t,\tau)[0, \mathfrak{R}_{(\Psi);\lambda}(\tau)] \|_{L^2_t([\tau,t_{k+1}])L^\infty_x} \, d\tau \\ &\lesssim \int_{t_k}^{t_{k+1}} |t_{k+1}-\tau|^\delta \| P_\lambda \pmb{\partial} W(t,\tau)[0, \mathfrak{R}_{(\Psi);\lambda}(\tau)] \|_{L^q_t([\tau,t_{k+1}])L^\infty_x} \, d\tau \\ &\lesssim |t_{k+1}-t_k|^\delta \lambda^{1+\delta} \| \mathfrak{R}_{(\Psi);\lambda} \|_{L^1([t_k,t_{k+1}])L^2_x}. \end{aligned} \tag{113}$$

Using (111), (112), and (113), and recalling that $|t_{k+1}-t_k| \lesssim \lambda^{-8\epsilon_0} T_*$, we find that

$$\| P_\lambda \pmb{\partial} \Psi \|_{L^2([t_k,t_{k+1}])L^\infty_x} \lesssim \lambda^{1+\delta(1-8\epsilon_0)} T_*^\delta \left\{ \| \pmb{\partial} P_\lambda \Psi \|_{L^2(\Sigma_0)} + \| \mathfrak{R}_{(\Psi);\lambda} \|_{L^1([0,T_*])L^2_x} \right\}. \tag{114}$$

Next, we square (114), sum over all intervals $[t_k, t_{k+1}]$, recall that there are $\lesssim \lambda^{8\epsilon_0}$ such intervals, and multiply the resulting inequality by $\lambda^{2\delta_1}$ (where $\delta_1 > 0$ is defined in (35e)), thereby obtaining:

$$\begin{aligned} &\lambda^{2\delta_1} \| P_\lambda \pmb{\partial} \Psi \|^2_{L^2([0,T_*])L^\infty_x} \\ &\quad \lesssim \lambda^{2\delta_1} \lambda^{8\epsilon_0} \lambda^{2+2\delta(1-8\epsilon_0)} T_*^{2\delta} \left\{ \| \pmb{\partial} P_\lambda \Psi \|^2_{L^2(\Sigma_0)} + \| \mathfrak{R}_{(\Psi);\lambda} \|^2_{L^1([0,T_*])L^2_x} \right\} \\ &\quad \lesssim T_*^{2\delta} \left\{ \| \lambda^{N-1} \pmb{\partial} P_\lambda \Psi \|^2_{L^2(\Sigma_0)} + \| \lambda^{N-1} \mathfrak{R}_{(\Psi);\lambda} \|^2_{L^1([0,T_*])L^2_x} \right\}. \end{aligned} \tag{115}$$

We now sum (115) over dyadic frequencies $\lambda \geq \Lambda_0$ and use the Hölder-in-time estimate

$$\| \lambda^{N-1} \mathfrak{R}_{(\Psi);\lambda} \|^2_{L^1([0,T_*])L^2_x} \lesssim T_* \| \lambda^{N-1} \mathfrak{R}_{(\Psi);\lambda} \|^2_{L^2([0,T_*])L^2_x}$$

to deduce that

$$\begin{aligned} &\sum_{\nu \geq \Lambda_0} \nu^{2\delta_1} \| P_\nu \pmb{\partial} \Psi \|^2_{L^2([0,T_*])L^\infty_x} \\ &\quad \lesssim T_*^{2\delta} \left\{ \| (\Psi, \partial_t \Psi) \|^2_{H^N(\Sigma_0) \times H^{N-1}(\Sigma_0)} + T_* \| \nu^{N-1} \mathfrak{R}_{(\Psi);\nu} \|^2_{L^2([0,T_*])\ell^2_\nu L^2_x} \right\}. \end{aligned} \tag{116}$$

Using the estimate (84), the Strichartz-type bootstrap assumption (41a), and the top-order energy estimate (69), we deduce that $\|\nu^{N-1}\mathfrak{R}_{(\Psi);\nu}\|^2_{L^2([0,T_*])\ell^2_\nu L^2_x} \lesssim 1$. Inserting this estimate and the trivial bound $\|(\Psi, \partial_t\Psi)\|^2_{H^N(\Sigma_0)\times H^{N-1}(\Sigma_0)} \lesssim 1$ into RHS (116), we find that

$$\sum_{\nu\geq\Lambda_0} \nu^{2\delta_1}\|P_\nu\boldsymbol{\partial}\Psi\|^2_{L^2([0,T_*])L^\infty_x} \lesssim T_*^{2\delta}. \tag{117}$$

Next, we note that Sobolev embedding and the energy estimate (69) yield that $\|P_{\leq\Lambda_0}\boldsymbol{\partial}\Psi\|_{L^\infty_x(\Sigma_t)} \lesssim \|P_{\leq\Lambda_0}\boldsymbol{\partial}\Psi\|_{H^2(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\Psi\|_{L^2(\Sigma_t)} \lesssim 1$ (where the implicit constants are allowed to depend on Λ_0) and thus

$$\|P_{\leq\Lambda_0}\boldsymbol{\partial}\Psi\|^2_{L^2([0,T_*])L^\infty_x} \lesssim T_* \lesssim T_*^{2\delta}. \tag{118}$$

We are now ready to bound the term $\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])L^\infty_x}$ on LHS (105). To proceed, we use the triangle inequality, the Cauchy–Schwarz inequality, and the fact that $\sum_{\nu\geq\Lambda_0}\nu^{-2\delta_1} < \infty$ to deduce that

$$\begin{aligned}\|\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_t)} &\lesssim \|P_{\leq\Lambda_0}\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_t)} + \sum_{\nu\geq\Lambda_0}\nu^{-\delta_1}\|\nu^{\delta_1}P_\nu\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_t)} \\ &\lesssim \|P_{\leq\Lambda_0}\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_t)} + \sqrt{\sum_{\nu\geq\Lambda_0}\nu^{2\delta_1}\|P_\nu\boldsymbol{\partial}\Psi\|^2_{L^\infty(\Sigma_t)}}. \end{aligned} \tag{119}$$

Squaring (119), integrating the resulting inequality over the interval $[0, T_*]$, and using (117) and (118), we conclude the desired bound for the term $\|\boldsymbol{\partial}\vec{\Psi}\|^2_{L^2([0,T_*])L^\infty_x}$ on LHS (105). From this bound, (117), and the basic inequality $\|P_\nu\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\Psi\|_{L^\infty(\Sigma_t)}$, the desired bound for the sum on LHS (105) readily follows. This completes the proof of Theorem 7.1. ◻

8 Schauder-transport estimates in Hölder spaces for the first derivatives of the specific vorticity and the second derivatives of the entropy

Our main goal in this section is to derive improvements of the mixed spacetime norm bootstrap assumptions (41b) for $\partial\vec{\Omega}$ and $\partial\vec{S}$. The main result is Theorem 8.1. We also derive a strict improvement of the bootstrap assumption (40). Before proving the theorem, we first derive two fundamentally important precursor results: (i) Schauder estimates for div-curl systems; (ii) Estimates that yield control of the characteristics of the transport operator $\mathbf{B}$ (i.e., over the integral curves of $\mathbf{B}$); and (ii)' With the help of (ii), we derive a priori estimates in Hölder spaces for solutions φ to transport equations $\mathbf{B}\varphi = \mathfrak{F}$ with $\mathfrak{F} \in L^1_tC^{0,\delta_1}_x$ (see Lemma 8.4). Thanks to these three preliminary ingredients, Theorem 8.1 will follow from a Grönwall inequality estimate.

8.1 Statement of Theorem 8.1 and proof of an improvement of the basic L^∞-type bootstrap assumption

We now state the main theorem of this section. Its proof is located in Sect. 8.5.

Theorem 8.1 (Lebesgue-Hölder norm estimates for the specific vorticity and entropy gradient and improvements of the bootstrap assumptions). *Under the initial data and bootstrap assumptions of Sect. 3, the following estimates hold:*

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^\infty([0,T_*])C_x^{0,\delta_1}} \lesssim 1, \tag{120}$$

$$\|\partial(\vec{\Omega}, \vec{S})\|^2_{L^2([0,T_*])C_x^{0,\delta_1}} \lesssim T_*^{2\delta}. \tag{121}$$

Moreover,

$$\sum_{\nu \geq 1} \nu^{\delta_1} \|P_\nu \partial(\vec{\Omega}, \vec{S})\|^2_{L^2([0,T_*])L_x^\infty} \lesssim T_*^{2\delta}. \tag{122}$$

Before initiating the proof of Theorem 8.1, we first use it as an ingredient in deriving a strict improvement of the bootstrap assumption (40).

Corollary 8.1 (Improvement of the basic L^∞-type bootstrap assumption). *Let $\mathfrak{K}$ be the compact set appearing in the bootstrap assumption* (40). *Under the initial data and bootstrap assumptions of Sect. 3, the following containment holds whenever T_* is sufficiently small:*

$$(\rho, s, \vec{v}, \vec{\Omega}, \vec{S})([0, T_*] \times \mathbb{R}^3) \subset int\mathfrak{K}. \tag{123}$$

Proof Let $\vec{\varphi}$ denote the following array of scalar functions: $\vec{\varphi} := (\rho, s, \vec{v}, \vec{\Omega}, \vec{S})$. Using (14) and the bootstrap assumption (40), we deduce that $|\partial_t \vec{\varphi}| \lesssim |\boldsymbol{\partial}\vec{\Psi}| + |\partial\vec{\Omega}| + |\partial\vec{S}| + 1$. Hence, from the fundamental theorem of calculus, the estimates (105) and (121), and the Cauchy–Schwarz inequality with respect to t, we deduce that the following estimate holds for $t \in [0, T_*]$: $|\vec{\varphi}(t,x) - \vec{\varphi}(0,x)| \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{L^1([0,t])L_x^\infty} + \|\partial\vec{\Omega}\|_{L^1([0,t])L_x^\infty} + \|\partial\vec{S}\|_{L^1([0,t])L_x^\infty} + t \lesssim T_*^{1/2+\delta}$. It follows that we can guarantee that $\vec{\varphi}(t,x)$ is arbitrarily close to $\vec{\varphi}(0,x)$ by choosing T_* to be sufficiently small. From this fact and (39), we conclude (123). □

8.2 Schauder estimates for div-curl systems

In the next lemma, we provide a standard Schauder estimate for div-curl systems on Euclidean space $\mathbb{R}^3$.

Lemma 8.2 (Schauder estimates for div-curl systems). *Let V be a vectorfield on $\mathbb{R}^3$ such that $V \in C^2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$, and let $\delta_1 > 0$ be the parameter from* (35e). *Then the following estimate holds:*[30]

[30] Our proof of the estimate (124) goes through for $\delta_1 \in (0, 1/2)$, but in practice, we need the estimate only for the value of δ_1 specified in (35e).

$$\|\partial V\|_{C^{0,\delta_1}(\mathbb{R}^3)} \lesssim \|divV\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|curlV\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|V\|_{H^2(\mathbb{R}^3)}. \tag{124}$$

Proof Let $z \in \mathbb{R}^3$ and let $B_2(z)$ be the ball of Euclidean radius 2 centered at z. As a first step, we will show that if $W \in C^2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ is a vectorfield on $\mathbb{R}^3$ that is supported in $B_2(z)$, then we have (with implicit constants that are independent of z):

$$\|\partial W\|_{C^{0,\delta_1}(B_2(z))} \lesssim \|\mathrm{div} W\|_{C^{0,\delta_1}(B_2(z))} + \|\mathrm{curl} W\|_{C^{0,\delta_1}(B_2(z))}. \tag{125}$$

To prove (125), we let $\Phi(x) := \frac{-1}{4\pi|x|}$ denote the fundamental solution of the Euclidean Laplacian on $\mathbb{R}^3$. The standard Helmholtz decomposition yields the following identity, where ϵ^{ijk} is the fully antisymmetric symbol normalized by $\epsilon^{123} = 1$:

$$W^j = \mathrm{div} W * \delta^{jc}\partial_c \Phi - \epsilon^{jcd}\delta_{ca}(\mathrm{curl} W)^a * \partial_d \Phi. \tag{126}$$

The desired estimate (125) now follows from standard estimates for the first derivatives of the convolutions on RHS (126); see, for example, the proofs of [14, Lemma 4.2] and [14, Lemma 4.4].

To prove (124), let $B_1(z) \subset \mathbb{R}^3$ be the Euclidean ball with radius 1 centered at z. Let $\chi \geq 0$ be a C^∞ spherically symmetric cut-off function on $\mathbb{R}^3$ with $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$, and let $\chi_z(x) := \chi(x - z)$. It follows that $\chi_z(x) = 1$ for $x \in B_1(z)$ and thus $\|\partial V\|_{C^{0,\delta_1}(B_1(z))} = \|\partial(\chi_z V)\|_{C^{0,\delta_1}(B_1(z))} \leq \|\partial(\chi_z V)\|_{C^{0,\delta_1}(B_2(z))}$. From this estimate, (125) with $\chi_z V$ in the role of W (this estimate is valid since $\chi_z V$ is compactly supported in $B_2(z)$), the standard estimate $\|F \cdot G\|_{C^{0,\delta_1}(B_2(z))} \leq 2\|F\|_{C^{0,\delta_1}(B_2(z))}\|G\|_{C^{0,\delta_1}(B_2(z))}$, and the simple estimates (which are uniform in z) $\|\chi_z\|_{C^{0,\delta_1}(B_1(z))} \leq \|\chi\|_{C^{0,\delta_1}(\mathbb{R}^3)} \lesssim 1$ and $\|\partial\chi_z\|_{C^{0,\delta_1}(B_1(z))} \leq \|\partial\chi\|_{C^{0,\delta_1}(\mathbb{R}^3)} \lesssim 1$, we obtain

$$\begin{aligned}\|\partial V\|_{C^{0,\delta_1}(B_1(z))} &\lesssim \|\mathrm{div}(\chi_z V)\|_{C^{0,\delta_1}(B_2(z))} + \|\mathrm{curl}(\chi_z V)\|_{C^{0,\delta_1}(B_2(z))} \\ &\lesssim \|\mathrm{div} V\|_{C^{0,\delta_1}(B_2(z))} + \|\mathrm{curl} V\|_{C^{0,\delta_1}(B_2(z))} + \|V\|_{C^{0,\delta_1}(B_2(z))}.\end{aligned} \tag{127}$$

From (127) and the Sobolev embedding result $H^2(\mathbb{R}^3) \hookrightarrow C^{0,\delta_1}(\mathbb{R}^3)$ (which is valid since $\delta_1 < 1/2$), we deduce that

$$\sup_{x,y \in B_1(z), 0<|x-y|} \frac{|\partial V(x) - \partial V(y)|}{|x-y|^{\delta_1}} \lesssim \|\mathrm{div} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|\mathrm{curl} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|V\|_{H^2(\mathbb{R}^3)}. \tag{128}$$

Moreover, since $\|\partial V\|_{L^2(B_1(z))} \leq \|V\|_{H^1(\mathbb{R}^3)}$ and since $B_1(z)$ has Euclidean volume greater than 1, there must be a point $p \in B_1(z)$ such that $|\partial V(p)| \leq \|V\|_{H^1(\mathbb{R}^3)}$. From this simple fact and (128), we conclude that

$$\sup_{x \in B_1(z)} |\partial V(x)| \lesssim \|\mathrm{div} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|\mathrm{curl} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|V\|_{H^2(\mathbb{R}^3)}. \tag{129}$$

Since z is arbitrary in (129), we conclude that

$$\|\partial V\|_{L^\infty(\mathbb{R}^3)} \lesssim \|\mathrm{div} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|\mathrm{curl} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|V\|_{H^2(\mathbb{R}^3)}. \tag{130}$$

From (130), it easily follows that

$$\begin{aligned} \sup_{|x-y|\geq 1} \frac{|\partial V(x) - \partial V(y)|}{|x-y|^{\delta_1}} &\leq 2\|\partial V\|_{L^\infty(\mathbb{R}^3)} \\ &\lesssim \|\mathrm{div} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|\mathrm{curl} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|V\|_{H^2(\mathbb{R}^3)}. \end{aligned} \tag{131}$$

Next, if $0 < |x-y| \leq 1$, then $y \in B_1(x)$, which, in view of (127) with x in the role of z and the Sobolev embedding result $H^2(\mathbb{R}^3) \hookrightarrow C^{0,\delta_1}(\mathbb{R}^3)$, implies that

$$\sup_{0<|x-y|\leq 1} \frac{|\partial V(x) - \partial V(y)|}{|x-y|^{\delta_1}} \lesssim \|\mathrm{div} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|\mathrm{curl} V\|_{C^{0,\delta_1}(\mathbb{R}^3)} + \|V\|_{H^2(\mathbb{R}^3)}. \tag{132}$$

Finally, in view of definition (32), we see that the desired estimate (124) follows from (130), (131), and (132). ◻

8.3 Estimates for the flow map of the material derivative vectorfield

Our proof of Theorem 8.1 is through a Grönwall inequality estimate that relies on having sufficient control of the flow map of the material derivative vectorfield $\mathbf{B}$. In the next lemma, we derive the estimates for the flow map.

Lemma 8.3 (Estimates for the flow map of the material derivative vectorfield). *Let $\gamma : [0, T_*] \times \mathbb{R}^3 \to [0, T_*] \times \mathbb{R}^3$ be the flow map of $\mathbf{B}$, that is, the solution to the following transport initial value problem for the Cartesian component functions $\gamma^\alpha(t; x)$:*

$$\frac{d}{dt}\gamma^\alpha(t; x) = \mathbf{B}^\alpha \circ \gamma(t; x), \tag{133a}$$

$$\gamma^0(0; x) = 0, \gamma^i(0; x) = x^i. \tag{133b}$$

Then under the bootstrap assumptions, for every fixed $x \in \mathbb{R}^3$, there exists a unique solution $t \to \gamma(t; x)$ to the system (133a)–(133b). Moreover, γ is a smooth function of t and x. In addition, there exists a constant $C > 0$ such that for $t \in [0, T_*]$ and all $x, y \in \mathbb{R}^3$, we have

$$\gamma^0(t;x) = t, \tag{134a}$$

$$|\gamma^i(t;x) - x^i| \le C, \tag{134b}$$

$$\sum_{i=1}^{3} |\gamma^i(t;x) - \gamma^i(t;y)| \approx |x-y|. \tag{134c}$$

In particular, for each fixed $t \in [0, T_*]$*, the map* $x \to \big(\gamma^1(t,x), \gamma^2(t,x), \gamma^3(t,x)\big)$ *is a smooth global diffeomorphism from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$.

Proof The identity (134a) follows easily from considering the 0 component of (133a)–(133b).

Since the components $\mathbf{B}^\alpha$ are smooth on $[0, T_*] \times \mathbb{R}^3$ and satisfy[31] $\sup_{t\in[0,T_*]} \|\partial^{\le 1}\mathbf{B}^\alpha\|_{L^\infty(\Sigma_t)} < \infty$, the existence and uniqueness of solutions $\gamma(t;x)$ to (133a)–(133b) that depend smoothly on t and x is a standard result from ODE theory, as is the fact that the map $x \to \big(\gamma^1(t,x), \gamma^2(t,x), \gamma^3(t,x)\big)$ is a smooth global diffeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ for each fixed $t \in [0, T_*]$.

Next, we use the fundamental theorem of calculus and the fact that $\mathbf{B}^i = v^i$ (see (2)) to deduce

$$\gamma^i(t;x) - \gamma^i(t;y) = x^i - y^i + \int_0^t \Big\{ v^i \circ \gamma(\tau;x) - v^i \circ \gamma(\tau;y) \Big\}\, d\tau. \tag{135}$$

Let $\underline{\gamma}(t,x) := \big(\gamma^1(t,x), \gamma^2(t,x), \gamma^3(t,x)\big)$. Since ∂v and γ are smooth, we deduce from (135) and the mean value theorem that

$$|\Big(\underline{\gamma}(t;x) - \underline{\gamma}(t;y)\Big) - (x-y)| \le C \int_0^t \|\partial\vec{v}\|_{L^\infty(\Sigma_\tau)} |\underline{\gamma}(\tau;x) - \underline{\gamma}(\tau;y)|\, d\tau. \tag{136}$$

From (136) and Grönwall's inequality (more precisely, a straightforward extension of the standard Grönwall inequality to yield upper and lower bounds), we deduce that

$$\exp\left(-C\int_0^t \|\partial\vec{v}\|_{L^\infty(\Sigma_\tau)}\, d\tau\right) \le \frac{|\underline{\gamma}(t;x) - \underline{\gamma}(t;y)|}{|x-y|} \le \exp\left(C\int_0^t \|\partial\vec{v}\|_{L^\infty(\Sigma_\tau)}\, d\tau\right). \tag{137}$$

From (137) and the bootstrap assumption (41a), we conclude the desired bounds (134c).

The estimate (134b) follows from a similar argument based on the simple bound $\|\vec{v}\|_{L^1([0,T_*])L^\infty_x} \lesssim 1$; we omit the details. □

[31] Here, we are only using the *qualitative* finiteness property $\sup_{t\in[0,T_*]} \|\partial\mathbf{B}^\alpha\|_{L^\infty(\Sigma_t)} < \infty$ to guarantee the existence and uniqueness of the solution to (133a)–(133b). In contrast, the constants in (134a)–(134c) are controlled by the bootstrap assumptions, such as (41a).

8.4 Estimates for transport equations in Hölder spaces

With the help of Lemma 8.3, we now derive estimates for transport equations with Hölder-class initial data and source terms.

Lemma 8.4 (Estimates for transport equations in Hölder spaces). *Let $\mathfrak{F}$ be a smooth function on $[0, T_*] \times \mathbb{R}^3$ and let $\mathring{\varphi}$ be a smooth function on $\mathbb{R}^3$. Let φ be a smooth solution to the following inhomogeneous transport equation initial value problem:*

$$\mathbf{B}^{\alpha} \partial_{\alpha} \varphi = \mathfrak{F}, \tag{138a}$$

$$\varphi|_{\Sigma_0} = \mathring{\varphi}. \tag{138b}$$

Then the following estimate holds for $t \in [0, T_*]$, where $\delta_1 > 0$ is the parameter from (35e):

$$\|\varphi\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|\mathring{\varphi}\|_{C^{0,\delta_1}(\Sigma_0)} + \int_0^t \|\mathfrak{F}\|_{C^{0,\delta_1}(\Sigma_\tau)} \, d\tau. \tag{139}$$

Proof Let $\gamma(t; x)$ be the flow map of $\mathbf{B}$, as in Lemma 8.3. Then equation (138a) can be rewritten as $\frac{d}{dt}(\varphi \circ \gamma(t; x)) = \mathfrak{F}$. Integrating in time and using (133b), we find that

$$\varphi \circ \gamma(t; x) - \varphi \circ \gamma(t; y) = \mathring{\varphi}(x) - \mathring{\varphi}(y) + \int_0^t \{\mathfrak{F}(\tau, x) - \mathfrak{F}(\tau, y)\} \, d\tau, \tag{140}$$

from which it easily follows that

$$|\varphi \circ \gamma(t; x) - \varphi \circ \gamma(t; y)| \leq \|\mathring{\varphi}\|_{C_x^{0,\delta_1}} |x - y|^{\delta_1} + |x - y|^{\delta_1} \int_0^t \|\mathfrak{F}\|_{C^{0,\delta_1}(\Sigma_\tau)} \, d\tau. \tag{141}$$

From (134c) and (141), we deduce that

$$\begin{aligned} |\varphi \circ \gamma(t; x) - \varphi \circ \gamma(t; y)| &\lesssim \|\mathring{\varphi}\|_{C_x^{0,\delta_1}} |\gamma(t; x) - \gamma(t; y)|^{\delta_1} \\ &\quad + |\gamma(t; x) - \gamma(t; y)|^{\delta_1} \int_0^t \|\mathfrak{F}\|_{C^{0,\delta_1}(\Sigma_\tau)} \, d\tau. \end{aligned} \tag{142}$$

Since Lemma 8.3 guarantees that the map $x \to \big(\gamma^1(t, x), \gamma^2(t, x), \gamma^3(t, x)\big)$ is a smooth global diffeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ for each fixed $t \in [0, T_*]$, we conclude from (142) that

$$\sup_{0 < |x-y|} \frac{|\varphi(t, x) - \varphi(t, y)|}{|x - y|^{\delta_1}} \leq \|\mathring{\varphi}\|_{C_x^{0,\delta_1}} + \|\mathfrak{F}\|_{L^1([0,t]) C_x^{0,\delta_1}}. \tag{143}$$

Using a similar but simpler argument, based on the fundamental theorem of calculus, we find that

$$\|\varphi\|_{L^\infty(\Sigma_t)} \lesssim \|\mathring{\varphi}\|_{L^\infty(\Sigma_0)} + \|\mathfrak{F}\|_{L^1([0,t])L^\infty_x}$$

which, in view of definition (32) and (143), yields (139). □

8.5 Proof of Theorem 8.1

From Eqs. (15a)–(16b), the bootstrap assumption (40), the energy-elliptic estimate (69), the standard estimates

$$\|F \cdot G\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|F\|_{C^{0,\delta_1}(\Sigma_t)} \|G\|_{C^{0,\delta_1}(\Sigma_t)} \text{ and}$$
$$\|[\mathrm{f} \circ \vec{\varphi}] \cdot G\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|\vec{\varphi}\|_{C^{0,\delta_1}(\Sigma_t)} \|G\|_{C^{0,\delta_1}(\Sigma_t)}$$

(where the latter estimate is valid for any fluid variable array $\vec{\varphi}$ comprised of elements of $\{\rho, s, \vec{v}, \vec{\Omega}, \vec{S}\}$ and any function f that is smooth on the domain of $\vec{\varphi}$ values corresponding to the set $\mathfrak{K}$ from (40)), the standard embedding result $H^2(\Sigma_t) \hookrightarrow C^{0,\delta_1}(\Sigma_t)$ (which is valid since $\delta_1 < 1/2$), and Young's inequality, we deduce that

$$\|\mathbf{B}\vec{\mathcal{C}}\|_{C^{0,\delta_1}(\Sigma_t)} + \|\mathbf{B}\mathcal{D}\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|^2_{C^{0,\delta_1}(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_1}(\Sigma_t)} \|\partial(\vec{\Omega}, \vec{S})\|_{C^{0,\delta_1}(\Sigma_t)} + \|\partial\vec{\Omega}\|_{C^{0,\delta_1}(\Sigma_t)} + 1, \tag{144}$$

$$\|\mathrm{div}\Omega\|_{C^{0,\delta_1}(\Sigma_t)} + \|\mathrm{curl} S\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_1}(\Sigma_t)}. \tag{145}$$

Using Definition 1.2 to algebraically solve for curlΩ and div$S$ and using a similar argument, we deduce that

$$\|\mathrm{curl}\Omega\|_{C^{0,\delta_1}(\Sigma_t)} + \|\mathrm{div} S\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|(\vec{\mathcal{C}}, \mathcal{D})\|_{C^{0,\delta_1}(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_1}(\Sigma_t)}. \tag{146}$$

Next, from (139) with $(\vec{\mathcal{C}}, \mathcal{D})$ in the role of φ, the Hölder bounds (144)–(145), and the data-bound

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{C^{0,\delta_1}(\Sigma_0)} \lesssim 1,$$

(which follows from (35e) and (38b)), we deduce

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim 1 + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|^2_{C^{0,\delta_1}(\Sigma_t)} \, d\tau + \int_0^t \left\{\|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_1}(\Sigma_t)} + 1\right\} \|\partial(\vec{\Omega}, \vec{S})\|_{C^{0,\delta_1}(\Sigma_t)} \, d\tau. \tag{147}$$

Next, using the elliptic estimate (124) with Ω and S in the role of V, (145)–(146), and the energy estimate (69), we find that the following estimate holds for $t \in [0, T_*]$:

$$\|\partial(\vec{\Omega}, \vec{S})\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim \|(\vec{\mathcal{C}}, \mathcal{D})\|_{C^{0,\delta_1}(\Sigma_t)} + \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_1}(\Sigma_t)} + 1. \tag{148}$$

Using (148) to bound the factor $\|\partial(\vec{\Omega}, \vec{S})\|_{C^{0,\delta_1}(\Sigma_\tau)}$ on RHS (147), applying Grönwall's inequality in the term $\|(\vec{\mathcal{C}}, \mathcal{D})\|_{C^{0,\delta_1}(\Sigma_t)}$, and using (106), we find that

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{C^{0,\delta_1}(\Sigma_t)} \lesssim 1. \tag{149}$$

We have therefore proved (120). Then, using (149) to bound the first term on RHS (148), squaring the resulting inequality and integrating it in time, and using (106), we arrive at the desired estimate (121).

(122) then follows from (121), the following well-known estimate (see, for example, [48]*Equation (A.1.2) and the discussion surrounding it), valid for scalar functions f: $\sup_{\nu \geq 1} \nu^{\delta_1} \|P_\nu f\|_{L^\infty(\Sigma_t)} \lesssim \|f\|_{C^{0,\delta_1}(\Sigma_t)}$, and the fact that the dyadic sum $\sum_{\nu \geq 1} \nu^{-\delta_1}$ is finite. This completes the proof of Theorem 8.1. □

9 The setup of the proof of Theorem 7.2: the rescaled solution and construction of the eikonal function

To complete our bootstrap argument and finish the proof of Theorem 1.2, we have one remaining arduous task: proving Theorem 7.2. We accomplish this in Sects. 9–11. In this section, we set up the geometric and analytic framework that we use in the rest of the paper. As in the works [18,21,54], the main ingredients are an appropriate rescaling of the solution,[32] an eikonal function u with suitable initial conditions, and a collection of geometric tensorfields constructed out of u. Compared to previous works, the main new contribution of the present section is located in Sect. 9.9.3, where we derive various PDEs satisfied by the geometric tensorfields; there, one explicitly sees how the source terms in these geometric PDEs depend on the vorticity and entropy, and some of the precise structures in these PDEs are crucial for our analysis.

9.1 The rescaled quantities and the radius *R*

9.1.1 The rescaled quantities

Let $\{[t_k, t_{k+1}]\}_{k=1,2,\cdots}$ be the (finite collection of) time intervals introduced in Sect. 7.2, and let $\Lambda_0 > 0$ be the large parameter introduced there. For any fixed dyadic frequency $\lambda \geq \Lambda_0$, let

$$T_{*;(\lambda)} := \lambda(t_{k+1} - t_k). \tag{150}$$

[32] In [54], instead of rescaling the solution, the author worked with rescaled coordinates. These two approaches are equivalent.

Note that since (by construction) $|t_{k+1} - t_k| \leq \lambda^{-8\epsilon_0} T_*$, it follows that

$$0 \leq T_{*;(\lambda)} \leq \lambda^{1-8\epsilon_0} T_*. \tag{151}$$

We now define the "rescaled" solution variables that we will analyze in the rest of the paper.

Definition 9.1 *(Rescaled quantities)* We define the array of scalar functions

$$\vec{\Psi}_{(\lambda)} = (\rho_{(\lambda)}, v^1_{(\lambda)}, v^2_{(\lambda)}, v^3_{(\lambda)}, s_{(\lambda)})$$

and the Cartesian components of the Σ_t-tangent vectorfields $\Omega_{(\lambda)}$ and $S_{(\lambda)}$ as follows, ($i = 1, 2, 3$):

$$\vec{\Psi}_{(\lambda)}(t,x) := \vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x), \qquad \Omega^i_{(\lambda)}(t,x) := \Omega^i(t_k + \lambda^{-1}t, \lambda^{-1}x),$$
$$S^i_{(\lambda)}(t,x) := S^i(t_k + \lambda^{-1}t, \lambda^{-1}x). \tag{152}$$

Similarly, we define the Cartesian components of the Σ_t-tangent vectorfield $\mathcal{C}_{(\lambda)}$ and the scalar function $\mathcal{D}_{(\lambda)}$ as follows:

$$\mathcal{C}^i_{(\lambda)} := \exp(-\rho_{(\lambda)})(\text{curl}\,\Omega_{(\lambda)})^i + \exp(-3\rho_{(\lambda)})c^{-2}(\vec{\Psi}_{(\lambda)})\frac{p_{;s}(\vec{\Psi}_{(\lambda)})}{\bar{\varrho}} S^a_{(\lambda)}\partial_a v^i_{(\lambda)}$$
$$- \exp(-3\rho_{(\lambda)})c^{-2}(\vec{\Psi}_{(\lambda)})\frac{p_{;s}(\vec{\Psi}_{(\lambda)})}{\bar{\varrho}}(\partial_a v^a_{(\lambda)})S^i_{(\lambda)}, \tag{153a}$$

$$\mathcal{D}_{(\lambda)} := \exp(-2\rho_{(\lambda)})\text{div}\,S_{(\lambda)} - \exp(-2\rho_{(\lambda)})S^a_{(\lambda)}\partial_a\rho_{(\lambda)}. \tag{153b}$$

Finally, we let $\mathbf{g}_{(\lambda)}$, $g_{(\lambda)}$, and $\mathbf{B}_{(\lambda)}$ be the "rescaled" tensorfields whose Cartesian components are as follows, ($\alpha, \beta = 0, 1, 2, 3$ and $i, j = 1, 2, 3$):

$$(\mathbf{g}_{(\lambda)})_{\alpha\beta}(t,x) := \mathbf{g}_{\alpha\beta}\left(\vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x)\right), \quad (g_{(\lambda)})_{ij}(t,x) := g_{ij}\left(\vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x)\right), \tag{154a}$$

$$\mathbf{B}^\alpha_{(\lambda)}(t,x) := \mathbf{B}^\alpha\left(\vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x)\right). \tag{154b}$$

Remark 9.1 *(Remarks on the rescaling).* Note that the slab $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ for $\vec{\Psi}_{(\lambda)}(t,x)$ corresponds to the slab $[t_k, t_{k+1}] \times \mathbb{R}^3$ for $\vec{\Psi}(t,x)$. The same remark applies for the other rescaled quantities.

Note also that when we are controlling the rescaled quantities such as $\vec{\Psi}_{(\lambda)}$, the hypersurface that we denote by "Σ_t" in Sects. 9–11 corresponds to the hypersurface $\Sigma_{t_k+\lambda^{-1}t}$ for the non-rescaled quantities, which appear throughout Sects. 3–8.

Remark 9.2 Note that $S^i_{(\lambda)} \neq \partial_i s_{(\lambda)}$, but rather $S^i_{(\lambda)} = \lambda\partial_i s_{(\lambda)}$. This is merely a reflection of our choice of how to keep track of powers of λ in the equations and estimates. Similarly, we have $\Omega^i_{(\lambda)} = \lambda\frac{(\text{curl}\,v_{(\lambda)})^i}{\exp\rho_{(\lambda)}}$. We clarify that although we use the relationships $S^i_{(\lambda)} = \lambda \partial_i s_{(\lambda)}$ and $\Omega^i_{(\lambda)} = \lambda \frac{(\mathrm{curl}\, v_{(\lambda)})^i}{\exp \rho_{(\lambda)}}$ to derive the equations of Proposition 9.1, when we derive PDE estimates for solutions to these equations, we generally do not need these relationships; that is, for estimates, we generally treat $S_{(\lambda)}$, $s_{(\lambda)}$, $\Omega_{(\lambda)}$, $v_{(\lambda)}$, and $\rho_{(\lambda)}$ as if they were independent quantities.

9.1.2 The radius *R*

For any $t \in [0, T_{*;(\lambda)}]$, $p \in \Sigma_t$, and $r > 0$, let $B_r(p)$ denote the Euclidean ball of radius r in Σ_t centered at p and let $B_{r;g_{(\lambda)}(t,\cdot)}(p)$ denote the metric ball, with respect to the rescaled Riemannian metric $g_{(\lambda)}(t, \cdot)$, of radius r in Σ_t centered at p. The statement of Theorem 11.3 refers to a Euclidean radius R, which we now define. Specifically, in the rest of the article, R denotes a fixed number chosen such that

$$0 < R < 1, \tag{155a}$$

$$B_R(p) \subset B_{1/2;g_{(\lambda)}(t,\cdot)}(p), \qquad \forall t \in [0, T_{*;(\lambda)}] \text{ and } \forall p \in \Sigma_t. \tag{155b}$$

The existence of such an R (one that is independent of λ) is guaranteed by the formula (10) (which in particular shows that $g_{(\lambda)}(t, \cdot)$ is equal to c^{-2} times the Euclidean metric on Σ_t, with c the speed of sound) and the fact that, by virtue of the bootstrap assumption (40), c is uniformly bounded from above and below by positive constants.

9.2 The rescaled compressible Euler equations

In the next proposition, we provide the equations verified by the rescaled quantities. We omit the simple proof, which follows from scaling considerations.

Proposition 9.1 (The rescaled geometric wave-transport formulation of the compressible Euler equations). *For solutions to Proposition 1.1, the rescaled quantities defined in Sect. 9.1 verify the following equations.*

Wave equations: *For rescaled wave variables* $\Psi_{(\lambda)} \in \{\rho_{(\lambda)}, v^1_{(\lambda)}, v^2_{(\lambda)}, v^3_{(\lambda)}, s_{(\lambda)}\}$, *we have:*

$$\hat{\square}_{\mathbf{g}_{(\lambda)}} \Psi_{(\lambda)} = \lambda^{-1} \mathscr{L}(\vec{\Psi}_{(\lambda)})[\vec{\mathcal{C}}_{(\lambda)}, \mathcal{D}_{(\lambda)}] + \mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial} \vec{\Psi}_{(\lambda)}, \boldsymbol{\partial} \vec{\Psi}_{(\lambda)}]. \tag{156}$$

Transport equations:

$$\mathbf{B}_{(\lambda)} \Omega^i_{(\lambda)} = \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\Omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial} \vec{\Psi}_{(\lambda)}], \quad \mathbf{B}_{(\lambda)} S^i_{(\lambda)} = \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial} \vec{\Psi}_{(\lambda)}]. \tag{157}$$

Transport div-curl system for the specific vorticity:

$$div\Omega_{(\lambda)} = \mathscr{L}(\vec{\Omega}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}], \tag{158a}$$

$$\mathbf{B}_{(\lambda)}\mathcal{C}^i_{(\lambda)} = \lambda\mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \partial\vec{\Omega}_{(\lambda)}] + \lambda\mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \partial\vec{S}_{(\lambda)}]$$
$$quad + \lambda\mathscr{Q}(\vec{\Psi}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)}] + \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{\Omega}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}]. \tag{158b}$$

Transport div-curl system for the entropy gradient:

$$\mathbf{B}_{(\lambda)}\mathcal{D}_{(\lambda)} = \lambda\mathscr{Q}(\vec{\Psi}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \partial\vec{S}_{(\lambda)}] + \lambda\mathscr{Q}(\vec{\Psi}_{(\lambda)}, \vec{S}_{(\lambda)})[\boldsymbol{\partial}\vec{\Psi}_{(\lambda)}, \boldsymbol{\partial}\vec{\Psi}_{(\lambda)}]$$
$$+ \mathscr{L}(\vec{\Psi}_{(\lambda)}, \vec{S}_{(\lambda)})[\partial\vec{\Omega}_{(\lambda)}], \tag{159a}$$

$$(curlS_{(\lambda)})^i = 0. \tag{159b}$$

9.3 Key notational remark and the mixed spacetime norm bootstrap assumptions for the rescaled quantities

For notational convenience, in the remainder of the article, we drop the sub- and super-scripts "(λ)" introduced in Sect. 9.1, except for the rescaled time $T_{*,(\lambda)}$. That is, we write $\vec{\Psi}$ in place of $\vec{\Psi}_{(\lambda)}$, $\mathbf{g}$ in place of $\mathbf{g}_{(\lambda)}$, $\mathbf{g}_{\alpha\beta}(t,x)$ in place of $\mathbf{g}_{\alpha\beta}\left(\vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x)\right)$, etc. Nonetheless, our analysis will properly take into account the explicit factors of λ on the RHSs of the equations of Proposition 9.1.

9.4 $\mathcal{M}$, the point z, the eikonal function, and construction of the geometric coordinates

Let $\mathcal{M} := [0, T_{*;(\lambda)}] \times \mathbb{R}^3 \subset \mathbb{R}^{1+3}$ denote the slab on which the rescaled quantities of Sect. 9.1.1 are defined. In the rest of the paper, we will construct various geometric quantities and derive estimates on various subsets of $\mathcal{M}$.

The proof of Theorem 11.3 fundamentally relies on the *acoustic geometry*, that is, a solution u to the eikonal equation (where under the conventions of Sect. 9.3, "$\mathbf{g}$" denotes the rescaled metric):

$$(\mathbf{g}^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0. \tag{160}$$

Following the setup used in [54], we will construct u by patching an "interior solution" with an "exterior solution." More precisely, the results of Sects. 9.4.1–9.4.2 will yield an eikonal function u defined in subsets $\widetilde{\mathcal{M}} \subset \mathcal{M}$, which we will define to be the union of an interior region and an exterior region: $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}^{(Int)} \cup \widetilde{\mathcal{M}}^{(Ext)}$. Moreover, an exercise in Taylor expansions, omitted here, yields that the solution u is smooth in $\widetilde{\mathcal{M}}$ away from the cone-tip axis (which is a curve in $\widetilde{\mathcal{M}}^{(Int)}$ that we define in Sect. 9.4.1).

Throughout Sects. 9 and 10, $\mathbf{z}$ denotes a fixed (but arbitrary) point in Σ_0 (where here, "Σ_0" corresponds to the hypersurface that we denoted by "Σ_{t_k}" in Sects. 3–9)

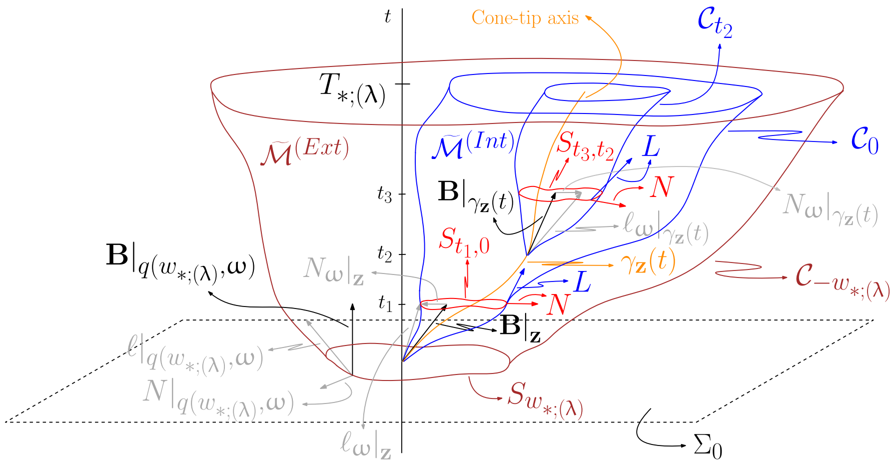

Fig. 2 The interior and exterior regions and related geometric constructions in the case $\mathbf{z} := \mathbf{0}$

that forms the bottom tip of $\widetilde{\mathcal{M}}^{(Int)}$. The point $\mathbf{z}$ will vary when one carries out the partition of unity argument that allows for a reduction of the proof of the desired Strichartz estimate (more precisely, the frequency localized estimates provided by Theorem 7.2) to that of Proposition 11.1. More precisely, the proof of Theorem 7.2 relies on partitioning the full slab $\mathcal{M}$ into various "localized" subsets of type $\widetilde{\mathcal{M}}$ and proving dispersive decay estimates on subsets of the $\widetilde{\mathcal{M}}$ for solutions φ to the linear wave equation $\square_{\mathbf{g}(\vec{\Psi})}\varphi = 0$. The spatially localized dispersive decay estimates (which correspond to a fixed $\widetilde{\mathcal{M}}$ and thus a fixed $\mathbf{z}$) are provided by Proposition 11.1. We refer readers to Sect. 11.3 for further discussion on the various standard reductions of the proof of the Strichartz estimates to spatially localized dispersive estimates (and ultimately to the proof of control over the growth rate of a conformal energy, provided by Theorem 11.3). We also remark that the varying of $\mathbf{z}$ during the partition of unity argument is a minor issue in the sense that estimates that we derive in Sects. 9 and 10 are independent of $\mathbf{z}$, and all of the constants and parameters in our analysis can be chosen to be independent of $\mathbf{z}$.

We provide a figure, Fig. 2, that exhibits many of the geometric objects that we will construct in Sect. 9.4. In the figure, for convenience, we have set $\mathbf{z}$ to be equal to the origin in Σ_0.

9.4.1 The interior solution emanating from the cone-tip axis and the region $\widetilde{\mathcal{M}}^{(Int)}$

We let $\gamma_{\mathbf{z}} = \gamma_{\mathbf{z}}(t)$ denote the future-directed integral curve of the vectorfield[33] $\mathbf{B}$ emanating from the point $\mathbf{z}$, i.e., $\gamma_{\mathbf{z}}(0) = \mathbf{z} \in \Sigma_0$. We refer to $\{\gamma_{\mathbf{z}}(t)\}_{t\in[0,T_{*;(\lambda)}]}$ as the *cone-tip axis*. Let $q = q(t) := \gamma_{\mathbf{z}}(t)$ be a point on the cone-tip axis. Let $\ell \in T_q\mathcal{M}$ be a null vector normalized by $\mathbf{g}|_q(\ell, \mathbf{B}|_q) = -1$. We denote the set of

[33] We again stress that by the conventions of Sect. 9.3, in the rest of the paper, we use the notation $\mathbf{B}^\alpha(t,x)$ to denote $\mathbf{B}^\alpha\left(\vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x)\right)$ and $\mathbf{g}_{\alpha\beta}(t,x)$ to denote $\mathbf{g}_{\alpha\beta}\left(\vec{\Psi}(t_k + \lambda^{-1}t, \lambda^{-1}x)\right)$.

all of these normalized null vectors $\ell \in T_q\mathcal{M}$ by $\mathscr{N}_q$. We now consider the case $q = \mathbf{z} \in \Sigma_0$. It is straightforward to see that $\mathscr{N}_{\mathbf{z}}$ is diffeomorphic to $\mathbb{S}^2$; we therefore fix a diffeomorphism from $\mathbb{S}^2$ onto $\mathscr{N}_{\mathbf{z}}$. For each $\omega \in \mathbb{S}^2$, we let $\ell_\omega \in \mathscr{N}_{\mathbf{z}}$ denote the corresponding (via the diffeomorphism) null vector. We will use parallel transport to construct a diffeomorphism from $\mathscr{N}_{\mathbf{z}}$ onto $\mathscr{N}_{\gamma_{\mathbf{z}}(t)}$. Ultimately, this diffeomorphism will allow us, upon pre-composing it with the fixed diffeomorphism $\omega \to \ell_\omega$ from $\mathbb{S}^2$ onto $\mathscr{N}_{\mathbf{z}}$ and post-composing it with a null geodesic flow,[34] to construct angular coordinates ω that are defined[35] in $\widetilde{\mathcal{M}}^{(Int)}$; see just below Eq. (162b).

To initiate the construction of the diffeomorphism from $\mathscr{N}_{\mathbf{z}}$ onto $\mathscr{N}_{\gamma_{\mathbf{z}}(t)}$, for each $\omega \in \mathbb{S}^2$, we define the vector $N_\omega \in T_{\mathbf{z}}\mathcal{M}$ as follows: $N_\omega := \ell_\omega - \mathbf{B}|_{\mathbf{z}}$. Considering the relations $\mathbf{g}|_{\mathbf{z}}(\ell_\omega, \mathbf{B}|_{\mathbf{z}}) = -1$ and $\mathbf{g}|_{\mathbf{z}}(\mathbf{B}|_{\mathbf{z}}, \mathbf{B}|_{\mathbf{z}}) = -1$, we find that $\mathbf{g}|_{\mathbf{z}}(\mathbf{B}|_{\mathbf{z}}, N_\omega) = 0$. Considering also that $\mathbf{g}|_{\mathbf{z}}(\ell_\omega, \ell_\omega) = 0$, we find that $\mathbf{g}|_{\mathbf{z}}(N_\omega, N_\omega) = 1$. Thus, $N_\omega \in UT_{\mathbf{z}}\Sigma_0$, where $UT_{\mathbf{z}}\Sigma_0$ denotes the g-unit tangent bundle of Σ_0 at $\mathbf{z}$, and g is the rescaled first fundamental form of Σ_0. It is straightforward to see that the map $\ell_\omega \to \ell_\omega - \mathbf{B}|_{\mathbf{z}}$ defines a diffeomorphism from $\mathscr{N}_{\mathbf{z}}$ onto $UT_{\mathbf{z}}\Sigma_0$. To propagate N_ω along the cone-tip axis, we solve the parallel transport equation[36] $\mathbf{D}_{\mathbf{B}}N_\omega = 0$, where $\mathbf{D}$ is the Levi-Civita connection of the rescaled spacetime metric $\mathbf{g}$. In Cartesian coordinates, for each $N_\omega^\alpha|_{\mathbf{z}} \in \mathscr{N}_{\mathbf{z}}$, the parallel transport equation takes the form of the following transport equation system, which is *linear* in the scalar Cartesian component functions N_ω^α:

$$\frac{d}{dt}N_\omega^\alpha + \mathbf{\Gamma}_{\kappa\ \lambda}^{\ \alpha}\mathbf{B}^\kappa N_\omega^\lambda = 0, \tag{161}$$

where the initial conditions for (161) are $N_\omega^\alpha|_{\mathbf{z}}$, $\mathbf{\Gamma}_{\alpha\ \beta}^{\ \nu} = \frac{1}{2}(\mathbf{g}^{-1})^{\sigma\nu}\left\{\partial_\alpha\mathbf{g}_{\sigma\beta} + \partial_\beta\mathbf{g}_{\alpha\sigma} - \partial_\sigma\mathbf{g}_{\alpha\beta}\right\}$ are the Cartesian Christoffel symbols of the rescaled metric $\mathbf{g}$, and it is understood that all quantities are evaluated along $\gamma_{\mathbf{z}}(t)$, e.g., $N_\omega^\alpha = N_\omega^\alpha \circ \gamma_{\mathbf{z}}(t)$ and $\mathbf{B}^\kappa = \mathbf{B}^\kappa \circ \vec{\Psi} \circ \gamma_{\mathbf{z}}(t)$, with $\vec{\Psi}$ the rescaled solution. It is straightforward to show, based on the normalization condition $\mathbf{g}|_{\gamma_{\mathbf{z}}(t)}(\mathbf{B}|_{\gamma_{\mathbf{z}}(t)}, \mathbf{B}|_{\gamma_{\mathbf{z}}(t)}) = -1$, (161), and the initial conditions $\mathbf{g}|_{\mathbf{z}}(\mathbf{B}|_{\mathbf{z}}, N_\omega|_{\mathbf{z}}) = 0$ and $\mathbf{g}|_{\mathbf{z}}(N_\omega|_{\mathbf{z}}, N_\omega|_{\mathbf{z}}) = 1$, that for $t \in [0, T_{*;(\lambda)}]$, the solution $N_\omega|_{\gamma_{\mathbf{z}}(t)}$ to equation (161) is an element of $UT_{\gamma_{\mathbf{z}}(t)}\Sigma_t$, where $UT_{\gamma_{\mathbf{z}}(t)}\Sigma_t$ denotes the g-unit tangent bundle of Σ_t at $\gamma_{\mathbf{z}}(t)$, and g is the rescaled first fundamental form of Σ_t. That is, we have $\mathbf{g}|_{\gamma_{\mathbf{z}}(t)}(\mathbf{B}|_{\gamma_{\mathbf{z}}(t)}, N_\omega|_{\gamma_{\mathbf{z}}(t)}) = 0$ and $\mathbf{g}|_{\gamma_{\mathbf{z}}(t)}(N_\omega|_{\gamma_{\mathbf{z}}(t)}, N_\omega|_{\gamma_{\mathbf{z}}(t)}) = 1$. In particular, $N_\omega|_{\gamma_{\mathbf{z}}(t)}$ is tangent to Σ_t at $\gamma_{\mathbf{z}}(t)$. From these relations and arguments similar to the ones given above, we find that $\ell_\omega|_{\gamma_{\mathbf{z}}(t)} := \mathbf{B}|_{\gamma_{\mathbf{z}}(t)} + N_\omega|_{\gamma_{\mathbf{z}}(t)} \in \mathscr{N}_{\gamma_{\mathbf{z}}(t)}$. Similar arguments that take into account standard ODE existence and uniqueness theory[37] for the equation (161) yield that the

34 The null curves, whose Cartesian components are solutions to the ODE system (162a), are not affine-parameterized.

35 More precisely, the angular coordinate functions (ω^1, ω^2) are uniquely defined away from the cone-tip axis, while each point on the cone-tip axis is associated with an entire $\mathbb{S}^2$ manifold worth of angles (i.e., the same degeneracy that occurs at the origin in $\mathbb{R}^3$ under the standard Euclidean spherical coordinates).

36 This is in fact parallel transport along geodesics since $\mathbf{D}_{\mathbf{B}}\mathbf{B} = 0$; this latter identity is straightforward to derive using that $\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1$ and the fact that $[\mathbf{B}, Z]$ is Σ_t-tangent (hence $\mathbf{g}$-orthogonal to $\mathbf{B}$) whenever Z is Σ_t-tangent.

37 Here we are using our qualitative assumption that the fluid solution is smooth.

map $N_\omega|_{\mathbf{z}} \to N_\omega|_{\gamma_{\mathbf{z}}(t)}$ is a diffeomorphism from $UT_{\mathbf{z}}\Sigma_0$ onto $UT_{\gamma_{\mathbf{z}}(t)}\Sigma_t$. Considering also that for each for $t \in [0, T_{*;(\lambda)}]$, the map $N_\omega|_{\gamma_{\mathbf{z}}(t)} \to \mathbf{B}|_{\gamma_{\mathbf{z}}(t)} + N_\omega|_{\gamma_{\mathbf{z}}(t)}$ (where $N_\omega|_{\gamma_{\mathbf{z}}(t)}$ is the solution to (161)) defines a diffeomorphism from $UT_{\gamma_{\mathbf{z}}(t)}\Sigma_t$ onto $\mathscr{N}_{\gamma_{\mathbf{z}}(t)}$, we conclude that the map $\ell_\omega|_{\mathbf{z}} \to \mathbf{B}|_{\gamma_{\mathbf{z}}(t)} + N_\omega|_{\gamma_{\mathbf{z}}(t)}$ is the desired diffeomorphism from $\mathscr{N}_{\mathbf{z}}$ onto $\mathscr{N}_{\gamma_{\mathbf{z}}(t)}$.

Next, for $u \in [0, T_{*;(\lambda)}]$, we let $q = q(u) := \gamma_{\mathbf{z}}(u)$ be the unique point[38] on the cone-tip axis with Cartesian component $q^0 = u$. Let $\omega \in \mathbb{S}^2$, and let $\ell_\omega := \mathbf{B}|_{\gamma_{\mathbf{z}}(u)} + N_\omega|_{\gamma_{\mathbf{z}}(u)} \in \mathscr{N}_{\gamma_{\mathbf{z}}(u)}$ denote the corresponding null vector that we constructed in the previous paragraph. We now let $\Upsilon_{u;\omega} = \Upsilon_{u;\omega}(t)$ be the null geodesic curve emanating from $q(u)$ with initial velocity ℓ_ω, parameterized by t (see Footnote 34), that is, $\Upsilon^0_{u;\omega}(t) = t$. Introducing the notation $\dot{\Upsilon}^\alpha_{u;\omega} := \frac{d}{dt}\Upsilon^\alpha_{u;\omega}$ and $\ddot{\Upsilon}^\alpha_{u;\omega} := \frac{d^2}{dt^2}\Upsilon^\alpha_{u;\omega}$, we note that standard arguments[39] yield that the four scalar functions $\{\Upsilon^\alpha_{u;\omega}(t)\}_{\alpha=0,1,2,3}$ are the solution to the following ODE system initial value problem (Footnote 37 also applies here) with data given at $t = u$, where on RHS (162a), $\mathcal{L}_{\mathbf{B}}$ denotes Lie differentiation with respect to $\mathbf{B}$:

$$\ddot{\Upsilon}^\alpha_{u;\omega}(t) = -\mathbf{\Gamma}_{\kappa\ \lambda}^{\ \alpha}|_{\Upsilon_{u;\omega}(t)}\dot{\Upsilon}^\kappa_{u;\omega}(t)\dot{\Upsilon}^\lambda_{u;\omega}(t) + \frac{1}{2}[\mathcal{L}_{\mathbf{B}}\mathbf{g}]_{\kappa\lambda}|_{\Upsilon_{u;\omega}(t)}\left(\dot{\Upsilon}^\kappa_{u;\omega}(t) - \mathbf{B}^\kappa|_{\Upsilon_{u;\omega}(t)}\right)\left(\dot{\Upsilon}^\lambda_{u;\omega}(t) - \mathbf{B}^\lambda|_{\Upsilon_{u;\omega}(t)}\right)\dot{\Upsilon}^\alpha_{u;\omega}(t), \tag{162a}$$

$$\Upsilon^\alpha_{u;\omega}(u) = q^\alpha(u) = \gamma^\alpha_{\mathbf{z}}(u), \qquad \dot{\Upsilon}^\alpha_{u;\omega}(u) = \ell^\alpha_\omega. \tag{162b}$$

We are now able to extend the angular coordinates by declaring that ω is constant along the null geodesic curve $t \to \Upsilon_{u;\omega}(t)$. Next, given any fixed $t \in [u, T_{*;(\lambda)}]$, we define the truncated cone

$$\mathcal{C}^t_u := \bigcup_{\tau\in[u,t],\,\omega\in\mathbb{S}^2} \Upsilon_{u;\omega}(\tau). \tag{163}$$

We then define a function u by the requirement that its level sets are precisely the cones (163), that is, along $\mathcal{C}^{T_{*;(\lambda)}}_{u'}$, the function u takes the value u'.

We then set

$$\widetilde{\mathcal{M}}^{(Int)} := \bigcup_{u\in[0,T_{*;(\lambda)}]} \mathcal{C}^{T_{*;(\lambda)}}_u. \tag{164}$$

At times, we will use the alternate notation

$$\mathcal{C}_u := \mathcal{C}^{T_{*;(\lambda)}}_u. \tag{165}$$

38 It is unique since $\mathbf{B}t = 1$.

39 (162a) is equivalent to equation (199b) for $\mathbf{D}_L L^\alpha$, where $\dot{\Upsilon}^\alpha_{u;\omega}$ can be identified with L^α, $\dot{\Upsilon}^\alpha_{u;\omega} - \mathbf{B}^\alpha$ can be identified with N^α, and $\frac{1}{2}[\mathcal{L}_{\mathbf{B}}\mathbf{g}]_{\kappa\lambda}(\dot{\Upsilon}^\kappa_{u;\omega} - \mathbf{B}^\kappa)(\dot{\Upsilon}^\lambda_{u;\omega} - \mathbf{B}^\lambda)\dot{\Upsilon}^\alpha_{u;\omega}$ can be identified with $-k_{NN}L^\alpha$.

As is described, for example, in [7], this construction provides a solution of (160) in the region $\widetilde{\mathcal{M}}^{(Int)}$ depicted in Fig. 2. Note that by construction, we have

$$u(\gamma_{\mathbf{z}}(t)) = t, \qquad \mathbf{B}[u(\gamma_{\mathbf{z}}(t))] = 1. \tag{166}$$

In total, we have constructed geometric coordinates (t, u, ω) in $\widetilde{\mathcal{M}}^{(Int)}$. More precisely, standard ODE theory yields that the map $(t, u, \omega) \to \left(\Upsilon^0_{u;\omega}(t), \Upsilon^1_{u;\omega}(t), \Upsilon^2_{u;\omega}(t), \Upsilon^3_{u;\omega}(t)\right)$ is smooth on $\{(t, u, \omega) \mid u \in [0, T_{*;(\lambda)}], t \in [u, T_{*;(\lambda)}], \omega \in \mathbb{S}^2\}$ and locally injective away from points with $t = u$ (which correspond to the cone-tip axis); note that here we are identifying $\Upsilon^\alpha_{u;\omega}(t)$ with the Cartesian coordinate x^α. Moreover, the continuity argument mentioned in Sect. 9.5 guarantees that in fact, this map is a global diffeomorphism from $\{(t, u, \omega) \mid u \in [0, T_{*;(\lambda)}], t \in [u, T_{*;(\lambda)}], \omega \in \mathbb{S}^2\}\backslash\{(u, u, \omega) \mid u \in [0, T_{*;(\lambda)}], \omega \in \mathbb{S}^2\}$ onto its image, i.e., onto $\widetilde{\mathcal{M}}^{(Int)}$ minus the cone-tip axis $\{\gamma_{\mathbf{z}}(t)\}_{t\in[0,T_{*;(\lambda)}]}$; see also Proposition 10.7 for a quantitative proof that the null curves $t \to \Upsilon_{u;\omega}(t)$ corresponding to distinct values of u and ω remain separated.[40]

9.4.2 The exterior solution and the region $\widetilde{\mathcal{M}}^{(Ext)}$

Let $\mathbf{z}$ be the point in Σ_0 from Sect. 9.4.1, i.e., the point $\gamma_{\mathbf{z}}(0)$, at which $t = u = 0$. The same arguments leading to [54, Proposition 4.3] guarantee that for T_* sufficiently small, there is a neighborhood $\mathscr{O}$ in Σ_0 contained in the metric ball $B_{T_{*;(\lambda)}}(\mathbf{z}, g)$ (with respect to the rescaled first fundamental form g of Σ_0) of radius $T_{*;(\lambda)}$ centered at $\mathbf{z}$ such that $\mathscr{O}$ can be foliated with the level sets of a function w on Σ_0, defined for $0 \leq w \leq w_{*;(\lambda)} := \frac{4}{5}T_{*;(\lambda)}$, where, away from $\mathbf{z}$, w is smooth and has level sets S_w diffeomorphic to $\mathbb{S}^2$, while $S_0 = \{\mathbf{z}\}$. To obtain suitable control of the geometry, we require w to have a variety of crucial properties, especially (280); see Proposition 9.8 for the existence of a function w with the desired properties.

Let $\omega \in \mathbb{S}^2$ be as in Sect. 9.4.1, let $\ell_\omega \in T_{\mathbf{z}}\mathcal{M}$ be the corresponding null vector, and let $N_\omega = \ell_\omega - \mathbf{B}|_{\mathbf{z}}$ be the corresponding element of $UT_{\mathbf{z}}\Sigma_0$. Let ∇ denote the Levi-Civita connection of g and let $a := |\nabla w|_g^{-1}$ denote the lapse, where $|\nabla w|_g = \sqrt{(g^{-1})^{cd}\partial_c w \partial_d w}$. In our forthcoming analysis, we will have $a(\mathbf{z}) = 1$ and $a \approx 1$; see Proposition 9.8. Let N be the outward g-unit normal to S_w in Σ_0, i.e., $N^i := a(g^{-1})^{ic}\partial_c w$, $N^0 = 0$, and $g_{cd}N^c N^d = 1$. Each fixed integral curve of N can be extended[41] to a smooth curve emanating from $\mathbf{z}$. More precisely, for each vector $N_\omega \in UT_{\mathbf{z}}\Sigma_0$, there is a unique integral curve $\Phi_\omega : [0, w_{*;(\lambda)}] \to \Sigma_0$ of aN parameterized by w (i.e., $\dot{\Phi}^i_\omega(w) = [aN^i]\circ\Phi_\omega(w)$, with a the lapse, where $\dot{\Phi}^i_\omega(w) = \frac{\partial}{\partial w}\Phi^i_\omega(w)$) that emanates from $\mathbf{z}$ with $\Phi_\omega(0) = \mathbf{z}$ and $\dot{\Phi}_\omega(0) = N_\omega$ (here we have used that $a(\mathbf{z}) = 1$). This yields a diffeomorphism from $\mathbb{S}^2$ to each S_w for

40 By "separated," in $\widetilde{\mathcal{M}}^{(Int)}$, we mean, of course, away from the cone-tip axis.

41 In particular, in the proof of Lemma 10.6, we show that along Σ_0, for $i = 1, 2, 3$, $\|\frac{\partial}{\partial u}N^i\|_{L^2_u L^\infty_\omega} < \infty$, where this norm is defined in Sect. 9.10; this implies the extendibility of each integral curve of N to $\mathbf{z}$, where $\mathbf{z}$ is the point at which $u = 0$.

$0 < w \leq w_{*;(\lambda)}$, defined such that ω is constant along the integral curve $w \to \Phi_\omega(w)$. In particular, if $\{\omega^A\}_{A=1,2}$ are local angular coordinates on $\mathbb{S}^2$, then for each fixed w with $0 < w \leq w_{*;(\lambda)}$, the map $\omega \to \Phi_\omega(w)$ yields angular coordinates $\{\omega^A\}_{A=1,2}$ on S_w. It is straightforward to see that on $\cup_{0<w\leq w_{*;(\lambda)}} S_w$, we have the vectorfield identity (where $\frac{\partial}{\partial w}$ denotes partial differentiation at fixed ω)

$$\frac{\partial}{\partial w} = aN, \tag{167}$$

and that the rescaled first fundamental form of Σ_0, denoted by g, can be expressed relative to the coordinates (w, ω) as follows:

$$g = a^2 dw \otimes dw + \not{g}\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) d\omega^A \otimes d\omega^B, \tag{168}$$

where $\not{g}$ is the Riemannian metric induced on S_w by g.

In view of the constructions provided above, to each point $q \in \cup_{0<w\leq w_{*;(\lambda)}} S_w \subset \Sigma_0$, we can associate the geometric coordinates $(0, w, \omega)$ (where "0" is the time coordinate). In particular, these points $q = q(w, \omega)$ are parameterized by the coordinates $(w, \omega) \in [0, w_{*;(\lambda)}] \times \mathbb{S}^2$. We then define the vector $\ell_{q(w,\omega)} := \mathbf{B}|_{q(w,\omega)} + N|_{q(w,\omega)} \in T_{q(w,\omega)}\mathcal{M}$. Since $\mathbf{g}|_{q(w,\omega)}(\mathbf{B}|_{q(w,\omega)}, \mathbf{B}|_{q(w,\omega)}) = -1$, $\mathbf{g}|_{q(w,\omega)}(\mathbf{B}|_{q(w,\omega)}, N|_{q(w,\omega)}) = 0$, and $\mathbf{g}|_{q(w,\omega)}(N|_{q(w,\omega)}, N|_{q(w,\omega)}) = 1$, it follows that $\mathbf{g}|_{q(w,\omega)}(\ell_{q(w,\omega)}, \ell_{q(w,\omega)}) = 0$, i.e., $\ell_{q(w,\omega)}$ is null. Next, we construct the null geodesic $\Upsilon_{q(w,\omega)} = \Upsilon_{q(w,\omega)}(t)$ by solving the ODE (162a) with initial conditions $\Upsilon^\alpha_{q(w,\omega)}(0) = q^\alpha(w, \omega)$ and $\dot{\Upsilon}^\alpha_{q(w,\omega)}(0) = \ell^\alpha_{q(w,\omega)}$. For each fixed $w \in [0, w_{*;(\lambda)}]$, the set $\{\Upsilon_{q(w,\omega)}(t) \mid (t, \omega) \in [0, T_{*;(\lambda)}] \times \mathbb{S}^2\}$ is a portion of a $\mathbf{g}$-null cone. We define the function u by declaring that along this null cone portion, it takes on the value $-w$. Thus, with $\mathcal{C}^t_u$ denoting the level set portion contained in $[0, t] \times \mathbb{R}^3$, we have $\mathcal{C}^t_u = \{\Upsilon_{q(-u,\omega)}(\tau) \mid (\tau, \omega) \in [0, t] \times \mathbb{S}^2\}$. As we do in the interior region, we sometimes use the alternate notation $\mathcal{C}_u := \mathcal{C}_u^{T_{*;(\lambda)}}$. We then set

$$\widetilde{\mathcal{M}}^{(Ext)} := \bigcup_{u\in[-w_{*;(\lambda)},0]} \mathcal{C}_u^{T_{*;(\lambda)}}. \tag{169}$$

This procedure yields a function u defined in the region $\widetilde{\mathcal{M}}^{(Ext)}$ depicted in Fig. 2. It is a standard result that u is a solution to the eikonal equation (160) in $\widetilde{\mathcal{M}}^{(Ext)}$. Finally, we extend the angular coordinates to $\widetilde{\mathcal{M}}^{(Ext)}$ by declaring that ω is constant along the null geodesic curve $t \to \Upsilon_{q(w,\omega)}(t)$. In total, we have constructed geometric coordinates (t, u, ω) in $\widetilde{\mathcal{M}}^{(Ext)}$.

9.4.3 Acoustical metric and first fundamental forms

We refer to Sect. 1.1.2 for discussion of the acoustical metric $\mathbf{g}$ and the first fundamental form g of Σ_t. We now define $\not{g}$ to be the first fundamental form of $S_{t,u} := \mathcal{C}_u \cap \Sigma_t$, that

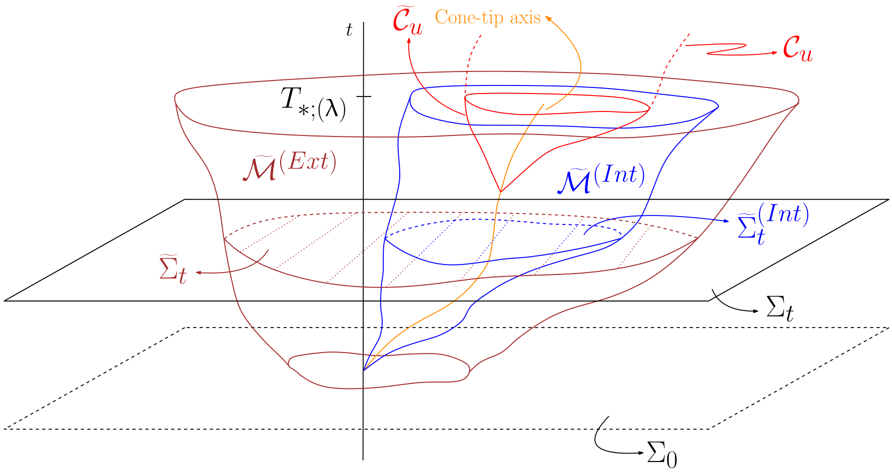

Fig. 3 Depiction of various subsets of spacetime in the case $\mathbf{z} := \mathbf{0}$

is, the Riemannian metric induced on $S_{t,u}$ by $\mathbf{g}$. We again clarify that we are working under the conventions of Sect. 9.3.

9.5 Geometric subsets of spacetime and the containment $B_R(\gamma_{\mathbf{z}}(1)) \subset \mathrm{Int}\widetilde{\Sigma}_1$

In the rest of the paper, we denote $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}^{(Int)} \cup \widetilde{\mathcal{M}}^{(Ext)}$. From the constructions in Sects. 9.4.1–9.4.2, it follows that

$$\widetilde{\mathcal{M}} = \bigcup_{u \in [-w_{*;(\lambda)}, T_{*;(\lambda)}]} \mathcal{C}_u^{T_{*;(\lambda)}}, \tag{170}$$

where $w_{*;(\lambda)} = \frac{4}{5} T_{*;(\lambda)}$. We also define a truncated version of $\widetilde{\mathcal{M}}^{(Int)}$, namely $\widetilde{\mathcal{M}}_1^{(Int)}$, as follows:

$$\widetilde{\mathcal{M}}_1^{(Int)} := \widetilde{\mathcal{M}}^{(Int)} \cap \left([1, T_{*;(\lambda)}] \times \mathbb{R}^3 \right). \tag{171}$$

We also define

$$\widetilde{\Sigma}_t := \Sigma_t \cap \widetilde{\mathcal{M}}, \qquad \widetilde{\Sigma}_t^{(Int)} := \Sigma_t \cap \widetilde{\mathcal{M}}^{(Int)}, \qquad \widetilde{\mathcal{C}}_u := \mathcal{C}_u \cap \widetilde{\mathcal{M}}. \tag{172}$$

See Fig. 3 for a depiction of these sets.

For the same reasons given in [54, Section 4], if T_* is small[42] (where the required smallness is controlled by our bootstrap assumptions and our assumptions on the data), then the results of Sects. 9.4.1–9.4.2 yield a complete system of *geometric coordinates* (t, u, ω), which are defined on $\widetilde{\mathcal{M}}$ and non-degenerate away from the cone-tip axis;

[42] Note that, as is explained on [54, pg. 24], it is only T_*, and not $T_{*;(\lambda)}$, that is required to be small; once we have fixed T_*, we have $T_{*;(\lambda)} \to \infty$ as $\lambda \to \infty$.

the proof is based on a continuity argument involving the bootstrap assumptions and the bounds (288a)–(297b) proved below; see also the proof of [50, Theorem 1.2] and [19,23] for additional details. In particular, for $u \in [-w_{*;(\lambda)}, T_{*;(\lambda)}]$ and $t \in [[u]_+, T_{*;(\lambda)}]$ such that[43] $t \neq u$ (where $[u]_+ := \max\{0, u\}$), the sets

$$S_{t,u} := \mathcal{C}_u \cap \Sigma_t \tag{173}$$

are embedded submanifolds that are diffeomorphic to $\mathbb{S}^2$, equipped with the (local) angular coordinates (ω^1, ω^2). We also note that

$$\widetilde{\mathcal{M}} = \bigcup_{u \in [-w_{*;(\lambda)}, T_{*;(\lambda)}], t \in [[u]_+, T_{*;(\lambda)}]} S_{t,u}, \tag{174a}$$

$$\widetilde{\mathcal{M}}^{(Int)} = \bigcup_{u \in [0, T_{*;(\lambda)}], t \in [u, T_{*;(\lambda)}]} S_{t,u}, \qquad \widetilde{\mathcal{M}}^{(Ext)} = \bigcup_{u \in [-w_{*;(\lambda)}, 0], t \in [0, T_{*;(\lambda)}]} S_{t,u}, \tag{174b}$$

$$\widetilde{\Sigma}_t^{(Int)} = \bigcup_{u \in [0, T_{*;(\lambda)}]} S_{t,u}. \tag{174c}$$

For future use, we also note that for the same reasons given on [54, page 25], based on (155b) and the estimate (297b) proved below, we have the following containments, where $B_R(\gamma_{\mathbf{z}}(1))$ denotes the Euclidean ball of radius R centered at $\gamma_{\mathbf{z}}(1)$ in $\widetilde{\Sigma}_1$ (with R is as in Sect. 9.1), and $B_{1/2;g(1,\cdot)}(\gamma_{\mathbf{z}}(1))$ is the metric ball of radius $1/2$ centered at $\gamma_{\mathbf{z}}(1)$ in $\widetilde{\Sigma}_1$ corresponding to the rescaled first fundamental form $g(1, \cdot)$:

$$B_R(\gamma_{\mathbf{z}}(1)) \subset B_{1/2;g}(\gamma_{\mathbf{z}}(1)) \subset \bigcup_{\frac{1}{3} \leq u \leq 1} S_{1,u} \subset \widetilde{\Sigma}_1^{(Int)}. \tag{175}$$

9.6 Geometric quantities constructed out of the eikonal function

We now define a collection of geometric quantities constructed out of u.

9.6.1 Geometric radial variable, null lapse, and the unit outward normal

We define the *geometric radial variable* $\tilde{r}$ as follows:

$$\tilde{r} = \tilde{r}(t, u) := t - u. \tag{176}$$

Since in $\widetilde{\mathcal{M}}$ we have that $t \in [0, T_{*;(\lambda)}]$ and $u \in [-w_{*;(\lambda)}, t]$, and since $w_{*;(\lambda)} := \frac{4}{5} T_{*;(\lambda)}$, it follows from (151) that

$$0 \leq \tilde{r} < 2T_{*;(\lambda)} = 2\lambda^{1-8\epsilon_0} T_*, \qquad -\frac{4}{5}\lambda^{1-8\epsilon_0} T_* \leq u \leq \lambda^{1-8\epsilon_0} T_*. \tag{177}$$

[43] Note that for $t \in [0, T_{*;(\lambda)}]$, $S_{t,t}$ is a single point on the cone tip axis.

Throughout the article, we will often silently use the inequalities in (177).

We define the *null lapse* b to be the following scalar function, where $|\nabla u|_g = \sqrt{(g^{-1})^{ab}\partial_a u \partial_b u}$:

$$b := \frac{1}{|\nabla u|_g}. \tag{178}$$

From (178), (168), and the fact that $u = -w$ along Σ_0, it follows that $b = a$ along Σ_0. Moreover, using (178), (10), (11), and (160), we see that

$$b = \frac{1}{\mathbf{B}u}. \tag{179}$$

Considering also (166), we see that for $t \in [0, T_{*;(\lambda)}]$, we have

$$b|_{\gamma_{\mathbf{z}}(t)} = 1, \tag{180}$$

where the curve $t \to \gamma_{\mathbf{z}}(t)$ is the cone-tip axis introduced in Sect. 9.4.1.

Let N denote the outward unit normal to $S_{t,u}$ in Σ_t, i.e., N is Σ_t-tangent, g-orthogonal to $S_{t,u}$, outward pointing, and normalized by $g(N, N) = 1$. From (178), it follows that

$$N^i = -b(g^{-1})^{ia}\partial_a u, \qquad Nu = -\frac{1}{b}. \tag{181}$$

9.6.2 Null frame and basic geometric constructions

We now define the following vectorfields:

$$L := \mathbf{B} + N, \qquad \underline{L} := \mathbf{B} - N. \tag{182}$$

Since $\mathbf{B}^0 = 1$ and $N^0 = 0$, it follows that

$$Lt = \underline{L}t = 1. \tag{183}$$

Moreover, from (10), (11), (178), (179), (181), and (182), we see that

$$L^\alpha = -b(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta u. \tag{184}$$

Since $\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1$, $\mathbf{g}(N, N) = 1$, and $\mathbf{g}(\mathbf{B}, N) = 0$, it follows that

$$\mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = 0, \qquad \mathbf{g}(L, \underline{L}) = -2. \tag{185}$$

In particular, (183) and (185) imply that L and $\underline{L}$ are future-directed and $\mathbf{g}$-null. Let now $\{e_A\}_{A=1,2}$ be a (locally-defined) $\mathbf{g}$-orthonormal frame on $S_{t,u}$, i.e., $\not{g}(e_A, e_B) = \delta_{AB}$, where δ_{AB} is the Kronecker delta. We note that since $\mathbf{B}$ and N are $\mathbf{g}$-orthogonal to $S_{t,u}$, it follows from (182) that $\mathbf{g}(L, e_A) = \mathbf{g}(\underline{L}, e_B) = 0$. We refer to

$$\{L, \underline{L}, e_1, e_2\} \tag{186}$$

as a *null frame*; see Fig. 1.

If $\boldsymbol{\xi}$ is a one-form, then $\boldsymbol{\xi}_L := \boldsymbol{\xi}_\alpha L^\alpha$, $\boldsymbol{\xi}_{\underline{L}} := \boldsymbol{\xi}_\alpha \underline{L}^\alpha$, and $\boldsymbol{\xi}_A := \boldsymbol{\xi}_\alpha e_A^\alpha$ denote contractions against the null frame elements. Similarly, if $\mathbf{X}$ is a vectorfield, then $\mathbf{X}_L := \mathbf{X}_\alpha L^\alpha$, $\mathbf{X}_{\underline{L}} := \mathbf{X}_\alpha \underline{L}^\alpha$, and $\mathbf{X}_A := \mathbf{X}_\alpha e_A^\alpha$. We use analogous notation to denote the components of higher order tensorfields as well as contractions against N, e.g., $\boldsymbol{\xi}_{AN} := \boldsymbol{\xi}_{\alpha\beta} e_A^\alpha N^\beta$.

It is straightforward to deduce from the above considerations that

$$(\mathbf{g}^{-1})^{\alpha\beta} = -\frac{1}{2} L^\alpha \underline{L}^\beta - \frac{1}{2} \underline{L}^\alpha L^\beta + (\not{g}^{-1})^{\alpha\beta}, \qquad (\not{g}^{-1})^{\alpha\beta} = \sum_{A=1,2} e_A^\alpha e_A^\beta. \tag{187}$$

Next, we define the $\mathbf{g}$-orthogonal projection $\not{\Pi}$ onto $S_{t,u}$ and the $\mathbf{g}$-orthogonal projection $\underline{\Pi}$ onto Σ_t to be, respectively, the following type $\binom{1}{1}$ tensorfields, where $\delta^\alpha_{\ \beta}$ is the Kronecker delta:

$$\not{\Pi}^\alpha_{\ \beta} := \delta^\alpha_{\ \beta} + \frac{1}{2} L^\alpha \underline{L}_\beta + \frac{1}{2} \underline{L}^\alpha L_\beta, \qquad \underline{\Pi}^\alpha_{\ \beta} := \delta^\alpha_{\ \beta} + \mathbf{B}^\alpha \mathbf{B}_\beta. \tag{188}$$

It is straightforward to check that $\not{\Pi}^0_{\ \alpha} = \underline{\Pi}^0_{\ \alpha} = 0$ for $\alpha = 0, 1, 2, 3$; we will silently use this simple fact throughout the article.

If $\boldsymbol{\xi}$ is a spacetime tensor, then $\not{\Pi}\boldsymbol{\xi}$ denotes its $\mathbf{g}$-orthogonal projection onto $S_{t,u}$, obtained by projecting every component of $\boldsymbol{\xi}$ onto $S_{t,u}$ via $\not{\Pi}$. For example, if $\mathbf{X}$ is a vectorfield, then $(\not{\Pi}\mathbf{X})^\alpha = \not{\Pi}^\alpha_{\ \beta}\mathbf{X}^\beta$, and if $\boldsymbol{\xi}$ is a type $\binom{0}{2}$ tensorfield, then $(\not{\Pi}\boldsymbol{\xi})_{\alpha\beta} = \not{\Pi}^\gamma_{\ \alpha}\not{\Pi}^\delta_{\ \beta}\boldsymbol{\xi}_{\gamma\delta}$. We say that a tensor $\boldsymbol{\xi}$ is $S_{t,u}$-tangent if $\not{\Pi}\boldsymbol{\xi} = \boldsymbol{\xi}$. We often denote $S_{t,u}$-tangent tensorfields in non-bold font, i.e., as X or ξ. We use the notation $|\xi|_{\not{g}}$ to denote the norm of the $S_{t,u}$-tangent tensorfield ξ with respect to the rescaled first fundamental form $\not{g}$. For example, if ξ is a type $\binom{0}{2}$ $S_{t,u}$-tangent tensorfield, then $|\xi|_{\not{g}} = \sqrt{(\not{g}^{-1})^{\alpha\gamma}(\not{g}^{-1})^{\beta\delta}\xi_{\alpha\beta}\xi_{\gamma\delta}} = \sqrt{\xi_{AB}\xi_{AB}}$, where the last relation holds relative to the $S_{t,u}$-frame $\{e_A\}_{A=1,2}$. If ξ is a symmetric type $\binom{0}{2}$ $S_{t,u}$-tangent tensorfield, then we define its $\not{g}$-trace to be the scalar $\mathrm{tr}_{\not{g}}\xi := (\not{g}^{-1})^{\alpha\beta}\xi_{\alpha\beta} = \xi_{AA}$, where the last relation holds relative to the $S_{t,u}$-frame $\{e_A\}_{A=1,2}$. We then define $\hat{\xi} := \xi - \frac{1}{2}(\mathrm{tr}_{\not{g}}\xi)\not{g}$ to be the trace-free part of ξ. Given a tensor whose components with respect to $\{e_A\}_{A=1,2}$ are known, we can extend ξ to an $S_{t,u}$-tangent spacetime tensor ξ (i.e., one verifying $\not{\Pi}\xi = \xi$) by declaring that all contractions of ξ against elements of $\{L, \underline{L}\}$ vanish; throughout the paper, we will often implicitly assume such an extension. Similarly, $\underline{\Pi}\boldsymbol{\xi}$ denotes the $\mathbf{g}$-orthogonal projection of $\boldsymbol{\xi}$ onto Σ_t, we say that $\boldsymbol{\xi}$ is Σ_t-tangent if $\underline{\Pi}\boldsymbol{\xi} = \boldsymbol{\xi}$, and we can extend tensors ξ whose Σ_t components are given to a Σ_t-tangent spacetime tensor by declaring that all contractions of ξ against $\mathbf{B}$ vanish. We also note that $\not{\Pi}L = \not{\Pi}\underline{L} = 0$, and $\underline{\Pi}\mathbf{B} = 0$.

Remark 9.3 We remark that we do not attribute a tensorial structure to $\vec{\Psi}$ or $\boldsymbol{\partial}\vec{\Psi}$. Therefore, whenever $\vec{\Psi}$ or $\boldsymbol{\partial}\vec{\Psi}$ appears under the $|\cdot|_{g\!\!/}$ norm, it should be interpreted as the standard Euclidean norm of the array $\vec{\Psi}$ or $\boldsymbol{\partial}\vec{\Psi}$. The only reason why we occasionally have $\vec{\Psi}$ or $\boldsymbol{\partial}\vec{\Psi}$ under $|\cdot|_{g\!\!/}$ is because, in our schematic notation, we sometimes group it with $S_{t,u}$-tangent tensorfields for which pointwise norms are taken with respect to $|\cdot|_{g\!\!/}$, such as, for example, in (337a).

Throughout, if $\mathbf{V}$ is a spacetime vectorfield and $\boldsymbol{\xi}$ is a spacetime tensorfield, then we define $\mathcal{L}\!\!\!/_{\mathbf{V}}\boldsymbol{\xi} := \Pi\!\!\!/\mathcal{L}_{\mathbf{V}}\boldsymbol{\xi}$ and $\underline{\mathcal{L}}_{\mathbf{V}}\boldsymbol{\xi} := \underline{\Pi}\mathcal{L}_{\mathbf{V}}\boldsymbol{\xi}$, where $\mathcal{L}_{\mathbf{V}}$ denotes Lie differentiation with respect to $\mathbf{V}$.

We use the following notation to denote the arrays of the Cartesian components of L, $\underline{L}$, N:

$$\vec{L} := (1, L^1, L^2, L^3), \quad \vec{\underline{L}} := (1, \underline{L}^1, \underline{L}^2, \underline{L}^3), \quad \vec{N} := (0, N^1, N^2, N^3). \tag{189}$$

From (182) and the fact that $\mathbf{B}^{\alpha}$ is a smooth function of $\vec{\Psi}$, it follows that there exist smooth functions, denoted schematically by f, such that both $\vec{\underline{L}}$ and $\vec{N}$ are of the form $\vec{L} - \mathrm{f}(\vec{\Psi})$. In the rest of the paper, we will often use this fact without explicitly mentioning it.

9.6.3 The metrics and volume forms relative to geometric coordinates, and the ratio υ

From the above considerations, it is straightforward to deduce that there exists an $S_{t,u}$-tangent vectorfield Y such that $\mathbf{g}$ and g can be expressed as follows relative to the geometric coordinates (see [42]*Lemma 3.45 for further details):

$$\begin{aligned}\mathbf{g} = &-b dt \otimes du - b du \otimes dt + b^2 du \otimes du \\ &+ g\!\!\!/\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right)(d\omega^A + Y^A du) \otimes (d\omega^B + Y^B du),\end{aligned} \tag{190a}$$

$$g = b^2 du \otimes du + g\!\!\!/\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right)(d\omega^A + Y^A du) \otimes (d\omega^B + Y^B du). \tag{190b}$$

The volume form $d\varpi_{g\!\!/}$ induced on $S_{t,u}$ by $g\!\!\!/$ can be expressed as follows relative to the geometric coordinates:

$$d\varpi_{g\!\!/(t,u,\omega)} = \sqrt{\det g\!\!\!/}\, d\omega^1 d\omega^2. \tag{191}$$

In addition, the volume form $d\varpi_g$ induced on Σ_t by g, which in Cartesian coordinates takes the form $d\varpi_g = c^{-3} dx^1 dx^2 dx^3$ (see (10)), can be expressed as follows relative to the geometric coordinates:

$$d\varpi_{g(t,u,\omega)} = b(t, u, \omega)\, du d\varpi_{g\!\!/(t,u,\omega)}. \tag{192}$$

Let $\not{e} = \not{e}(\omega)$ be the standard round metric on the Euclidean unit sphere $\mathbb{S}^2$, and let $d\varpi_{\not{e}(\omega)}$ denote the corresponding volume form. The following ratio[44] of volume forms will play a role in the ensuing discussion:

$$\upsilon(t, u, \omega) := \frac{d\varpi_{\not{g}(t,u,\omega)}}{d\varpi_{\not{e}(\omega)}} = \frac{\sqrt{\det \not{g}}(t, u, \omega)}{\sqrt{\det \not{e}}(\omega)}. \tag{193}$$

9.6.4 Levi-Civita connections, angular divergence and curl operators, and curvatures

We let $\mathbf{D}$ denote the Levi-Civita connection of the rescaled spacetime metric $\mathbf{g}$ and $\not{\nabla}$ denote the Levi-Civita connection of $\not{g}$. Our Christoffel symbol conventions for $\mathbf{g}$ are that $\mathbf{D}_\beta \mathbf{X}^\alpha = \partial_\beta \mathbf{X}^\alpha + \mathbf{\Gamma}_{\beta\ \gamma}^{\ \alpha} \mathbf{X}^\gamma$, where $\mathbf{\Gamma}_{\beta\ \gamma}^{\ \alpha} := (\mathbf{g}^{-1})^{\alpha\delta} \mathbf{\Gamma}_{\beta\delta\gamma}$ and $\mathbf{\Gamma}_{\beta\delta\gamma} := \frac{1}{2}\left\{\partial_\beta \mathbf{g}_{\delta\gamma} + \partial_\gamma \mathbf{g}_{\beta\delta} - \partial_\delta \mathbf{g}_{\beta\gamma}\right\}$.

If $\boldsymbol{V}$ is a vectorfield and $\boldsymbol{\xi}$ is a spacetime tensorfield, then $\mathbf{D}_{\boldsymbol{V}} \boldsymbol{\xi} := \boldsymbol{V}^\alpha \mathbf{D}_\alpha \boldsymbol{\xi}$ and $\not{\mathbf{D}}_{\boldsymbol{V}} \boldsymbol{\xi} := \not{\Pi} \mathbf{D}_{\boldsymbol{V}} \boldsymbol{\xi}$; note that $\not{\mathbf{D}}_V \xi := \not{\nabla}_V \xi$ when both V and ξ are $S_{t,u}$-tangent.

If ξ is an $S_{t,u}$-tangent one-form, then relative to an arbitrary $\not{g}$-orthonormal frame $\{e_{(1)}, e_{(2)}\}$, $\mathrm{d}\not{\mathrm{i}}\mathrm{v}\, \xi := \not{\nabla}_A \xi_A$ and $\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\, \xi := \epsilon^{AB} \not{\nabla}_A \xi_B$, where repeated capital Latin indices are summed from 1 to 2 and ϵ^{AB} is fully antisymmetric and normalized by $\epsilon^{12} = 1$. If f is a scalar function defined on $S_{t,u}$, then $\not{\Delta} f := \not{\nabla}^2_{AA} f$ denotes its covariant angular Laplacian. We clarify that above and in all of our subsequent formulas, frame contractions are taken after covariant differentiation. For example, relative to arbitrary local coordinates $\{y^1, y^2\}$ on $S_{t,u}$, we have $\not{\nabla}_A \xi_A := e_A^a e_A^b \not{\nabla}_a \xi_b$ and $\not{\nabla}^2_{AA} f := e_A^a e_A^b \not{\nabla}_a \not{\nabla}_b f$. Similarly, if ξ is a symmetric type $\binom{0}{2}$ $S_{t,u}$-tangent tensorfield, then $\mathrm{d}\not{\mathrm{i}}\mathrm{v}\, \xi_A := \not{\nabla}_B \xi_{AB}$ and $\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\, \xi_A := \epsilon^{BC} \not{\nabla}_B \xi_{CA} = \frac{1}{2} \epsilon^{BC} \left\{\not{\nabla}_B \xi_{CA} - \not{\nabla}_C \xi_{BA}\right\}$.

We let $\mathbf{Riem}_{\alpha\beta\gamma\delta}$ denote the Riemann curvature of $\mathbf{g}$ and $\mathbf{Ric}_{\alpha\beta} := (\mathbf{g}^{-1})^{\gamma\delta} \mathbf{Riem}_{\alpha\gamma\beta\delta}$ denote its Ricci curvature. We adopt the curvature sign convention $\mathbf{g}(\mathbf{D}^2_{\mathbf{XY}} \mathbf{W} - \mathbf{D}^2_{\mathbf{YX}} \mathbf{W}, \mathbf{Z}) := -\mathbf{Riem}(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z})$, where $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{W}$, and $\mathbf{Z}$ are arbitrary spacetime vectors, and $\mathbf{D}^2_{\mathbf{XY}} \mathbf{W} := \mathbf{X}^\alpha \mathbf{Y}^\beta \mathbf{D}_\alpha \mathbf{D}_\beta \mathbf{W}$.

9.6.5 Connection coefficients

Definition 9.2 *(Connection coefficients).* We define the second fundamental form k of Σ_t to be the type $\binom{0}{2}$ Σ_t-tangent tensorfield such that the following relation holds for all Σ_t-tangent vectorfields X and Y:

$$k(X, Y) := -\mathbf{g}(\mathbf{D}_X \mathbf{B}, Y). \tag{194}$$

[44] Note that RHS (193) is invariant under arbitrary diffeomorphisms on $\mathbb{S}^2$, i.e., diffeomorphisms corresponding to the geometric angular coordinates. This ratio is determined by the diffeomorphism from $\mathbb{S}^2$ to $S_{t,u}$ that we constructed in Sect. 9.4 (which in particular determine the component functions $\not{g}(t, u, \omega)\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right)$). That is, the ratio is determined by our construction of the geometric coordinates.

We define the second fundamental form θ of $S_{t,u}$, the null second fundamental form χ of $S_{t,u}$, and $\underline{\chi}$ to be the following type $\binom{0}{2}$ $S_{t,u}$-tangent tensorfields:

$$\theta_{AB} := \mathbf{g}(\mathbf{D}_A N, e_B), \tag{195a}$$

$$\chi_{AB} := \mathbf{g}(\mathbf{D}_A L, e_B), \underline{\chi}_{AB} := \mathbf{g}(\mathbf{D}_A \underline{L}, e_B). \tag{195b}$$

We define the torsion ζ and $\underline{\zeta}$ to be the following $S_{t,u}$-tangent one-forms:

$$\zeta_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, e_A), \qquad \underline{\zeta}_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, e_A). \tag{196}$$

In the next lemma, we provide some standard decompositions and identities. We omit the simple proof and instead refer readers to [18] for details.

Lemma 9.2 (Connection coefficients and relationships between various tensors). *k, θ, χ, and $\underline{\chi}$ are symmetric tensorfields. Moreover, the following relations hold:*

$$k = -\frac{1}{2}\mathcal{L}_{\mathbf{B}} g = -\frac{1}{2}\mathcal{L}_{\mathbf{B}} \mathbf{g}, \tag{197a}$$

$$\chi = \frac{1}{2}\not{\mathcal{L}}_L \not{g} = \frac{1}{2}\not{\mathcal{L}}_L \mathbf{g}, \qquad \underline{\chi} = \frac{1}{2}\not{\mathcal{L}}_{\underline{L}} \not{g} = \frac{1}{2}\not{\mathcal{L}}_{\underline{L}} \mathbf{g}, \tag{197b}$$

$$\not{\mathbf{D}}_N N = -\not{\nabla} \ln b, \qquad \not{\mathbf{D}}_A N_B = \theta_{AB}, \tag{198}$$

$$\mathbf{D}_A L = \chi_{AB} e_B - k_{AN} L, \qquad \mathbf{D}_A \underline{L} = \underline{\chi}_{AB} e_B + k_{AN} \underline{L}, \tag{199a}$$

$$\mathbf{D}_L L = -k_{NN} L, \qquad \mathbf{D}_L \underline{L} = 2\underline{\zeta}_A e_A + k_{NN} \underline{L}, \tag{199b}$$

$$\mathbf{D}_{\underline{L}} L = 2\zeta_A e_A + k_{NN} L, \qquad \mathbf{D}_L e_A = \not{\mathbf{D}}_L e_A + \underline{\zeta}_A L, \tag{199c}$$

$$\mathbf{D}_B e_A = \not{\nabla}_B e_A + \frac{1}{2}\chi_{AB} \underline{L} + \frac{1}{2}\underline{\chi}_{AB} L, \quad \mathbf{D}_{\underline{L}} \underline{L} = -2(\not{\nabla}_A \ln b) e_A - k_{NN} \underline{L}, \tag{199d}$$

$$\chi_{AB} = \theta_{AB} - k_{AB}, \quad \underline{\chi}_{AB} = -\theta_{AB} - k_{AB}, \quad \underline{\zeta}_A = -k_{AN}, \quad \zeta_A = \not{\nabla}_A \ln b + k_{AN}. \tag{200}$$

9.7 Modified acoustical quantities

As we explained at the end of Sect. 2.1.3, to obtain suitable control of the acoustic geometry, we must work with modified quantities and a metric equal to a conformal rescaling of $\mathbf{g}$. In this subsection, we define the relevant quantities.

9.7.1 The conformal metric in $\widetilde{\mathcal{M}}^{(Int)}$

Definition 9.3 *(The conformal factor and conformal metric in the interior region $\widetilde{\mathcal{M}}^{(Int)}$).* We define σ to be the solution to the following transport initial value problem (with data given on the cone-tip axis defined in Sect. 9.4.1):

$$L\sigma(t,u,\omega) = \frac{1}{2}[\mathbf{\Gamma}_L](t,u,\omega), \quad u \in [0, T_{*;(\lambda)}],\ t \in [u, T_{*;(\lambda)}],\ \omega \in \mathbb{S}^2, \tag{201a}$$

$$\sigma(u,u,\omega) = 0, \quad u \in [0, T_{*;(\lambda)}],\ \omega \in \mathbb{S}^2, \tag{201b}$$

where $\mathbf{\Gamma}_L := \mathbf{\Gamma}_\alpha L^\alpha$ and $\mathbf{\Gamma}_\alpha := (\mathbf{g}^{-1})^{\kappa\lambda}\mathbf{\Gamma}_{\kappa\alpha\lambda}$ is a contracted (and lowered) Cartesian Christoffel symbol of $\mathbf{g}$.

We define

$$\widetilde{\mathbf{g}} := e^{2\sigma}\mathbf{g}, \qquad \widetilde{\not{g}} := e^{2\sigma}\not{g} \tag{202}$$

to be, respectively, the conformal spacetime metric and the Riemannian metric that it induces on $S_{t,u}$.

Definition 9.4 *(Null second fundamental forms of the conformal metric).* We define the null second fundamental forms of the conformal metric to be the following symmetric $S_{t,u}$-tangent tensorfields:

$$\widetilde{\chi} := \frac{1}{2}\not{\mathcal{L}}_L\widetilde{\not{g}}, \qquad \widetilde{\underline{\chi}} := \frac{1}{2}\not{\mathcal{L}}_{\underline{L}}\widetilde{\not{g}}. \tag{203}$$

From straightforward computations, taking into consideration definition (203) and the PDE (201a), we deduce the following relations:

$$\widetilde{\chi} = e^{2\sigma}\left\{\chi + (L\sigma)\not{g}\right\}, \qquad \widetilde{\underline{\chi}} = e^{2\sigma}\left\{\underline{\chi} + (\underline{L}\sigma)\not{g}\right\}, \tag{204a}$$

$$\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi} = \mathrm{tr}_{\not{g}}\chi + 2L\sigma = \mathrm{tr}_{\not{g}}\chi + \mathbf{\Gamma}_L, \qquad \mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\underline{\chi}} = \mathrm{tr}_{\not{g}}\underline{\chi} + 2\underline{L}\sigma, \tag{204b}$$

$$\chi = \frac{1}{2}\left\{\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\chi} - \mathbf{\Gamma}_L\right\}\not{g} + \hat{\chi}, \qquad \underline{\chi} = \frac{1}{2}\left\{\mathrm{tr}_{\widetilde{\not{g}}}\widetilde{\underline{\chi}} - 2\underline{L}\sigma\right\}\not{g} + \hat{\underline{\chi}}. \tag{204c}$$

Moreover, above and throughout, if ξ is a symmetric type $\binom{0}{2}$ $S_{t,u}$-tangent tensorfield, then $\mathrm{tr}_{\widetilde{\not{g}}}\xi := (\widetilde{\not{g}}^{-1})^{\alpha\beta}\xi_{\alpha\beta} = e^{-2\sigma}(\not{g}^{-1})^{\alpha\beta}\xi_{\alpha\beta} = e^{-2\sigma}\mathrm{tr}_{\not{g}}\xi$ denotes its trace with respect to $\widetilde{\not{g}}$.

9.7.2 Average values on $S_{t,u}$

Some of our forthcoming constructions refer to the average values of scalar functions f on $S_{t,u}$. Specifically, we define the average value of f, denoted by $\overline{f}$, as follows:

$$\overline{f} = \overline{f}(t,u) := \frac{1}{|S_{t,u}|_{g\!\!/}} \int_{S_{t,u}} f \, d\varpi_{g\!\!/}, \qquad |S_{t,u}|_{g\!\!/} := \int_{S_{t,u}} 1 \, d\varpi_{g\!\!/}. \tag{205}$$

In the next lemma, we connect the evolution equation for $\overline{f}$ along integral curves of L to that of f. We omit the standard proof, which is based on the identity (212a) below.

Lemma 9.3 (Evolution equation for the average value on $S_{t,u}$). *For scalar functions f, we have*

$$L\overline{f} + tr_{g\!\!/}\chi\overline{f} = \left\{tr_{g\!\!/}\chi - \overline{tr_{g\!\!/}\chi}\right\}\overline{f} + \overline{Lf + tr_{g\!\!/}\chi f}. \tag{206}$$

9.7.3 Definitions of the modified acoustical quantities

Definition 9.5 *(Modified acoustical quantities).* In the interior region $\widetilde{\mathcal{M}}^{(Int)}$, we define $\mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}$ to be[45] $-\frac{2}{\tilde{r}}$ plus the trace of the $S_{t,u}$-tangent tensorfield $\widetilde{\chi}$ defined in (203) with respect to the conformal metric $\tilde{g\!\!/}$ defined in (202). That is, in view of (204b), in $\widetilde{\mathcal{M}}^{(Int)}$, we have:

$$\mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)} = \mathrm{tr}_{g\!\!/}\chi + \mathbf{\Gamma}_L - \frac{2}{\tilde{r}} = \mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi} - \frac{2}{\tilde{r}}, \tag{207}$$

where $\mathbf{\Gamma}_L := \mathbf{\Gamma}_\alpha L^\alpha$, and $\mathbf{\Gamma}_\alpha := (\mathbf{g}^{-1})^{\kappa\lambda}\mathbf{\Gamma}_{\kappa\alpha\lambda}$ is a contracted (and lowered) Cartesian Christoffel symbol of $\mathbf{g}$. We then extend the definition of $\mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}$ to all of $\widetilde{\mathcal{M}}$ by declaring that the first equality in (207) holds in all of $\widetilde{\mathcal{M}}$.

In $\widetilde{\mathcal{M}}$, we define the mass aspect function μ to be the following scalar function:

$$\mu := \underline{L}\mathrm{tr}_{g\!\!/}\chi + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathrm{tr}_{g\!\!/}\underline{\chi}. \tag{208}$$

In $\widetilde{\mathcal{M}}^{(Int)}$, we define the modified mass aspect function[46] $\check{\mu}$ to be the following scalar function:

$$\check{\mu} := 2\Delta\!\!\!/\sigma + \underline{L}\mathrm{tr}_{g\!\!/}\chi + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathrm{tr}_{g\!\!/}\underline{\chi} - \mathrm{tr}_{g\!\!/}\chi k_{NN} + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathbf{\Gamma}_{\underline{L}}, \tag{209}$$

where $\mathbf{\Gamma}_{\underline{L}} := \mathbf{\Gamma}_\alpha \underline{L}^\alpha$.

45 In [54], $\mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}$ was denoted by "z" and $\mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi}$ was denoted by "$\mathrm{tr}\widetilde{\chi}$."

46 The idea of working with quantities in the spirit of the mass aspect function and the modified mass aspect function originates in [7]. As in [7,54], we use these quantities to avoid the loss of a derivative when controlling the $\underline{L}$ derivative of $\mathrm{tr}_{g\!\!/}\chi$.

In $\widetilde{\mathcal{M}}^{(Int)}$, we define $\not{\mu}$ to be[47] the $S_{t,u}$-tangent one-form that satisfies the following Hodge system on $S_{t,u}$:

$$\mathrm{d}\!\!\not{\mathrm{i}}\mathrm{v}\not{\mu} = \frac{1}{2}(\check{\mu} - \overline{\check{\mu}}), \qquad \mathrm{cu}\!\!\not{\mathrm{r}}\mathrm{l}\not{\mu} = 0. \tag{210}$$

In $\widetilde{\mathcal{M}}^{(Int)}$, we define the modified torsion $\widetilde{\zeta}$ to be the following $S_{t,u}$-tangent one-form:

$$\widetilde{\zeta} := \zeta + \not{\nabla}\sigma. \tag{211}$$

9.8 PDEs verified by geometric quantities - a preliminary version

To control the acoustic geometry, we will derive estimates for the PDEs that various geometric quantities solve. In the next lemma, we provide a first version of these PDEs. The results are standard and are independent of the compressible Euler equations. In Proposition 9.7, we use the compressible Euler equations to re-express various terms in the PDEs, which will lead to the form of the equations that we use in our analysis.

Lemma 9.4 [18, PDEs verified by the $S_{t,u}$ volume element ratio, null lapse, and connection coefficients, without regard for the compressible Euler equations] *The following evolution equations hold*[48] *relative to a null frame:*

$$L\upsilon = \upsilon tr_{\not{g}}\chi, \tag{212a}$$

$$Lb = -bk_{NN}, \tag{212b}$$

$$Ltr_{\not{g}}\chi + \frac{1}{2}(tr_{\not{g}}\chi)^2 = -|\hat{\chi}|_{\not{g}}^2 - k_{NN}tr_{\not{g}}\chi - \mathbf{Ric}_{LL}, \tag{212c}$$

$$\not{\mathbf{D}}_L\hat{\chi}_{AB} + (tr_{\not{g}}\chi)\hat{\chi}_{AB} = -k_{NN}\hat{\chi}_{AB} - \left\{\mathbf{Riem}_{LALB} - \frac{1}{2}\mathbf{Ric}_{LL}\delta_{AB}\right\}, \tag{212d}$$

[47] Existence and uniqueness for the system (210) is standard, given the smoothness of the source terms.

[48] In [54]*Equation (5.28), the terms in braces on the last line of RHS (212g) were omitted. However, equation (212g) is needed only to derive the evolution equation (237) for $\check{\mu}$, and the omitted terms have the same schematic structure as other error terms that were bounded in [54]; i.e., the omitted terms are harmless. Moreover, in [54], the second term on LHS (212d) was listed as $\frac{1}{2}(\mathrm{tr}_{\not{g}}\chi)\hat{\chi}_{AB}$. Fortunately, correcting the coefficient from $\frac{1}{2}$ to 1 does not lead to any changes in the estimates, as we further explain in the discussion surrounding equation (374). In our statement of Lemma 9.4, we also corrected index-placement/sign errors in some curvature terms, specifically the term $\frac{1}{2}\mathbf{Riem}_{AL\underline{L}L}$ on RHS (212e), the term $\mathbf{Riem}_{A\underline{L}LB}$ on RHS (212g), and the term $\frac{1}{2}\epsilon^{AB}\mathbf{Riem}_{AL\underline{L}B}$ on RHS (213c). These corrections are harmless in the sense that in practice, when deriving estimates, we only need to know the schematic structure of the first and third of these curvature terms, which is provided by (221a) and (222a) and which is insensitive to signs. In particular, these corrections do not affect the schematic form of the equations of Proposition 9.7, which is what we use when deriving estimates for the acoustic geometry.

$$\not{\mathcal{D}}_L \zeta_A + \frac{1}{2}(tr_{\not{g}}\chi)\zeta_A = -\{k_{BN} + \zeta_B\}\hat{\chi}_{AB} - \frac{1}{2}tr_{\not{g}}\chi k_{AN} + \frac{1}{2}\mathbf{Riem}_{ALL\underline{L}}, \quad (212e)$$

$$Ltr_{\not{g}}\underline{\chi} + \frac{1}{2}(tr_{\not{g}}\chi)tr_{\not{g}}\underline{\chi} = 2\mathrm{d}\not{\mathrm{i}}\mathrm{v}\underline{\zeta} + k_{NN}tr_{\not{g}}\underline{\chi} - \hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + 2|\underline{\zeta}|^2_{\not{g}} + \mathbf{Riem}_{A\underline{L}LA}, \quad (212f)$$

$$\begin{aligned} \not{\mathcal{D}}_{\underline{L}}\hat{\chi}_{AB} + \frac{1}{2}(tr_{\not{g}}\underline{\chi})\hat{\chi}_{AB} = &-\frac{1}{2}(tr_{\not{g}}\chi)\hat{\underline{\chi}}_{AB} + 2\not{\nabla}_A\zeta_B - \mathrm{d}\not{\mathrm{i}}\mathrm{v}\zeta\delta_{AB} + k_{NN}\hat{\chi}_{AB} \\ &+ \left\{2\zeta_A\zeta_B - |\zeta|^2_{\not{g}}\delta_{AB}\right\} - \left\{\hat{\underline{\chi}}_{AC}\hat{\chi}_{CB} - \frac{1}{2}\hat{\underline{\chi}}_{CD}\hat{\chi}_{CD}\delta_{AB}\right\} \\ &+ \mathbf{Riem}_{A\underline{L}LB} - \frac{1}{2}\mathbf{Riem}_{CL\underline{L}C}\delta_{AB}, \end{aligned} \quad (212g)$$

$$\mathrm{d}\not{\mathrm{i}}\mathrm{v}\hat{\chi}_A + \hat{\chi}_{AB}k_{BN} = \frac{1}{2}\left\{\not{\nabla}_A tr_{\not{g}}\chi + k_{AN}tr_{\not{g}}\chi\right\} + \mathbf{Riem}_{BLBA}, \quad (213a)$$

$$\mathrm{d}\not{\mathrm{i}}\mathrm{v}\zeta = \frac{1}{2}\left\{\mu - k_{NN}tr_{\not{g}}\chi - 2|\zeta|^2_{\not{g}} - |\hat{\chi}|^2_{\not{g}} - 2k_{AB}\hat{\chi}_{AB}\right\} - \frac{1}{2}\mathbf{Riem}_{A\underline{L}LA}, \quad (213b)$$

$$\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\zeta = \frac{1}{2}\epsilon^{AB}\hat{\underline{\chi}}_{AC}\hat{\chi}_{BC} + \frac{1}{2}\epsilon^{AB}\mathbf{Riem}_{AL\underline{L}B}. \quad (213c)$$

9.9 Main version of the PDEs verified by the acoustical quantities, including the modified ones

The main result of this subsection is Proposition 9.7, in which we derive, with the help of the compressible Euler equations, the main PDEs that we use to control the acoustic geometry. The proposition in particular shows how the source terms in the compressible Euler equations influence the evolution of the acoustic geometry. Before proving the proposition, we first introduce some additional schematic notation and, in Lemma 9.6, provide some decompositions of various null components of the acoustical curvature, that is, the curvature of **g**.

Remark 9.4 Compared to previous works, what is new are the terms in Lemma 9.6 and Proposition 9.7 that are multiplied by λ^{-1}; these terms capture, in particular, how the top-order derivatives of the vorticity and entropy affect the acoustical curvature.

9.9.1 Additional schematic notation and a simple lemma

Let U and ξ be scalar functions or $S_{t,u}$-tangent tensorfields. In the rest of the paper, we will use the schematic notation

$$U = \mathrm{f}_{(\vec{L})} \cdot \xi, \quad (214)$$

to mean that the *Cartesian* components of U can be expressed as linear combinations of products of the Cartesian components of ξ and scalar functions of type "$\mathrm{f}_{(\vec{L})}$," which by definition are linear combinations of products of **i)** smooth functions of $\vec{\Psi}$ and **ii)** the Cartesian components of vectorfields whose Cartesian components are polynomials in the components of $\vec{L}$ with coefficients that are smooth functions of $\vec{\Psi}$. Expressions such as $\mathrm{f}_{(\vec{L})} \cdot \xi_{(1)} \cdot \xi_{(2)}$ have the obvious analogous meaning. If $\xi = (\xi_{(1)}, \cdots, \xi_{(m)})$ is an array of scalar functions $S_{t,u}$-tangent tensorfields, then $\mathrm{f}_{(\vec{L})} \cdot \xi$ means sums of terms of type $\mathrm{f}_{(\vec{L})} \cdot \xi_{(i)}$, $1 \leq i \leq m$. As examples, we note (in view of the discussion below (189)) that $N_a \mathcal{C}^a = \mathrm{f}_{(\vec{L})} \cdot \vec{\mathcal{C}}$, while $\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \boldsymbol{\partial}\vec{\Psi}$ denotes a scalar function or an $S_{t,u}$-tangent tensorfield whose Cartesian components are products of $\mathrm{f}_{(\vec{L})}$ and a term that is quadratic in elements the array $\boldsymbol{\partial}\vec{\Psi}$. As another example, we note that (188) and the discussion below (189) imply that the $S_{t,u}$-tangent tensorfield $\Pi\!\!\!/$ has Cartesian components of the form $\mathrm{f}_{(\vec{L})}$, which we indicate by writing $\Pi\!\!\!/ = \mathrm{f}_{(\vec{L})}$. Finally, we note that since (by (197a)) the Cartesian components of the second fundamental form k of Σ_t verify $k_{ij} = \mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\vec{\Psi}$, it follows that the $S_{t,u}$-tangent tensorfield with components $k_{AN} := k(e_A, N)$, $(A = 1, 2)$, is of the form $k_{AN} = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$.

We will use the following simple lemma in our proof of Proposition 9.7.

Lemma 9.5 (Identities for the derivatives of some scalar functions). *With df denoting the spacetime gradient of the scalar function f (and thus $\Pi\!\!\!/ \cdot df = \nabla\!\!\!\!/ f$), we have the following identities (where in (215b) and (216b), the terms "$\mathrm{f}_{(\vec{L})}$" on the LHSs are not the same as the terms "$\mathrm{f}_{(\vec{L})}$" on the RHSs):*

$$\Pi\!\!\!/ \cdot d(\vec{L}, \vec{\underline{L}}, \vec{N}) = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}), \tag{215a}$$

$$\Pi\!\!\!/ \cdot d\mathrm{f}_{(\vec{L})} = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}). \tag{215b}$$

Moreover,

$$d(\vec{L}, \vec{\underline{L}}, \vec{N}) = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \tilde{r}^{-1}), \tag{216a}$$

$$d\mathrm{f}_{(\vec{L})} = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \tilde{r}^{-1}). \tag{216b}$$

Proof To prove (215a), we first note the schematic relation $\mathbf{D}L^\alpha = dL^\alpha + \boldsymbol{\Gamma} \cdot L = dL^\alpha + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, where $\boldsymbol{\Gamma}$ denotes a Cartesian Christoffel symbol of the rescaled metric $\mathbf{g}$. Viewing L^α as a scalar function, we can interpret this relation as an identity in which the term on the left and the two terms on the right are one-forms. Projecting these one-forms onto $S_{t,u}$ with the tensorfield $\Pi\!\!\!/$, and using the first identity in (199a), the fact that $k_{ij} = \mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\vec{\Psi}$, and the fact that $\vec{\underline{L}} = \mathrm{f}_{(\vec{L})}$ and $\vec{N} = \mathrm{f}_{(\vec{L})}$, we deduce that for $\alpha = 0, 1, 2, 3$, we have the following schematic identity for the scalar function L^α: $\Pi\!\!\!/ dL^\alpha = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \chi)$. Considering also that $\chi = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})$ (as can be seen by decomposing $\chi = \hat{\chi} + \frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\chi)g\!\!\!/$ and using (207)), we conclude (215a) for $\vec{L}$. In addition, taking into account that $\vec{\underline{L}} = \mathrm{f}_{(\vec{L})}$ and $\vec{N} = \mathrm{f}_{(\vec{L})}$, and using the chain and product rules, we also deduce the identity (215a) for $\vec{\underline{L}}$ and $\vec{N}$. (215b) follows from similar arguments, and we omit the details.

The identities (216a)–(216b) from from a similar argument, but we also take into account (199b) and (199c). Note that the right-hand side of the identity (199c) for $\mathbf{D}_{\underline{L}} L$ leads to the presence of ζ on RHSs (216a)–(216b). □

9.9.2 Curvature component decompositions

In the next lemma, we provide some expressions for various components of the curvatures of the acoustical metric $\mathbf{g}$. These expressions will be important for controlling the acoustic geometry, since curvature components appear as source terms in the PDEs that they satisfy; see Lemma 9.4. Moreover, some of the curvature components can be expressed with the help of the equations of Proposition 9.1, thus tying the evolution of the acoustic geometry to the fluid evolution; see Remark 9.6 and Proposition 9.7.

Lemma 9.6 (Curvature component decompositions). *Relative to the Cartesian coordinates, the following identity holds, where on RHS (217), the component $\mathbf{g}_{\alpha\beta}(\vec{\Psi})$ is treated as a scalar function under covariant differentiation and $\mathbf{\Gamma}_\alpha := (\mathbf{g}^{-1})^{\kappa\lambda}\mathbf{g}_{\alpha\beta}\mathbf{\Gamma}_{\kappa\ \lambda}^{\ \beta}$ is treated as a one-form under covariant differentiation:*

$$\mathbf{Ric}_{\alpha\beta} = -\frac{1}{2}\square_{\mathbf{g}(\vec{\Psi})}\mathbf{g}_{\alpha\beta}(\vec{\Psi}) + \frac{1}{2}\left\{\mathbf{D}_\alpha\mathbf{\Gamma}_\beta + \mathbf{D}_\beta\mathbf{\Gamma}_\alpha\right\} + \mathscr{Q}(\vec{\Psi})[\boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}]. \tag{217}$$

Moreover,

$$\mathbf{Ric}_{LL} = L(\mathbf{\Gamma}_L) + k_{NN}\mathbf{\Gamma}_L + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}. \tag{218}$$

Finally, there exist scalar functions on $S_{t,u}$, $S_{t,u}$-tangent one-forms, and symmetric type $\binom{0}{2}$ $S_{t,u}$-tangent tensorfields, all schematically denoted by ξ and verifying $\xi = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$ (in the sense of Sect. 9.9.1), such that

$$\mathbf{Ric}_{LL} - L(\mathbf{\Gamma}_L) = \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{219}$$

$$\mathbf{Ric}_{L\underline{L}} - \frac{1}{2}\left\{L(\mathbf{\Gamma}_{\underline{L}}) + \underline{L}(\mathbf{\Gamma}_L)\right\} = \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, \zeta)\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{220}$$

$$\begin{aligned}\mathbf{Ric}_{LA},\ \mathbf{Riem}_{AL L\underline{L}} = (\nabla\!\!\!/, \mathbf{D}\!\!\!\!/_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) \\ + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, tr_{g\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi},\end{aligned} \tag{221a}$$

$$\mathbf{Riem}_{AL\underline{L}A} = \mathrm{d}i\!\!\!/v\,\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, tr_{g\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{221b}$$

$$\epsilon^{AB}\mathbf{Riem}_{AL\underline{L}B} = \mathrm{cu}r\!\!\!/\mathrm{l}\,\xi + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, tr_{g\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{222a}$$

$$\mathbf{Riem}_{LALB} = (\nabla\!\!\!/, \mathbf{D}\!\!\!\!/_L)\xi + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, tr_{g\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}, \tag{222b}$$

$$\mathbf{Riem}_{ABLB} = \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi_A + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\not{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{222c}$$

$$\mathbf{Riem}_{CALB} = \not{\nabla}\xi + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\not{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{222d}$$

$$\mathbf{Riem}_{ABAB} = \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\xi + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\not{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi}. \tag{222e}$$

Remark 9.5 The curvature identities of Lemma 9.6 are crucial for the proof of Proposition 9.7 below. In turn, the structure of the equations of Proposition 9.7 is crucial for our derivation of estimates for the acoustic geometry.

Remark 9.6 The proofs of the identities (218)–(221b) rely on the compressible Euler equations, while the proofs of the remaining identities in Lemma 9.6 do not. This explains why the former identities feature λ^{-1}-dependent source terms (which arise from RHS (156)).

Proof (Discussion of the proofs) The identities (217) and (222a)–(222e) are the same as in [54, Lemma 5.12], whose proofs can be found in [20]. The identities (218)–(221b) also mirror those given in [54, Lemma 5.12], except here there are new terms of type $\lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathcal{C}}, \mathcal{D})$, which arise when one uses equation (156) to substitute for the terms $\square_{\mathbf{g}(\vec{\Psi})}\Psi$ that are generated by the term $-\frac{1}{2}\square_{\mathbf{g}(\vec{\Psi})}\mathbf{g}_{\alpha\beta}(\vec{\Psi})$ on RHS (217). □

9.9.3 Main version of the PDEs verified by the acoustical quantities

We now provide the main result of Sect. 9.9.

Proposition 9.7 (PDEs verified by the modified acoustical quantities, assuming a compressible Euler solution). *Assume that the Cartesian component functions $(\vec{\Psi}, \vec{\Omega}, \vec{S}, \vec{\mathcal{C}}, \mathcal{D})$ are solutions to the rescaled compressible Euler equations of Proposition 9.1 (under the conventions of Sect. 9.3). There exist $S_{t,u}$-tangent one-forms and symmetric type $\binom{0}{2}$ $S_{t,u}$-tangent tensorfields, all schematically denoted by ξ and verifying $\xi = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$ (see Sect. 9.9.1 regarding the notation "$\mathrm{f}_{(\vec{L})}\cdot$"), such that the following schematic identities hold, where all terms on the left-hand sides are displayed exactly and terms on the right-hand sides are displayed schematically (in particular, we have ignored numerical constants and minus signs on the right-hand sides).*

***Transport equations involving the Cartesian components L^i and N^i**: The following evolution equations hold in $\widetilde{\mathcal{M}}$:*

$$LL^i = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}, \qquad LN^i = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}. \tag{223}$$

Moreover, along Σ_0 (where $w = \tilde{r} = -u$ and $a = b$), we have

$$\frac{\partial}{\partial w}L^i = a \cdot \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} + \not{\nabla}a, \qquad \frac{\partial}{\partial w}N^i = a \cdot \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} + \not{\nabla}a. \tag{224}$$

Transport equations involving the Cartesian components $\Theta^i_{(A)}$*: For* $A = 1, 2$ *and* $i = 1, 2, 3$*, let* $\left(\frac{\partial}{\partial \omega^A}\right)^i$ *denote a Cartesian component of* $\frac{\partial}{\partial \omega^A}$ *(i.e.,* $\left(\frac{\partial}{\partial \omega^A}\right)^i = \frac{\partial}{\partial \omega^A} x^i$*), and let* $\Theta_{(A)}$ *be the* Σ_t*-tangent vectorfield with Cartesian components defined by*

$$\Theta^i_{(A)} := \frac{1}{\tilde{r}} \left(\frac{\partial}{\partial \omega^A}\right)^i. \tag{225}$$

Then the following evolution equation holds in $\widetilde{\mathcal{M}}$*:*

$$L\Theta^i_{(A)} = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}) \cdot \vec{\Theta}_{(A)}. \tag{226}$$

Moreover, along Σ_0 *(where* $w = \tilde{r} = -u$ *and* $a = b$*), the following evolution equation holds for* $(w, \omega) \in (0, w_{*;(\lambda)}] \times \mathbb{S}^2$*:*

$$\frac{\partial}{\partial w}\Theta^i_{(A)} = a \cdot \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}) \cdot \vec{\Theta}_{(A)} + \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ a \cdot \vec{\Theta}_{(A)}, \tag{227}$$

where $\vec{\Theta}_{(A)} := (\Theta^1_{(A)}, \Theta^2_{(A)}, \Theta^3_{(A)})$*.*

Transport equations connected to the trace of χ*:*

$$\begin{aligned} &Ltr_{\tilde{g}}\widetilde{\chi}^{(Small)} + \frac{2}{\tilde{r}} tr_{\tilde{g}}\widetilde{\chi}^{(Small)} = \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathcal{C}}, \mathcal{D}) \\ &\quad + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi} + |\hat{\chi}|^2_{g\!\!\!/} + tr_{\tilde{g}}\widetilde{\chi}^{(Small)} \cdot tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \end{aligned} \tag{228a}$$

$$\begin{aligned} &\mathbf{D}\!\!\!\!/_L \nabla\!\!\!/ tr_{\tilde{g}}\widetilde{\chi}^{(Small)} + \frac{3}{\tilde{r}}\nabla\!\!\!/ tr_{g\!\!\!/}\widetilde{\chi}^{(Small)} = \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/(\vec{\mathcal{C}}, \mathcal{D}) \\ &\quad + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{S} \cdot \boldsymbol{\partial}\vec{\Psi}, \boldsymbol{\partial}\vec{\Psi}, \partial\vec{\Omega}, \partial\vec{S}) \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \\ &\quad + \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ \boldsymbol{\partial}\vec{\Psi} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \tilde{r}^{-1}) \\ &\quad + \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ \hat{\chi} \cdot \hat{\chi} + \mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!/ tr_{\tilde{g}}\widetilde{\chi}^{(Small)} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}) \\ &\quad + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi}. \end{aligned} \tag{228b}$$

Above and throughout, we use $\vec{S} \cdot \boldsymbol{\partial}\vec{\Psi}$ *to schematically denote terms of the form* $S^a \partial_\alpha \Psi_\iota$*, where* $a = 1, 2, 3$*,* $\alpha = 0, 1, 2, 3$ *and* $\iota = 0, 1, 2, 3, 4$*.*

Moreover,

$$\begin{aligned} &L\left\{\frac{1}{2} tr_{\tilde{g}}\widetilde{\chi}\upsilon\right\} - \frac{1}{4}\left(tr_{g\!\!\!/}\chi\right)^2 \upsilon + \frac{1}{2}\{L \ln b\} tr_{\tilde{g}}\widetilde{\chi}\upsilon - |\nabla\!\!\!/ \sigma|^2_{g\!\!\!/}\upsilon \\ &= \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathcal{C}}, \mathcal{D}) \cdot \upsilon + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \upsilon + |\hat{\chi}|^2_{g\!\!\!/} \cdot \upsilon + |\nabla\!\!\!/ \sigma|^2_{g\!\!\!/} \cdot \upsilon. \end{aligned} \tag{229}$$

PDEs involving $\hat{\chi}$*:*

$$\mathrm{div}\!\!\!/\, \hat{\chi} = \nabla\!\!\!/ tr_{\tilde{g}}\widetilde{\chi}^{(Small)} + \mathrm{div}\!\!\!/\, \xi + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi}, \tag{230}$$

$$\not{\mathbf{D}}_L\hat{\chi} + (tr_{\not{g}}\chi)\hat{\chi} = (\not\nabla, \not{\mathbf{D}}_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi}. \tag{231}$$

The transport equation for ζ*:*

$$\not{\mathbf{D}}_L\zeta + \frac{1}{2}(tr_{\not{g}}\chi)\zeta = (\not\nabla, \not{\mathbf{D}}_L)\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi} + \mathrm{f}_{(\vec{L})} \cdot \zeta \cdot \hat{\chi}. \tag{232}$$

The transport equation for b*:*

$$Lb = b \cdot \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}. \tag{233}$$

Transport equation for $\not{g}$*: Along the integral curves of* L*, parameterized by* t*, we have, with* $\not{e}$ *the standard round metric on the Euclidean unit sphere* $\mathbb{S}^2$*, the following identity:*

$$\begin{aligned}
&\frac{d}{dt}\left\{\tilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not{e}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right\} \\
&= \left\{tr_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)} - \mathbf{\Gamma}_L\right\}\left\{\tilde{r}^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not{e}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right)\right\} \\
&\quad + \left\{tr_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)} - \mathbf{\Gamma}_L\right\}\not{e}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) + \frac{2}{\tilde{r}^2}\hat{\chi}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right).
\end{aligned} \tag{234}$$

Transport equations for υ ***and*** $\not\nabla\upsilon$*:*

$$L\ln\left(\tilde{r}^{-2}\upsilon\right) = tr_{\not{g}}\chi - \frac{2}{\tilde{r}} = tr_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)} - \mathbf{\Gamma}_L, \tag{235a}$$

$$L\not\nabla\ln\left(\tilde{r}^{-2}\upsilon\right) + \frac{1}{2}(tr_{\not{g}}\chi)\not\nabla\ln\left(\tilde{r}^{-2}\upsilon\right) = \mathrm{f}_{(\vec{L})} \cdot \hat{\chi} \cdot \not\nabla\ln\left(\tilde{r}^{-2}\upsilon\right) + \not\nabla tr_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)} - \not\nabla(\mathbf{\Gamma}_L). \tag{235b}$$

An algebraic identity for μ*: The mass aspect function* μ *defined in* (208) *verifies the following identity:*

$$\begin{aligned}
\mu &= \lambda^{-1}\mathrm{f}_{(\vec{L})} \cdot (\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi + \mathrm{f}_{(\vec{L})} \cdot \hat{\chi} \cdot \hat{\chi} + \mathrm{f}_{(\vec{L})} \cdot \not\nabla\ln\left(\tilde{r}^{-2}\upsilon\right) \cdot (\boldsymbol{\partial}\vec{\Psi}, \zeta) \\
&\quad + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi}.
\end{aligned} \tag{236}$$

The transport equation for $\check{\mu}$*: The modified mass aspect function* $\check{\mu}$ *defined by* (209) *verifies the following transport equation:*

$$L\check{\mu} + (tr_{\not{g}}\chi)\check{\mu} = \mathfrak{I}_{(1)} + \mathfrak{I}_{(2)}, \tag{237}$$

$$\mathfrak{I}_{(1)} = \tilde{r}^{-1}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi + \tilde{r}^{-2}\xi, \tag{238a}$$

$$\begin{aligned}\mathfrak{I}_{(2)} &= \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\\&\quad+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\widetilde{\zeta}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\sigma\cdot(\nabla\!\!\!/\boldsymbol{\partial}\vec{\Psi},\nabla\!\!\!/ tr_{\not g}\widetilde{\chi}^{(Small)})\\&\quad+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\sigma\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/ tr_{\not g}\widetilde{\chi}^{(Small)}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\\&\quad+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta)\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi})\\&\quad+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta)\cdot\boldsymbol{\partial}^2\vec{\Psi}.\end{aligned}\tag{238b}$$

The Hodge system for ζ*: The torsion* ζ *defined in* (196) *satisfies the following Hodge system on* $S_{t,u}$*:*

$$\begin{aligned}\mathrm{d}\!\!\!/\mathrm{iv}\,\zeta &= \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})+\mathrm{d}\!\!\!/\mathrm{iv}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}\\&\quad+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}+\mathrm{f}_{(\vec{L})}\cdot\nabla\!\!\!/\ln\left(\tilde{r}^{-2}\upsilon\right)\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta),\end{aligned}\tag{239a}$$

$$\mathrm{cu}\!\!\!/\mathrm{rl}\,\zeta=\mathrm{cu}\!\!\!/\mathrm{rl}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}.\tag{239b}$$

The Hodge system for $\widetilde{\zeta}$*: The modified torsion* $\widetilde{\zeta}$ *defined by* (211) *satisfies the following Hodge system on* $S_{t,u}$*:*

$$\begin{aligned}\mathrm{d}\!\!\!/\mathrm{iv}\,\widetilde{\zeta}-\frac{1}{2}\check{\mu} &= \mathrm{d}\!\!\!/\mathrm{iv}\,\xi+\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})+\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}\\&\quad+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi},\end{aligned}\tag{240a}$$

$$\mathrm{cu}\!\!\!/\mathrm{rl}\,\widetilde{\zeta}=\mathrm{cu}\!\!\!/\mathrm{rl}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}.\tag{240b}$$

The Hodge system for $\widetilde{\zeta}-\mu\!\!\!/$*: The difference* $\widetilde{\zeta}-\mu\!\!\!/$ *(where* $\widetilde{\zeta}$ *is defined by* (211) *and* $\mu\!\!\!/$ *is defined by* (210)*) verifies the following Hodge system on* $S_{t,u}$ *(see definition* (205) *regarding "overline" notation):*

$$\begin{aligned}\mathrm{d}\!\!\!/\mathrm{iv}(\widetilde{\zeta}-\mu\!\!\!/) &= \mathrm{d}\!\!\!/\mathrm{iv}\,\xi+\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})-\lambda^{-1}\overline{\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})}\right\}\\&\quad+\left\{\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta-\overline{\mathrm{f}_{(\vec{L})}\cdot\zeta\cdot\zeta}\right\}+\left\{\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}-\overline{\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}}\right\}\\&\quad+\left\{\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}-\overline{\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}}\right\},\end{aligned}\tag{241a}$$

$$\mathrm{cu}\!\!\!/\mathrm{rl}(\widetilde{\zeta}-\mu\!\!\!/)=\mathrm{cu}\!\!\!/\mathrm{rl}\,\xi+\mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\hat{\chi}+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},tr_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}.\tag{241b}$$

A decomposition of $\mu\!\!\!/$ ***and a Hodge-transport system for the constituent parts****: Let* $\mathfrak{I}_{(1)}$ *and* $\mathfrak{I}_{(2)}$ *be the inhomogeneous terms from* (238a)–(238b)*. Then in* $\widetilde{\mathcal{M}}^{(Int)}$ *(see*

(174b)*), we can decompose the solution* $\not\mu$ *to* (210) *as follows:*

$$\not\mu = \not\mu_{(1)} + \not\mu_{(2)}, \tag{242}$$

where $\not\mu_{(1)}$ *and* $\not\mu_{(2)}$ *verify the following Hodge-transport PDE systems:*

$$\mathrm{d}\not{\mathrm{i}}\mathrm{v}\left\{\not{\mathbf{D}}_L \not\mu_{(1)} + \frac{1}{2}(tr_{\not g}\chi)\not\mu_{(1)}\right\} = \mathfrak{I}_{(1)} - \overline{\mathfrak{I}_{(1)}}, \tag{243a}$$

$$\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\left\{\not{\mathbf{D}}_L \not\mu_{(1)} + \frac{1}{2}(tr_{\not g}\chi)\not\mu_{(1)}\right\} = 0, \tag{243b}$$

$$\begin{aligned}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\left\{\not{\mathbf{D}}_L \not\mu_{(2)} + \frac{1}{2}(tr_{\not g}\chi)\not\mu_{(2)}\right\} &= \mathfrak{I}_{(2)} - \overline{\mathfrak{I}_{(2)}} + \hat{\chi}\cdot\not\nabla\not\mu + (\not\nabla\boldsymbol{\partial}\vec{\Psi}, \not\nabla tr_{\tilde{\not g}}\widetilde{\chi}^{(Small)})\cdot\not\mu \\ &\quad + (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{\not g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot(\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{\not g}}\widetilde{\chi}^{(Small)}, \hat{\chi})\cdot\not\mu \\ &\quad + (tr_{\not g}\chi - \overline{tr_{\not g}\chi})\overline{\not\mu},\end{aligned} \tag{244a}$$

$$\begin{aligned}\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\left\{\not{\mathbf{D}}_L \not\mu_{(2)} + \frac{1}{2}(tr_{\not g}\chi)\not\mu_{(2)}\right\} &= \hat{\chi}\cdot\not\nabla\not\mu + (\not\nabla\boldsymbol{\partial}\vec{\Psi}, \not\nabla tr_{\tilde{\not g}}\widetilde{\chi}^{(Small)})\cdot\not\mu \\ &\quad + (\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{\not g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot(\boldsymbol{\partial}\vec{\Psi}, tr_{\tilde{\not g}}\widetilde{\chi}^{(Small)}, \hat{\chi})\cdot\not\mu,\end{aligned} \tag{244b}$$

subject to the following initial conditions along the cone-tip axis for $u \in [0, T_{*;(\lambda)}]$*:*

$$|\not\mu_{(1)} - \not\mu|_{\not g}(t,u,\omega) \to 0 \text{ as } t \downarrow u, \qquad |\not\mu_{(2)}|_{\not g}(t,u,\omega) \to 0 \text{ as } t \downarrow u. \tag{245}$$

Proof (Proof sketch) Throughout, we will silently use the identities provided by Lemma 9.5.

The equations in (223) are a straightforward consequence of the first equation in (199b) and the relation $L^i = N^i + \mathrm{f}(\vec{\Psi})$.

To prove (224), we first note that along Σ_0, we have the vectorfield identity $\frac{\partial}{\partial w} = aN$ (see (167)). Also using the identity $L^i = \mathbf{B}^i + N^i$ and the fact that $\mathbf{B}^i = \mathrm{f}(\vec{\Psi})$, we deduce that $\frac{\partial}{\partial w}L^i = a\cdot\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi} + \frac{\partial}{\partial w}N^i$. Thus, to conclude both equations in (224), it suffices to derive the equation for $\frac{\partial}{\partial w}N^i$ stated in (224). The desired result is a straightforward consequence of the identity $\frac{\partial}{\partial w} = aN$ and the identity (198).

(228a) is essentially proved as [54, Equation (5.75)]. The only difference is that in the present work, we have the λ^{-1}-multiplied terms on RHS (228a), which arise when one uses equation (219) to algebraically substitute for the term $\mathbf{Ric}_{LL}$ on RHS (212c). Similarly, (228b) was essentially proved as [54, Equation (5.76)], the only difference being that we take into account Lemma 9.5 and the expressions (153a) and (153b) for the rescaled $\mathcal{C}^i$ and $\mathcal{D}$ when computing $\nabla\!\!\!/$ applied to the λ^{-1}-multiplied terms on RHS (228a).

The identity (229) follows from the same arguments used to prove (228a), based on (218), (212a), (212b), and (212c); see the proof of [54, Proposition 7.22] for the analogous identity in the context of scalar wave equations.

Based on (222c) and (213a) (and the standard properties of $\mathbf{Riem}_{\alpha\beta\gamma\delta}$ under exchanges of indices), the identity (230) was proved as [54, Equation (5.77)].

The identity (231) is essentially proved as [54, Equation (5.68)] based on Lemma 9.6 and equation (212d). The only difference (modulo Footnote 48) is that in the present work, we have the λ^{-1}-multiplied terms on RHS (231), which arise when one uses equation (219) to algebraically substitute for the term $\mathbf{Ric}_{LL}$ on RHS (212d). Similar remarks apply to equation (232), which follows from (212e) and (221a).

(233) follows from (212b).

(234) was proved just below [54, Equation (5.68)].

(235a) and (235b) were derived in the proof of [54] Lemma 5.15, where $\ln\left(\tilde{r}^{-2}\upsilon\right)$ was denoted by "φ."

(236) is essentially proved as [54, Equation (5.92)], where $\tilde{r}^{-2}\upsilon$ was denoted by "φ." The only difference is that in the present work, we have the λ^{-1}-multiplied terms on RHS (236), which arise when one uses equation (221b) to algebraically substitute for the term $\mathbf{Riem}_{A\underline{L}LA}$ on RHS (212f). We remark that equation (212f) is relevant for the proof since the argument relies on deriving an expression for $L\mathrm{tr}_{g\!\!\!/}\underline{\chi} - \underline{L}\mathrm{tr}_{g\!\!\!/}\chi$.

To prove (239a)–(239b), we use (221b)–(222a) to substitute for the curvature terms on RHSs (213b)–(213c), and we use (236) to substitute for the term μ on RHS (213b). Similarly, (240a)–(240b) follow from (213b)–(213c), the definitions of $\widetilde{\zeta}$ and $\check{\mu}$, and the curvature identities (221b)–(222a). (241a)–(241b) then follow easily from (210), (240a)–(240b), and the fact that $\mathrm{d}\!\!\!/\mathrm{iv}$ of an $S_{t,u}$-tangent one-form must have vanishing average value on $S_{t,u}$ (in the sense of (205)).

To prove (242)–(244b), one commutes equation (210) with L and uses the same arguments used in the proof of [54, Equation (6.34)], which in particular rely on Lemma 9.3 as well as equation (237), derived independently below. We clarify the following new feature of the present work: in [54, Equation (6.34)], the author derived equations of the form $\mathrm{d}\!\!\!/\mathrm{iv}\left\{\mathbf{D}\!\!\!/_L \mu\!\!\!/ + \frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\chi)\mu\!\!\!/\right\} = \cdots$, $\mathrm{cu}\!\!\!/\mathrm{rl}\left\{\mathbf{D}\!\!\!/_L \mu\!\!\!/ + \frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\chi)\mu\!\!\!/\right\} = \cdots$, whereas for mathematical convenience, we have split these equations into similar equations for $\mu\!\!\!/_{(1)}$ and $\mu\!\!\!/_{(2)}$, the point being that later, we will use distinct arguments to control the $\mu\!\!\!/_{(i)}$. The splitting is possible since equation (210) is linear in $\mu\!\!\!/$.

To prove (245), we first clarify that the $\mu\!\!\!/_{(i)}$ are solved for by first solving their Hodge systems (243a)–(244b) to obtain $\mathbf{D}\!\!\!/_L \mu\!\!\!/_{(i)} + \frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\chi)\mu\!\!\!/_{(i)}$ and then integrating the corresponding inhomogeneous transport equations to obtain $\mu\!\!\!/_{(i)}$. However, there is freedom in how we relate the "initial conditions" of $\mu\!\!\!/$ along the cone-tip axis to those of $\mu\!\!\!/_{(1)}$ and $\mu\!\!\!/_{(2)}$, where the only constraint is that (242) must hold. Thus, (245) merely represents a choice of vanishing initial conditions for $\mu\!\!\!/_{(2)}$.

To prove (227), we first note that since $\frac{\partial}{\partial w}|_{\Sigma_0} = [aN]|_{\Sigma_0}$ and since $\frac{\partial}{\partial w}$ commutes with $\frac{\partial}{\partial\omega^A}$, we have the following evolution equation for the Cartesian components $\left(\frac{\partial}{\partial\omega^A}\right)^i$: $\frac{\partial}{\partial w}\left(\frac{\partial}{\partial\omega^A}\right)^i = a\frac{\partial}{\partial\omega^A}N^i + N^i\frac{\partial a}{\partial\omega^A}$. From this evolution equation and the second equation in (198), we find, after splitting θ into its trace and trace-free parts, that the evolution equation can be expressed in the following schematic form:

$$\begin{aligned}\frac{\partial}{\partial u}\left(\frac{\partial}{\partial \omega^A}\right)^i &= a \cdot \mathbf{D\!\!\!/}_{\frac{\partial}{\partial \omega^A}} N^i + a \cdot \mathrm{f}_{(\vec{L})} \cdot \partial \vec{\Psi} \cdot \left\{\left(\frac{\partial}{\partial \omega^A}\right)^j\right\}_{j=1,2,3} + \mathrm{f}_{(\vec{L})} \cdot \frac{\partial a}{\partial \omega^A} \\ &= \frac{1}{2} a \mathrm{tr}_{g\!\!\!/} \theta \left(\frac{\partial}{\partial \omega^A}\right)^i + a \cdot \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial} \vec{\Psi}, \hat{\theta}) \cdot \left\{\left(\frac{\partial}{\partial \omega^A}\right)^j\right\}_{j=1,2,3} + \mathrm{f}_{(\vec{L})} \cdot \frac{\partial a}{\partial \omega^A},\end{aligned} \tag{246}$$

where the first product on RHS (246) is precisely depicted and the last two are schematically depicted. Using (280) to substitute for the term $\mathrm{tr}_{g\!\!\!/}\theta$ and using (200), we find that (246) can be expressed as

$$\frac{\partial}{\partial w}\left(\frac{\partial}{\partial \omega^A}\right)^i = \frac{1}{w}\left(\frac{\partial}{\partial \omega^A}\right)^i + a \cdot \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial} \vec{\Psi}, \hat{\chi}) \cdot \left\{\left(\frac{\partial}{\partial \omega^A}\right)^j\right\}_{j=1,2,3} + \mathrm{f}_{(\vec{L})} \cdot \frac{\partial a}{\partial \omega^A}, \tag{247}$$

where the first product on RHS (247) is precisely depicted and the last two are schematically depicted. From (247) and the fact that $\frac{\partial}{\partial w}\tilde{r} = 1$ (because $\tilde{r}|_{\Sigma_0} = w$), we easily conclude the desired equation (227).

To prove (226), we first note that since $\frac{\partial}{\partial t} = L$ relative to the geometric coordinates, and since $\frac{\partial}{\partial t}$ commutes with $\frac{\partial}{\partial \omega^A}$, we have the following evolution equation for the Cartesian components $\left(\frac{\partial}{\partial \omega^A}\right)^i$: $\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \omega^A}\right)^i = \frac{\partial}{\partial \omega^A} L^i$. From this evolution equation and the first equation in (199a), we find, after splitting χ into its trace and trace-free parts, that the evolution equation can be expressed in the following schematic form:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \omega^A}\right)^i &= \mathbf{D\!\!\!/}_{\frac{\partial}{\partial \omega^A}} L^i + \mathrm{f}_{(\vec{L})} \cdot \partial \vec{\Psi} \cdot \left\{\left(\frac{\partial}{\partial \omega^A}\right)^j\right\}_{j=1,2,3} \\ &= \frac{1}{2} \mathrm{tr}_{g\!\!\!/} \chi \left(\frac{\partial}{\partial \omega^A}\right)^i + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial} \vec{\Psi}, \hat{\chi}) \cdot \left\{\left(\frac{\partial}{\partial \omega^A}\right)^j\right\}_{j=1,2,3},\end{aligned} \tag{248}$$

where the first product on RHS (248) is precisely depicted and the second one is schematically depicted. Using (207) to substitute for the term $\mathrm{tr}_{g\!\!\!/}\chi$, we find that (248) can be expressed as

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \omega^A}\right)^i = \frac{1}{\tilde{r}}\left(\frac{\partial}{\partial \omega^A}\right)^i + \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial} \vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!\!/}} \widetilde{\chi}^{(Small)}, \hat{\chi}) \cdot \left\{\left(\frac{\partial}{\partial \omega^A}\right)^j\right\}_{j=1,2,3}, \tag{249}$$

where the first product on RHS (249) is precisely depicted and the last one is schematically depicted. From (249) and the fact that $\frac{\partial}{\partial t}\tilde{r} = 1$, we easily conclude the desired equation (226).

Finally, we provide the lengthy derivation of (237). Throughout the analysis, we will silently use the following identities, valid for scalar functions φ:

$$L\underline{L}\varphi - \underline{L}L\varphi = 2(\underline{\zeta}_A - \zeta_A)\nabla\!\!\!/_A\varphi + k_{NN}\underline{L}\varphi - k_{NN}L\varphi, \tag{250}$$

$$\begin{aligned} L\Delta\!\!\!/\varphi - \Delta\!\!\!/L\varphi = &-\mathrm{tr}_{g\!\!\!/}\chi\Delta\!\!\!/\varphi - 2\hat{\chi}_{AB}\nabla\!\!\!/^2_{AB}\varphi - (\mathrm{d}i\!\!\!/v\chi_A)\nabla\!\!\!/_A\varphi \\ &+ \left\{\mathrm{tr}_{g\!\!\!/}\chi\underline{\zeta}_B - \chi_{AB}\underline{\zeta}_A - \mathbf{Riem}_{BCLC}\right\}\nabla\!\!\!/_B\varphi, \end{aligned} \tag{251}$$

$$\square_{\mathbf{g}(\vec{\Psi})}\varphi = -L\underline{L}\varphi + \Delta\!\!\!/\varphi - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\underline{L}\varphi - \frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi L\varphi + 2\underline{\zeta}_A\nabla\!\!\!/_A\varphi + k_{NN}\underline{L}\varphi. \tag{252}$$

The identities (250)–(252) follow from Lemma 9.2, (187), and straightforward calculations. We will also often silently use the identity (see (200)) $\underline{\chi}_{AB} = -\chi_{AB} - 2k_{AB}$ to eliminate $\underline{\chi}_{AB}$ from various equations.

We now apply L to the definition (208) and use the evolution equations (212c), (212f), and (212g), and Lemma 9.2 to deduce:

$$\begin{aligned} &L\upmu + \mathrm{tr}_{g\!\!\!/}\chi\upmu \\ &= -\underline{L}(\mathbf{Ric}_{LL}) - \frac{1}{2}\mathbf{Ric}_{LL}\mathrm{tr}_{g\!\!\!/}\underline{\chi} - (\underline{L}k_{NN})\mathrm{tr}_{g\!\!\!/}\chi - (L\mathrm{tr}_{g\!\!\!/}\chi)k_{NN} + 2(\underline{\zeta}_A - \zeta_A)\nabla\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi \\ &\quad + \mathrm{tr}_{g\!\!\!/}\chi(\mathrm{d}i\!\!\!/v\underline{\zeta} + |\underline{\zeta}|^2_{g\!\!\!/} + \frac{1}{2}\mathbf{Riem}_{A\underline{L}LA}) + \frac{1}{2}\left(\mathrm{tr}_{g\!\!\!/}\chi\hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + \mathrm{tr}_{g\!\!\!/}\underline{\chi}|\hat{\chi}|^2_{g\!\!\!/}\right) \\ &\quad - 2\hat{\chi}_{AB}\left(2\nabla\!\!\!/_A\zeta_B + k_{NN}\hat{\chi}_{AB} + 2\zeta_A\zeta_B - \hat{\underline{\chi}}_{AC}\hat{\chi}_{CB} + \mathbf{Riem}_{AL\underline{L}B}\right). \end{aligned} \tag{253}$$

We will now re-express the factor $\underline{L}(k_{NN})$ that appears on RHS (253). To this end, we set $X = Y := N$ in (194), apply $\mathbf{D_B}$ to both sides (so that the LHS of the resulting identity is the scalar function $\mathbf{B}(k_{NN})$), commute $\mathbf{D_B}$ with $\mathbf{D}_N$ on the RHS of the resulting identity using the definition of curvature, use the relation $\mathbf{B} = \frac{1}{2}(L + \underline{L})$ (see (182)), use the relation $\mathbf{D_B}\mathbf{B} = 0$ (which is straightforward to derive using that $\mathbf{g}(\mathbf{B}, \mathbf{B}) = -1$ and the fact that $[\mathbf{B}, Z]$ is Σ_t-tangent—hence $\mathbf{g}$-orthogonal to $\mathbf{B}$—whenever Z is Σ_t-tangent), and use Lemma 9.2 to derive the following "second variation" identity:

$$\underline{L}(k_{NN}) = -L(k_{NN}) + 2k_{AN}k_{AN} - 2(k_{NN})^2 + 4k_{AN}\zeta_A + \frac{1}{2}\mathbf{Riem}_{L\underline{L}L\underline{L}}. \tag{254}$$

Since

$$\begin{aligned} \mathbf{Ric}_{\underline{L}L} &= \underline{L}^\alpha L^\beta(\mathbf{g}^{-1})^{\mu\nu}\mathbf{Riem}_{\alpha\mu\beta\nu}, \\ \mathbf{Riem}_{A\underline{L}AL} &= \underline{L}^\alpha L^\beta(g\!\!\!/^{-1})^{\mu\nu}\mathbf{Riem}_{\alpha\mu\beta\nu}, \end{aligned}$$

we have, in view of (187),

$$\begin{aligned}\mathbf{Ric}_{\underline{L}L} - \mathbf{Riem}_{A\underline{L}AL} &= \underline{L}^{\alpha} L^{\beta}\left[(\mathbf{g}^{-1})^{\mu\nu} - (\not{g}^{-1})^{\mu\nu}\right]\mathbf{Riem}_{\alpha\mu\beta\nu} \\ &= -\frac{1}{2}\underline{L}^{\alpha} L^{\beta} L^{\mu}\underline{L}^{\nu}\mathbf{Riem}_{\alpha\mu\beta\nu} - \frac{1}{2}\underline{L}^{\alpha} L^{\beta}\underline{L}^{\mu} L^{\nu}\mathbf{Riem}_{\alpha\mu\beta\nu} \\ &= -\frac{1}{2}(\mathbf{Riem}_{\underline{L}LL\underline{L}} + \mathbf{Riem}_{\underline{L}\underline{L}LL}) = \frac{1}{2}\mathbf{Riem}_{L\underline{L}L\underline{L}}.\end{aligned}$$

From this identity, the symmetries of the Riemann curvature tensor, and (221b), we find that

$$\begin{aligned}\frac{1}{2}\mathbf{Riem}_{L\underline{L}L\underline{L}} &= \mathbf{Ric}_{\underline{L}L} - \delta^{AB}\mathbf{Riem}_{A\underline{L}BL} \\ &= \mathbf{Ric}_{\underline{L}L} + \not{\mathrm{div}}\xi + \lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}.\end{aligned}$$

Combining the above calculations, we can rewrite (253) as follows:

$$\begin{aligned}L\mu + \mathrm{tr}_{\not{g}}\chi\mu = &-\underline{L}(\mathbf{Ric}_{LL}) - \frac{1}{2}\mathbf{Ric}_{LL}\mathrm{tr}_{\not{g}}\underline{\chi} \\ &+ \mathrm{tr}_{\not{g}}\chi\left\{L(k_{NN}) - \mathbf{Ric}_{\underline{L}L} - \not{\mathrm{div}}\xi - 2k_{AN}k_{AN} + 2(k_{NN})^2 - 4k_{AN}\zeta_A\right\} \\ &- \mathrm{tr}_{\not{g}}\chi\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}\right\} \\ &- (L\mathrm{tr}_{\not{g}}\chi)k_{NN} + 2(\underline{\zeta}_A - \zeta_A)\not{\nabla}_A\mathrm{tr}_{\not{g}}\chi \\ &+ \mathrm{tr}_{\not{g}}\chi\left(\not{\mathrm{div}}\underline{\zeta} + |\underline{\zeta}|_{\not{g}}^2 + \frac{1}{2}\mathbf{Riem}_{A\underline{L}LA}\right) + \frac{1}{2}\left(\mathrm{tr}_{\not{g}}\chi\hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + \mathrm{tr}_{\not{g}}\underline{\chi}|\hat{\chi}|_{\not{g}}^2\right) \\ &- 2\hat{\chi}_{AB}\left(2\not{\nabla}_A\zeta_B + k_{NN}\hat{\chi}_{AB} + 2\zeta_A\zeta_B - \hat{\underline{\chi}}_{AC}\hat{\chi}_{CB} + \mathbf{Riem}_{AL\underline{L}B}\right).\end{aligned}$$

With the help of (200), we can rearrange the RHS to rewrite this identity as follows:

$$\begin{aligned}L\mu + \mathrm{tr}_{\not{g}}\chi\mu = &-\underline{L}(\mathbf{Ric}_{LL}) - \frac{1}{2}\mathbf{Ric}_{LL}\mathrm{tr}_{\not{g}}\underline{\chi} - \mathrm{tr}_{\not{g}}\chi\mathbf{Ric}_{\underline{L}L} \\ &- (L\mathrm{tr}_{\not{g}}\chi)k_{NN} + 2(\underline{\zeta}_A - \zeta_A)\not{\nabla}_A\mathrm{tr}_{\not{g}}\chi + \frac{1}{2}\left(\mathrm{tr}_{\not{g}}\chi\hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + \mathrm{tr}_{\not{g}}\underline{\chi}|\hat{\chi}|_{\not{g}}^2\right) \\ &+ \mathrm{tr}_{\not{g}}\chi\left\{\not{\mathrm{div}}\underline{\zeta} - \not{\mathrm{div}}\xi + \frac{1}{2}\mathbf{Riem}_{A\underline{L}LA} + L(k_{NN}) - |\underline{\zeta}|_{\not{g}}^2 + 2(k_{NN})^2 + 4\underline{\zeta}_A\zeta_A\right\} \\ &- 2\hat{\chi}_{AB}\left(2\not{\nabla}_A\zeta_B + k_{NN}\hat{\chi}_{AB} + 2\zeta_A\zeta_B - \hat{\underline{\chi}}_{AC}\hat{\chi}_{CB} + \mathbf{Riem}_{AL\underline{L}B}\right) \\ &- \mathrm{tr}_{\not{g}}\chi\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}\right\}.\end{aligned} \tag{255}$$

We will now uncover the structure of the terms on RHS (255). To help the reader navigate the calculations in the remainder of the proof of (237), we also recall that we treat $\mathbf{\Gamma}_{\alpha} := (\mathbf{g}^{-1})^{\kappa\lambda}\mathbf{g}_{\alpha\beta}\mathbf{\Gamma}_{\kappa\ \lambda}^{\ \beta}$ as a one-form under covariant differentiation (as in Lemma 9.6), that $\mathbf{\Gamma}_{\underline{L}} := \underline{L}^{\alpha}\mathbf{\Gamma}_{\alpha}$, and that $\mathbf{\Gamma}_A := e_A^{\alpha}\mathbf{\Gamma}_{\alpha}$. First, invoking (218) and (220), we find that

$$
\begin{aligned}
&- \underline{L}(\mathbf{Ric}_{LL}) - \frac{1}{2}\mathbf{Ric}_{LL}\mathrm{tr}_{g\!\!/}\underline{\chi} - \mathrm{tr}_{g\!\!/}\chi\mathbf{Ric}_{\underline{L}L} \\
&= -\underline{L}L(\boldsymbol{\Gamma}_L) - \frac{1}{2}\mathrm{tr}_{g\!\!/}\underline{\chi}L(\boldsymbol{\Gamma}_L) - \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi L(\boldsymbol{\Gamma}_{\underline{L}}) - \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\underline{L}(\boldsymbol{\Gamma}_L) \\
&\quad - (\underline{L}(k_{NN}))\boldsymbol{\Gamma}_L - k_{NN}\underline{L}(\boldsymbol{\Gamma}_L) - \frac{1}{2}\mathrm{tr}_{g\!\!/}\underline{\chi}k_{NN}\boldsymbol{\Gamma}_L - \underline{L}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\} \\
&\quad - \mathrm{tr}_{g\!\!/}\underline{\chi}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\} - \mathrm{tr}_{g\!\!/}\chi\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\cdot\boldsymbol{\partial}\vec{\Psi}\right\}.
\end{aligned}
\tag{256}
$$

A key observation is that the first, second, and fourth terms on RHS (256) produce $\square_{\mathbf{g}(\vec{\Psi})}(\boldsymbol{\Gamma}_L)$ (up to lower-order terms) when added to[49] $\Delta\!\!\!/(\boldsymbol{\Gamma}_L)$; this can be seen from the expression (252). Next, we apply the operator $2\Delta\!\!\!/$ to (201a) and use (251) to commute it through the operator L. Also using the identity $\mathrm{div}\!\!\!/\,\chi_A = -\hat{\chi}_{AB}k_{BN} + \nabla\!\!\!/_A\mathrm{tr}_{g\!\!/}\chi + \frac{1}{2}k_{AN}\mathrm{tr}_{g\!\!/}\chi + \mathbf{Riem}_{BLBA}$ (see (213a)) and (200), we obtain the following identity:

$$
\begin{aligned}
2L\Delta\!\!\!/\sigma + 2\mathrm{tr}_{g\!\!/}\chi\Delta\!\!\!/\sigma &= \Delta\!\!\!/(\boldsymbol{\Gamma}_L) - 4\hat{\chi}_{AB}\nabla\!\!\!/^2_{AB}\sigma - 2(\nabla\!\!\!/_A\mathrm{tr}_{g\!\!/}\chi)\nabla\!\!\!/_A\sigma \\
&\quad - 4\mathbf{Riem}_{ABLB}\nabla\!\!\!/_A\sigma + \mathrm{tr}_{g\!\!/}\chi\underline{\zeta}_A\nabla\!\!\!/_A\sigma - 4\hat{\chi}_{AB}\underline{\zeta}_A\nabla\!\!\!/_B\sigma.
\end{aligned}
\tag{257}
$$

Adding (255) and (257), using (256) and (252), and rearranging the terms, we deduce that

$$
\begin{aligned}
&L(\mu + 2\Delta\!\!\!/\sigma) + \mathrm{tr}_{g\!\!/}\chi(\mu + 2\Delta\!\!\!/\sigma) \\
&= \square_{\mathbf{g}(\vec{\Psi})}(\boldsymbol{\Gamma}_L) - 2\underline{\zeta}_A\nabla\!\!\!/_A(\boldsymbol{\Gamma}_L) - 2k_{NN}\underline{L}(\boldsymbol{\Gamma}_L) - (\underline{L}(k_{NN}))\boldsymbol{\Gamma}_L \\
&\quad - 4\hat{\chi}_{AB}\nabla\!\!\!/^2_{AB}\sigma - 4\hat{\chi}_{AB}\nabla\!\!\!/_A\zeta_B \\
&\quad - 2(\nabla\!\!\!/_A\mathrm{tr}_{g\!\!/}\chi)\nabla\!\!\!/_A\sigma + 2(\underline{\zeta}_A - \zeta_A)\nabla\!\!\!/_A\mathrm{tr}_{g\!\!/}\chi \\
&\quad - 4\mathbf{Riem}_{ABLB}\nabla\!\!\!/_A\sigma \\
&\quad - (k_{AN}\mathrm{tr}_{g\!\!/}\chi - 2\hat{\chi}_{AB}k_{BN})\nabla\!\!\!/_A\sigma - 2\chi_{AB}\underline{\zeta}_A\nabla\!\!\!/_B\sigma \\
&\quad - (L\mathrm{tr}_{g\!\!/}\chi)k_{NN} \\
&\quad + \mathrm{tr}_{g\!\!/}\chi\left\{\mathrm{div}\!\!\!/\,\underline{\zeta} - \mathrm{div}\!\!\!/\,\xi\right\} + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\mathbf{Riem}_{A\underline{L}LA} \\
&\quad + \mathrm{tr}_{g\!\!/}\chi L(k_{NN}) - \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi L(\boldsymbol{\Gamma}_{\underline{L}}) \\
&\quad + \mathrm{tr}_{g\!\!/}\chi\left\{-|\underline{\zeta}|^2_{g\!\!/} + 2(k_{NN})^2 + 4\underline{\zeta}_A\zeta_A\right\} \\
&\quad + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi\hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + \mathrm{tr}_{g\!\!/}\underline{\chi}|\hat{\chi}|^2_{g\!\!/} - \frac{1}{2}\mathrm{tr}_{g\!\!/}\underline{\chi}k_{NN}\boldsymbol{\Gamma}_L \\
&\quad - 2\hat{\chi}_{AB}\mathbf{Riem}_{AL\underline{L}B} - 2\hat{\chi}_{AB}\left\{k_{NN}\hat{\chi}_{AB} + 2\zeta_A\zeta_B\right\} \\
&\quad - \mathrm{tr}_{g\!\!/}\chi\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi} \\
&\quad - \underline{L}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\}
\end{aligned}
$$

[49] We recall that $\Delta\!\!\!/ f := \nabla\!\!\!/^2_{AA}f$; see the discussion in Sect. 9.6.4.

$$- \mathrm{tr}_{g\!\!\!/}\underline{\chi}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})+\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\}$$
$$- \mathrm{tr}_{g\!\!\!/}\chi\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\cdot\boldsymbol{\partial}\vec{\Psi}\right\}. \tag{258}$$

We next manipulate (258) as follows: we move the terms $\mathrm{tr}_{g\!\!\!/}\chi L(k_{NN})-\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi L(\boldsymbol{\Gamma}_{\underline{L}})$ from the RHS to the LHS; subtract $(L\mathrm{tr}_{g\!\!\!/}\chi)k_{NN}$ from both sides; add $\frac{1}{2}(L\mathrm{tr}_{g\!\!\!/}\chi)\boldsymbol{\Gamma}_{\underline{L}}$ to both sides; and finally, add $\frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\chi)^2\boldsymbol{\Gamma}_{\underline{L}}-(\mathrm{tr}_{g\!\!\!/}\chi)^2 k_{NN}$ to both sides. After these steps, in view of definition (209), we see that the LHS becomes

$$L\check{\mu}+\mathrm{tr}_{g\!\!\!/}\chi\check{\mu}.$$

With the help of definition (211), we now rearrange some other terms on RHS (258) as follows:

$$-4\hat{\chi}_{AB}\nabla\!\!\!/^2_{AB}\sigma-4\hat{\chi}_{AB}\nabla\!\!\!/_A\zeta_B=-4\hat{\chi}_{AB}\nabla\!\!\!/_A\widetilde{\zeta}_B,$$
$$-2(\nabla\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi)\nabla\!\!\!/_A\sigma+2(\underline{\zeta}_A-\zeta_A)\nabla\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi=2(\underline{\zeta}_A-\widetilde{\zeta}_A)\nabla\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi.$$

Combining the above calculations, we have thus far obtained the following equation:

$$\begin{aligned}
&L\check{\mu}+\mathrm{tr}_{g\!\!\!/}\chi\check{\mu}\\
&\quad=\square_{\mathbf{g}(\vec{\Psi})}(\boldsymbol{\Gamma}_L)-2\underline{\zeta}_A\nabla\!\!\!/_A(\boldsymbol{\Gamma}_L)-2k_{NN}\underline{L}(\boldsymbol{\Gamma}_L)-(\underline{L}(k_{NN}))\boldsymbol{\Gamma}_L\\
&\quad+\frac{1}{2}(\mathrm{tr}_{g\!\!\!/}\chi)^2\boldsymbol{\Gamma}_{\underline{L}}-(\mathrm{tr}_{g\!\!\!/}\chi)^2k_{NN}\\
&\quad-4\hat{\chi}_{AB}\nabla\!\!\!/_A\widetilde{\zeta}_B+2(\underline{\zeta}_A-\widetilde{\zeta}_A)\nabla\!\!\!/_A\mathrm{tr}_{g\!\!\!/}\chi\\
&\quad-4\mathbf{Riem}_{ABLB}\nabla\!\!\!/_A\sigma\\
&\quad-(k_{AN}\mathrm{tr}_{g\!\!\!/}\chi-2\hat{\chi}_{AB}k_{BN})\nabla\!\!\!/_A\sigma-2\chi_{AB}\underline{\zeta}_A\nabla\!\!\!/_B\sigma\\
&\quad-(2L\mathrm{tr}_{g\!\!\!/}\chi)k_{NN}+\frac{1}{2}(L\mathrm{tr}_{g\!\!\!/}\chi)\boldsymbol{\Gamma}_{\underline{L}}\\
&\quad+\mathrm{tr}_{g\!\!\!/}\chi\left\{\mathrm{div\!\!\!/}\,\underline{\zeta}-\mathrm{div\!\!\!/}\,\xi\right\}+\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\mathbf{Riem}_{A\underline{L}LA}\\
&\quad+\mathrm{tr}_{g\!\!\!/}\chi\left\{-|\underline{\zeta}|^2_{g\!\!\!/}+2(k_{NN})^2+4\underline{\zeta}_A\zeta_A\right\}\\
&\quad+\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\chi\hat{\chi}_{AB}\hat{\underline{\chi}}_{AB}+\mathrm{tr}_{g\!\!\!/}\underline{\chi}|\hat{\chi}|^2_{g\!\!\!/}-\frac{1}{2}\mathrm{tr}_{g\!\!\!/}\underline{\chi}k_{NN}\boldsymbol{\Gamma}_L\\
&\quad-2\hat{\chi}_{AB}\mathbf{Riem}_{AL\underline{L}B}-2\hat{\chi}_{AB}\left\{k_{NN}\hat{\chi}_{AB}+2\zeta_A\zeta_B\right\}\\
&\quad-\mathrm{tr}_{g\!\!\!/}\chi\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}\\
&\quad-\underline{L}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})+\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\}\\
&\quad-\mathrm{tr}_{g\!\!\!/}\underline{\chi}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})+\mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\}\\
&\quad-\mathrm{tr}_{g\!\!\!/}\chi\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})+\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\zeta)\cdot\boldsymbol{\partial}\vec{\Psi}\right\}.
\end{aligned}\tag{259}$$

We proceed to expand the term $\square_{\mathbf{g}(\vec{\Psi})}(\mathbf{\Gamma}_L)$ on RHS (259), where we recall from Definition 9.3 that $\mathbf{\Gamma}_L := L^\alpha \mathbf{\Gamma}_\alpha$. We therefore compute that[50]

$$\square_{\mathbf{g}(\vec{\Psi})}(\mathbf{\Gamma}_L) = \mathbf{\Gamma}_\alpha \square_{\mathbf{g}(\vec{\Psi})} L^\alpha + L^\alpha \square_{\mathbf{g}(\vec{\Psi})} \mathbf{\Gamma}_\alpha + 2(\mathbf{D}_\mu L^\alpha)\mathbf{D}^\mu \mathbf{\Gamma}_\alpha. \tag{260}$$

We first handle the term $\square_{\mathbf{g}(\vec{\Psi})} L^\alpha$ in (260). Using the decomposition of $\mathbf{g}^{-1}$ relative to a null frame (i.e., (187)) and Lemma 9.2, we compute that

$$\begin{aligned}
\square_{\mathbf{g}(\vec{\Psi})} L^\alpha &= -\mathbf{D}_{\underline{L}}(\mathbf{D}_L L^\alpha) + \mathbf{D}_{e_A}(\mathbf{D}_{e_A} L^\alpha) + \mathbf{D}_{\mathbf{D}_{\underline{L}} L} L^\alpha - \mathbf{D}_{\mathbf{D}_{e_A} e_A} L^\alpha - \frac{1}{2}\mathbf{Riem}^\alpha{}_{LL\underline{L}} \\
&= \mathrm{d}\!\!/\mathrm{iv}\chi_A e_A^\alpha + (\mathrm{d}\!\!/\mathrm{iv}\underline{\zeta})L^\alpha + (\underline{L}(k_{NN}))L^\alpha + \frac{1}{2}|\chi|^2_{g\!\!/}\underline{L}^\alpha + \frac{1}{2}\chi_{AB}\underline{\chi}_{AB}L^\alpha \\
&\quad - \mathrm{tr}_{g\!\!/}\chi\zeta_A e_A^\alpha - \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi k_{NN} L^\alpha + \frac{1}{2}\mathrm{tr}_{g\!\!/}\underline{\chi} k_{NN} L^\alpha + 2\zeta_A \chi_{AB} e_B^\alpha + 2k_{NN}\zeta_A e_A^\alpha \\
&\quad + \underline{\zeta}_A \chi_{AB} e_B^\alpha + |\underline{\zeta}|^2_{g\!\!/} L^\alpha - \frac{1}{2}\mathbf{Riem}^\alpha{}_{LL\underline{L}}.
\end{aligned} \tag{261}$$

Contracting (261) against $\mathbf{\Gamma}_\alpha$, we find that

$$\begin{aligned}
\mathbf{\Gamma}_\alpha \square_{\mathbf{g}(\vec{\Psi})} L^\alpha &= (\mathrm{d}\!\!/\mathrm{iv}\chi_A)\mathbf{\Gamma}_A + (\mathrm{d}\!\!/\mathrm{iv}\underline{\zeta})\mathbf{\Gamma}_L + (\underline{L}(k_{NN}))\mathbf{\Gamma}_L + \frac{1}{2}|\chi|^2_{g\!\!/}\mathbf{\Gamma}_{\underline{L}} + \frac{1}{2}\chi_{AB}\underline{\chi}_{AB}\mathbf{\Gamma}_L \\
&\quad - \mathrm{tr}_{g\!\!/}\chi\zeta_A\mathbf{\Gamma}_A - \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi k_{NN}\mathbf{\Gamma}_L + \frac{1}{2}\mathrm{tr}_{g\!\!/}\underline{\chi} k_{NN}\mathbf{\Gamma}_L + 2\zeta_A\chi_{AB}\mathbf{\Gamma}_B \\
&\quad + \underline{\zeta}_A\chi_{AB}\mathbf{\Gamma}_B + |\underline{\zeta}|^2_{g\!\!/}\mathbf{\Gamma}_L - \frac{1}{2}\mathbf{\Gamma}_\alpha \mathbf{Riem}^\alpha{}_{LL\underline{L}}.
\end{aligned} \tag{262}$$

Next, we again use (187) and Lemma 9.2 to compute the last product in (260):

$$\begin{aligned}
2(\mathbf{D}_\mu L^\alpha)\mathbf{D}^\mu \mathbf{\Gamma}_\alpha &= k_{NN} L^\alpha \mathbf{D}_L \mathbf{\Gamma}_\alpha - 2\zeta_A e_A^\alpha \mathbf{D}_L \mathbf{\Gamma}_\alpha - k_{NN} L^\alpha \mathbf{D}_{\underline{L}} \mathbf{\Gamma}_\alpha \\
&\quad - 2k_{AN} L^\alpha \mathbf{D}_A \mathbf{\Gamma}_\alpha + 2\chi_{AB} e_B^\alpha \mathbf{D}_A \mathbf{\Gamma}_\alpha.
\end{aligned} \tag{263}$$

Next we use the decomposition $\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi g\!\!/_{AB}$, (207), and (213a) to rewrite the first product on RHS (262) as follows:

$$\begin{aligned}
(\mathrm{d}\!\!/\mathrm{iv}\chi_A)\mathbf{\Gamma}_A &= (\nabla\!\!\!/_A \mathrm{tr}_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)})\mathbf{\Gamma}_A - (\nabla\!\!\!/_A(\mathbf{\Gamma}_L))\mathbf{\Gamma}_A - \hat{\chi}_{AB} k_{BN}\mathbf{\Gamma}_A \\
&\quad + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi k_{AN}\mathbf{\Gamma}_A + \mathbf{Riem}_{ALAB}\mathbf{\Gamma}_B.
\end{aligned} \tag{264}$$

Moreover, we use the decomposition $\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{g\!\!/}\chi g\!\!/_{AB}$ and (207) to rewrite the last product on RHS (263) as follows, where ξ denotes the $\mathbf{g}$-orthogonal projection onto $S_{t,u}$ of the one-form with Cartesian components $2\mathbf{\Gamma}_\alpha$:

[50] Since $\square_{\mathbf{g}(\vec{\Psi})}(\mathbf{\Gamma}_L) = \square_{\mathbf{g}(\vec{\Psi})}(L^\alpha \mathbf{\Gamma}_\alpha) = (\mathbf{g}^{-1})^{\alpha\beta}\mathbf{D}_\alpha \mathbf{D}_\beta (L^\alpha \mathbf{\Gamma}_\alpha)$, to obtain (260), we have expanded this expression using the Leibniz rule, where we treat L^α as a vectorfield under covariant differentiation and we treat $\mathbf{\Gamma}_\alpha$ as a one-form under covariant differentiation.

$$\begin{aligned}2\chi_{AB}e_B^\alpha \mathbf{D}_A\boldsymbol{\Gamma}_\alpha &= 2\hat{\chi}_{AB}e_B^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha + \mathrm{tr}_{\not{g}}\chi e_A^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha \\ &= \frac{2}{\tilde{r}}e_A^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha + \mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}e_A^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha - \boldsymbol{\Gamma}_L e_A^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha + 2\hat{\chi}_{AB}e_B^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha \\ &= \frac{1}{\tilde{r}}\mathrm{d}\!\!\not{\mathrm{i}}\mathrm{v}\xi + \mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}e_A^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha - \boldsymbol{\Gamma}_L e_A^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha + 2\hat{\chi}_{AB}e_B^\alpha \not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha.\end{aligned} \tag{265}$$

Moreover, using (212c), we derive the following identity for the two $L\mathrm{tr}_{\not{g}}\chi$-involving products on RHS (259):

$$-2(L\mathrm{tr}_{\not{g}}\chi)k_{NN} = (\mathrm{tr}_{\not{g}}\chi)^2 k_{NN} + 2|\hat{\chi}|_{\not{g}}^2 k_{NN} + 2\mathrm{tr}_{\not{g}}\chi(k_{NN})^2 + 2\mathbf{Ric}_{LL}k_{NN}, \tag{266}$$

$$\frac{1}{2}(L\mathrm{tr}_{\not{g}}\chi)\boldsymbol{\Gamma}_{\underline{L}} = -\frac{1}{4}(\mathrm{tr}_{\not{g}}\chi)^2\boldsymbol{\Gamma}_{\underline{L}} - \frac{1}{2}|\hat{\chi}|_{\not{g}}^2\boldsymbol{\Gamma}_{\underline{L}} - \frac{1}{2}\mathrm{tr}_{\not{g}}\chi k_{NN}\boldsymbol{\Gamma}_{\underline{L}} - \frac{1}{2}\mathbf{Ric}_{LL}\boldsymbol{\Gamma}_{\underline{L}}. \tag{267}$$

We now use (266)–(267) to substitute for the relevant products on RHS (259), we use (260) to substitute for the first term $\square_{\mathbf{g}(\vec{\Psi})}(\boldsymbol{\Gamma}_L)$ on RHS (259), we use (262)–(263) to substitute for the first and third products on RHS (260) (specifically, $\boldsymbol{\Gamma}_\alpha\square_{\mathbf{g}(\vec{\Psi})}L^\alpha$ and $2(\mathbf{D}_\mu L^\alpha)\mathbf{D}^\mu\boldsymbol{\Gamma}_\alpha$), and we use (264)–(265) to substitute for the relevant products on RHSs (262)–(263). Also using (200), in total, we compute that the following equation holds:

$$\begin{aligned}L\check{\mu} + \mathrm{tr}_{\not{g}}\chi\check{\mu} &= L^\alpha\square_{\mathbf{g}(\vec{\Psi})}\boldsymbol{\Gamma}_\alpha - 4\mathbf{Riem}_{ABLB}\not{\nabla}_A\sigma - 2\hat{\chi}_{AB}\mathbf{Riem}_{AL\underline{L}B} + \mathbf{Riem}_{ALAB}\boldsymbol{\Gamma}_B \\ &\quad - \frac{1}{2}\boldsymbol{\Gamma}_\alpha\mathbf{Riem}^\alpha{}_{LL\underline{L}} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\mathbf{Riem}_{A\underline{L}LA} + 2\mathbf{Ric}_{LL}k_{NN} - \frac{1}{2}\mathbf{Ric}_{LL}\boldsymbol{\Gamma}_{\underline{L}} + \check{\mathrm{Err}},\end{aligned} \tag{268}$$

where

$$\begin{aligned}\check{\mathrm{Err}} &= \frac{1}{\tilde{r}}\mathrm{d}\!\!\not{\mathrm{i}}\mathrm{v}\xi + \mathrm{tr}_{\not{g}}\chi\left\{\mathrm{d}\!\!\not{\mathrm{i}}\mathrm{v}\underline{\zeta} - \mathrm{d}\!\!\not{\mathrm{i}}\mathrm{v}\xi\right\} + \frac{1}{\tilde{r}^2}\xi + \frac{1}{4}(\mathrm{tr}_{\not{g}}\chi)^2\boldsymbol{\Gamma}_{\underline{L}} + \frac{1}{2}|\chi|_{\not{g}}^2\boldsymbol{\Gamma}_{\underline{L}} + \frac{1}{2}\chi_{AB}\underline{\chi}_{AB}\boldsymbol{\Gamma}_L \\ &\quad + (\not{\nabla}_A\mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)})\boldsymbol{\Gamma}_A + (\mathrm{d}\!\!\not{\mathrm{i}}\mathrm{v}\underline{\zeta})\boldsymbol{\Gamma}_L + k_{NN}L^\alpha\mathbf{D}_L\boldsymbol{\Gamma}_\alpha - 2\zeta_A e_A^\alpha\mathbf{D}_L\boldsymbol{\Gamma}_\alpha - k_{NN}L^\alpha\mathbf{D}_{\underline{L}}\boldsymbol{\Gamma}_\alpha \\ &\quad - 2k_{NN}\underline{L}(\boldsymbol{\Gamma}_L) - (\not{\nabla}_A(\boldsymbol{\Gamma}_L))\boldsymbol{\Gamma}_A - 2k_{AN}L^\alpha\mathbf{D}_A\boldsymbol{\Gamma}_\alpha + \mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}e_A^\alpha\not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha \\ &\quad - \boldsymbol{\Gamma}_L e_A^\alpha\not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha + 2\hat{\chi}_{AB}e_B^\alpha\not{\mathbf{D}}_A\boldsymbol{\Gamma}_\alpha - 2\underline{\zeta}_A\not{\nabla}_A(\boldsymbol{\Gamma}_L) - 4\hat{\chi}_{AB}\not{\nabla}_A\widetilde{\zeta}_B + 2(\underline{\zeta}_A - \widetilde{\zeta}_A)\not{\nabla}_A\mathrm{tr}_{\not{g}}\chi \\ &\quad - (k_{AN}\mathrm{tr}_{\not{g}}\chi - 2\hat{\chi}_{AB}k_{BN})\not{\nabla}_A\sigma - 2\chi_{AB}\underline{\zeta}_A\not{\nabla}_B\sigma + 2\chi_{AB}\zeta_A\boldsymbol{\Gamma}_B + \chi_{AB}\underline{\zeta}_A\boldsymbol{\Gamma}_B \\ &\quad + \mathrm{tr}_{\not{g}}\chi\left\{-|\underline{\zeta}|_{\not{g}}^2 + 4(k_{NN})^2 + 4\underline{\zeta}_A\zeta_A - \frac{1}{2}k_{NN}\boldsymbol{\Gamma}_L + \frac{1}{2}k_{AN}\boldsymbol{\Gamma}_A - \zeta_A\boldsymbol{\Gamma}_A - \frac{1}{2}k_{NN}\boldsymbol{\Gamma}_{\underline{L}}\right\} \\ &\quad + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\hat{\chi}_{AB}\hat{\underline{\chi}}_{AB} + \mathrm{tr}_{\not{g}}\underline{\chi}|\hat{\chi}|_{\not{g}}^2 + \hat{\chi}_{AB}\left\{-4\zeta_A\zeta_B + \underline{\zeta}_A\boldsymbol{\Gamma}_B\right\} - \frac{1}{2}|\hat{\chi}|_{\not{g}}^2\boldsymbol{\Gamma}_{\underline{L}} \\ &\quad + 2k_{NN}\zeta_A\boldsymbol{\Gamma}_A + |\underline{\zeta}|_{\not{g}}^2\boldsymbol{\Gamma}_L - \mathrm{tr}_{\not{g}}\chi\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi} \\ &\quad - \underline{L}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\} - \mathrm{tr}_{\not{g}}\underline{\chi}\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\boldsymbol{\partial}\vec{\Psi}\right\} \\ &\quad - \mathrm{tr}_{\not{g}}\chi\left\{\lambda^{-1}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}}, \mathcal{D}) + \mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi}, \zeta)\cdot\boldsymbol{\partial}\vec{\Psi}\right\}.\end{aligned} \tag{269}$$

With the help of the decomposition $\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}_{\not{g}}\chi \not{g}_{AB}$, definition (207) (which implies that schematically, we have $\mathrm{tr}_{\not{g}}\chi = \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)}, \tilde{r}^{-1}))$, definition (211), the identity $\chi_{AB} + \underline{\chi}_{AB} = -2k_{AB}$ (see (200)), and Lemma 9.5, we verify by direct inspection that all terms on RHS (269) can be accommodated into RHS (238b), aside from the terms on the first line of RHS (269), which split into terms of type RHS (238a) and of type RHS (238b).

To finish the proof of (237), it remains only for us to verify that the remaining terms on RHS (268) have the form of terms on either RHS (238a) or RHS (238b). First, using (222c), we see that the term $-4\mathbf{Riem}_{ABLB}\not{\nabla}_A\sigma$ can be accommodated into the terms on RHS (238b) featuring a factor of $\not{\nabla}\sigma$. Next, to handle the term $-2\hat{\chi}_{AB}\mathbf{Riem}_{AL\underline{L}B}$, we first note that since the Cartesian components $\mathbf{g}_{\alpha\beta}$ are of the schematic form $\mathbf{g}_{\alpha\beta} = \mathrm{f}(\vec{\Psi})$, the standard expression for the components of $\mathbf{Riem}$ in terms of the Christoffel symbols of $\mathbf{g}$ and their first derivatives yields that relative to the Cartesian coordinates, we have $\mathbf{Riem}_{\alpha\beta\gamma\delta} = \mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}^2\vec{\Psi} + \mathrm{f}(\vec{\Psi})\cdot(\boldsymbol{\partial}\vec{\Psi})^2$. It follows that, schematically, we have $-2\hat{\chi}_{AB}\mathbf{Riem}_{AL\underline{L}B} = \mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot\boldsymbol{\partial}^2\vec{\Psi} + \mathrm{f}_{(\vec{L})}\cdot\hat{\chi}\cdot(\boldsymbol{\partial}\vec{\Psi})^2$, which is of the form of the next-to-last and last products on RHS (238b). Using the schematic relations $\boldsymbol{\Gamma}_\alpha = \mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}$, $\boldsymbol{\Gamma}_{\underline{L}} = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$, and $k_{NN} = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$, we can handle the terms $\mathbf{Riem}_{ALAB}\boldsymbol{\Gamma}_B$, $-\frac{1}{2}\boldsymbol{\Gamma}_\alpha\mathbf{Riem}^\alpha{}_{LL\underline{L}}$, $2\mathbf{Ric}_{LL}k_{NN}$, and $-\frac{1}{2}\mathbf{Ric}_{LL}\boldsymbol{\Gamma}_{\underline{L}}$ using a similar argument, which allows us to incorporate these error terms into the next-to-last and last products on RHS (238b). To handle $\frac{1}{2}\mathrm{tr}_{\not{g}}\chi\mathbf{Riem}_{A\underline{L}LA}$, we use (221b) to substitute for $\mathbf{Riem}_{A\underline{L}LA}$ and (207); this leads to terms of the form RHSs (238a)–(238b). To treat the remaining term $L^\alpha\square_{\mathbf{g}}\boldsymbol{\Gamma}_\alpha$ on RHS (268), we first recall that $\boldsymbol{\Gamma}_\alpha = \mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}$. Thus, we can commute equation (156) with $\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}$ (recall that we have dropped the "λ" subscripts featured in (156)) to conclude that $L^\alpha\square_{\mathbf{g}}\boldsymbol{\Gamma}_\alpha$ can be accommodated into the terms on RHS (238b) as desired. We clarify that when one commutes equation (156) with $\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}$, a source term appears from the RHS (156) that is of the form $\lambda^{-1}\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}\cdot(\vec{\mathcal{C}},\mathcal{D})$. One then uses equations (153a) and (153b) to express $\vec{\mathcal{C}} = \mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Omega} + \mathrm{f}(\vec{\Psi})\cdot\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi}$ and $\mathcal{D} = \mathrm{f}(\vec{\Psi})\boldsymbol{\partial}\vec{S} + \mathrm{f}(\vec{\Psi})\cdot\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi}$. In particular, this leads to the presence of the terms of type $\lambda^{-1}\mathrm{f}(\vec{\Psi})\cdot\boldsymbol{\partial}\vec{\Psi}\cdot\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi}$. This finishes the proof of (237) and completes our proof sketch of the proposition. □

9.10 Norms

In this subsection, we define the norms that we will use to control the acoustic geometry. These norms are stated in terms of the volume forms defined in Sect. 9.6.3.

Definition 9.6 *(Norms).* For $S_{t,u}$-tangent tensorfields ξ and $q \in [1,\infty)$, we define

$$\|\xi\|_{L^q_{\not{g}}(S_{t,u})} := \left\{\int_{\omega\in\mathbb{S}^2} |\xi(t,u,\omega)|^q_{\not{g}}\, d\varpi_{\not{g}(t,u,\omega)}\right\}^{1/q}, \tag{270a}$$

$$\|\xi\|_{L^q_{\omega}(S_{t,u})} := \left\{\int_{\omega\in\mathbb{S}^2} |\xi(t,u,\omega)|^q_{\not{g}}\, d\varpi_{\not{e}(\omega)}\right\}^{1/q}, \quad \|\xi\|_{L^\infty_{\omega}(S_{t,u})} := \mathrm{ess\,sup}_{\omega\in\mathbb{S}^2}|\xi(t,u,\omega)|_{\not{g}}. \tag{270b}$$

Moreover, if $q_1 \in [1,\infty)$ and $q_2 \in [1,\infty]$, then with $[u]_+ := \max\{0,u\}$ denoting the minimum value of t along $\widetilde{\mathcal{C}}_u$, we define

$$\|\xi\|_{L_t^{q_1}L_\omega^{q_2}(\widetilde{\mathcal{C}}_u)} := \left\{\int_{[u]_+}^{T_{*;(\lambda)}} \|\xi\|_{L_\omega^{q_2}(S_{\tau,u})}^{q_1}\, d\tau\right\}^{1/q_1},$$
$$\|\xi\|_{L_t^{\infty}L_\omega^{q_2}(\widetilde{\mathcal{C}}_u)} := \text{ess sup}_{\tau\in[[u]_+,T_{*;(\lambda)}]}\|\xi\|_{L_\omega^{q_2}(S_{t,u})}. \tag{271}$$

Moreover, if $q_1 \in [1,\infty)$ and $q_2 \in [1,\infty]$, then noting that $-\frac{4}{5}T_{*;(\lambda)} \le u \le t$ along $\widetilde{\Sigma}_t$, we define

$$\|\xi\|_{L_u^{q_1}L_\omega^{q_2}(\widetilde{\Sigma}_t)} := \left\{\int_{-\frac{4}{5}T_{*;(\lambda)}}^{t} \|\xi\|_{L_\omega^{q_2}(S_{t,u})}^{q_1}\, du\right\}^{1/q_1},$$
$$\|\xi\|_{L_u^{\infty}L_\omega^{q_2}(\widetilde{\Sigma}_t)} := \text{ess sup}_{u\in[-\frac{4}{5}T_{*;(\lambda)},t]}\|\xi\|_{L_\omega^{q_2}(S_{\tau,u})}. \tag{272}$$

Similarly, if $q_1, q_2 \in [1,\infty)$, then

$$\|\xi\|_{L_u^{q_1}L_{g\!\!/}^{q_2}(\widetilde{\Sigma}_t)} := \left\{\int_{-\frac{4}{5}T_{*;(\lambda)}}^{t} \|\xi\|_{L_{g\!\!/}^{q_2}(S_{t,u})}^{q_1}\, du\right\}^{1/q_1},$$
$$\|\xi\|_{L_u^{\infty}L_{g\!\!/}^{q_2}(\widetilde{\Sigma}_t)} := \text{ess sup}_{u\in[-\frac{4}{5}T_{*;(\lambda)},t]}\|\xi\|_{L_{g\!\!/}^{q_2}(S_{t,u})}^{q_1}. \tag{273}$$

Similarly, if $q_1, q_2, q_3 \in [1,\infty)$, then we define

$$\|\xi\|_{L_t^{q_1}L_u^{q_2}L_\omega^{q_3}(\widetilde{\mathcal{M}})} := \left\{\int_0^{T_{*;(\lambda)}} \|\xi\|_{L_u^{q_2}L_\omega^{q_3}(\widetilde{\Sigma}_\tau)}^{q_1}\, d\tau\right\}^{1/q_1}, \tag{274a}$$

$$\|\xi\|_{L_u^{q_1}L_t^{q_2}L_\omega^{q_3}(\widetilde{\mathcal{M}})} := \left\{\int_{-\frac{4}{5}T_{*;(\lambda)}}^{T_{*;(\lambda)}} \|\xi\|_{L_t^{q_2}L_\omega^{q_3}(\widetilde{\mathcal{C}}_u)}^{q_1}\, du\right\}^{1/q_1}, \tag{274b}$$

$$\|\xi\|_{L_t^{q}L_x^{\infty}(\widetilde{\mathcal{M}})} := \left\{\int_0^{T_{*;(\lambda)}} \|\xi\|_{L^\infty(\widetilde{\Sigma}_\tau)}^{q}\, d\tau\right\}^{1/q}, \tag{274c}$$

$$\|\xi\|_{L^\infty(\widetilde{\mathcal{M}})} := \text{ess sup}_{t\in[0,T_{*;(\lambda)}],\, u\in[-\frac{4}{5}T_{*;(\lambda)},t],\, \omega\in\mathbb{S}^2}|\xi(t,u,\omega)|_{g\!\!/}. \tag{274d}$$

We also extend the definitions (274a)–(274b) to allow $q_1, q_2, q_3 \in [1,\infty]$ by making the obvious modifications. We also define, by making the obvious modifications in (274a)–(274d), norms in which the set $\widetilde{\mathcal{M}}$ is replaced with the set $\widetilde{\mathcal{M}}^{(Int)}$ (see (174b)). For example, if $q_1, q_2, q_3 \in [1,\infty)$, then $\|\xi\|_{L_t^{q_1}L_u^{q_2}L_\omega^{q_3}(\widetilde{\mathcal{M}}^{(Int)})} := \left\{\int_0^{T_{*;(\lambda)}}\left\{\int_0^{\tau}\left\{\int_{\omega\in\mathbb{S}^2}|\xi(t,u,\omega)|_{g\!\!/}^{q_3}\, d\varpi_{e\!\!/(\omega)}\right\}^{q_2/q_3} du\right\}^{q_1/q_2} d\tau\right\}^{1/q_1}$.

Next, for $q \in [1, \infty)$, we define the following norms, where (275a) and (275c) involve υ (see definition (193)), such that an L^{∞} norm in t or u acts first:

$$\|\xi\|_{L^q_{\not g} L^\infty_t(\widetilde{\mathcal{C}}_u)} := \left\{ \int_{\mathbb{S}^2} \text{ess sup}_{t \in [[u]_+, T_{*;(\lambda)}]} \left(\upsilon(t,u,\omega) |\xi(t,u,\omega)|^q_{\not g} \right) d\varpi_{\not e}(\omega) \right\}^{1/q}, \tag{275a}$$

$$\|\xi\|_{L^q_\omega L^\infty_t(\widetilde{\mathcal{C}}_u)} := \left\{ \int_{\mathbb{S}^2} \text{ess sup}_{t \in [[u]_+, T_{*;(\lambda)}]} |\xi(t,u,\omega)|^q_{\not g} \, d\varpi_{\not e}(\omega) \right\}^{1/q}, \tag{275b}$$

$$\|\xi\|^q_{L^q_{\not g} L^\infty_u(\widetilde{\Sigma}_t)} := \left\{ \int_{\omega \in \mathbb{S}^2} \text{ess sup}_{u \in [-\frac{4}{5} T_{*;(\lambda)}, t]} \left(\upsilon(t,u,\omega) |\xi(t,u,\omega)|^q_{\not g} \right) d\varpi_{\not e}(\omega) \right\}^{1/q}. \tag{275c}$$

9.11 The fixed number *p*

In the rest of the article, $p > 2$ denotes a fixed number with

$$0 < \delta_0 < 1 - \frac{2}{p} < N - 2, \tag{276}$$

where δ_0 is the parameter that we fixed in (35c). p will appear in many of our ensuing estimates.

9.12 Hölder norms in the geometric angular variables

Some of our elliptic estimates for $\hat{\chi}$ involve Hölder norms in the geometric angular variables, which we define in this subsection. We remind the reader that $\not e$ denotes the standard round metric on the Euclidean unit sphere $\mathbb{S}^2$. In the rest of the paper, for points $\omega_{(1)}, \omega_{(2)} \in \mathbb{S}^2$, we denote their distance with respect to $\not e$ by $d_{\not e}(\omega_{(1)}, \omega_{(2)})$ In particular, $d_{\not e}(\omega_{(1)}, \omega_{(2)}) \leq \pi$.

To proceed, for each pair of points $\omega_{(1)}, \omega_{(2)} \in \mathbb{S}^2$ with $d_{\not e}(\omega_{(1)}, \omega_{(2)}) < \pi$ and for each pair m, n of non-negative integers, let $\Phi^m_n(\omega_{(1)}; \omega_{(2)}) : (T^m_n)_{\omega_{(1)}}(\mathbb{S}^2) \to (T^m_n)_{\omega_{(2)}}(\mathbb{S}^2)$, $\xi \to \Phi^m_n(\omega_{(1)}; \omega_{(2)})[\xi]$, denote the parallel transport operator with respect to $\not e$, where $(T^m_n)_\omega(\mathbb{S}^2)$ denotes the vector space of type $\binom{m}{n}$ tensors at $\omega \in \mathbb{S}^2$. Note that $\Phi^m_n(\omega_{(1)}; \omega_{(2)})$ provides a linear isomorphism between type $\binom{m}{n}$ tensors ξ at $\omega_{(1)}$ and type $\binom{m}{n}$ tensors at $\omega_{(2)}$ by parallel transport along the unique $\not e$-geodesic connecting $\omega_{(1)}$ and $\omega_{(2)}$. From the basic properties of parallel transport, it follows that Φ^m_n respects tensor products and contractions. That is, if $\xi_{(1)} \cdot \xi_{(2)}$ schematically denotes the tensor product of $\xi_{(1)}$ and $\xi_{(2)}$ possibly followed by some contractions, then $\Phi^m_n(\omega_{(1)}; \omega_{(2)})[\xi_{(1)} \cdot \xi_{(2)}] = \Phi^m_n(\omega_{(1)}; \omega_{(2)})[\xi_{(1)}] \cdot \Phi^m_n(\omega_{(1)}; \omega_{(2)})[\xi_{(2)}]$. If $\xi = \xi(\omega)$ is a type $\binom{m}{n}$ tensorfield on $\mathbb{S}^2$ and $d_{\not e}(\omega_{(1)}; \omega_{(2)}) < \pi$, then we define[51] $\xi^{\parallel}(\omega_{(1)};\omega_{(2)}) := \Phi_n^m(\omega_{(1)};\omega_{(2)})[\xi(\omega_{(1)})] \in (T_n^m)_{\omega_{(2)}}(\mathbb{S}^2)$. Note that $(\xi_{(1)}\cdot\xi_{(2)})^{\parallel}(\omega_{(1)};\omega_{(2)}) = \xi_{(1)}^{\parallel}(\omega_{(1)};\omega_{(2)})\cdot\xi_{(2)}^{\parallel}(\omega_{(1)};\omega_{(2)})$.

Definition 9.7 *(Hölder norms in the geometric angular variables)* For constants $\beta \in (0,1)$, we define

$$\|\xi\|_{\dot{C}_{\omega}^{0,\beta}(S_{t,u})} := \sup_{0<d_{\not g}(\omega_{(2)},\omega_{(1)})<\frac{\pi}{2}} \frac{\tilde{r}^{(m-n)}\left|\xi(t,u,\omega_{(1)}) - \xi^{\parallel}(t,u,\omega_{(2)};\omega_{(1)})\right|_{\not e(\omega_{(1)})}}{d_{\not e}^{\beta}(\omega_{(1)};\omega_{(2)})}, \tag{277a}$$

$$\|\xi\|_{C_{\omega}^{0,\beta}(S_{t,u})} := \|\xi\|_{L_{\omega}^{\infty}(S_{t,u})} + \|\xi\|_{\dot{C}_{\omega}^{0,\beta}(S_{t,u})}. \tag{277b}$$

Note that our bootstrap assumption (308a) below implies that if ξ is type $\binom{m}{n}$, then the denominator on RHS (277a) satisfies

$$\begin{aligned}&\tilde{r}^{(m-n)}\left|\xi(t,u,\omega_{(1)}) - \xi^{\parallel}(t,u,\omega_{(2)};\omega_{(1)})\right|_{\not e(\omega_{(1)})}\\ &\quad\approx \left|\xi(t,u,\omega_{(1)}) - \xi^{\parallel}(t,u,\omega_{(2)};\omega_{(1)})\right|_{\not g(t,u,\omega_{(1)})}.\end{aligned} \tag{278}$$

In Sect. 10, we will also use mixed norms that are defined by replacing the L_{ω}^{∞} norm from Sect. 9.10 with the C_{ω}^{0,δ_0} norm. For example, for $q \in [1,\infty)$, we define

$$\|\xi\|_{L_t^q L_u^{\infty} C_{\omega}^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})} := \left\{\int_0^{T_{*;(\lambda)}} \text{ess sup}_{u\in[0,\tau]} \|\xi\|^q_{C_{\omega}^{0,\delta_0}(S_{\tau,u})}\, d\tau\right\}^{1/q}, \tag{279}$$

and we extend definition (279) to the case $q=\infty$ by making the obvious modification.

9.13 The initial foliation on Σ_0

In this subsection, we state Proposition 9.8, which yields the existence of an initial condition for the eikonal function u (see Sect. 9.4) featuring a variety of properties that we exploit in our analysis. More precisely, as we mentioned in Sect. 9.4.2, we set $u|_{\Sigma_0} := -w$, where w is the function yielded by the proposition. The proof of the proposition is the same as in [54] and we therefore omit it. In particular, the key equation (280) stated below is exactly the same as in [54]. The proof of Proposition 9.8 relies on the regularity of the Ricci curvature of the spatial metric induced on Σ_0, and the regularity is exactly the same as in [54]. More precisely, since the spatial metric g satisfies $g_{ij} = g_{ij}(\vec{\Psi})$, the regularity of the spatial Ricci curvature on Σ_0 is controlled

[51] For example, if $\xi = \xi(\omega)$ is a scalar function, then $\xi^{\parallel}(\omega_{(1)};\omega_{(2)}) = \xi(\omega_{(1)})$. As a second example, if $\xi = \xi(\omega)$ is a one-form, then in a local angular coordinate chart containing the point $\omega_{(2)}$, we have, for $\omega_{(1)}$ close to $\omega_{(2)}$: $\xi^{\parallel}(\omega_{(1)};\omega_{(2)})(\frac{\partial}{\partial\omega^A}|_{\omega_{(2)}}) = M_A^B(\omega_{(1)};\omega_{(2)})\xi(\omega_{(1)})(\frac{\partial}{\partial\omega^B}|_{\omega_{(1)}})$, where the $M_A^B(\omega_{(1)};\omega_{(2)})$ are smooth functions of $\omega_{(1)}$ and $\omega_{(2)}$ such that for $A,B,C=1,2$, we have $M_A^B(\omega_{(C)},\omega_{(C)}) = \delta_A^B$, where δ_A^B is the Kronecker delta. That is, for $C=1,2$, $\xi^{\parallel}(\omega_{(C)};\omega_{(C)}) = \xi(\omega_{(C)})$.

by the energy estimates[52] we already derived in Propositions 4.1 and 5.1, and we stress that our energy estimates for $\vec{\Psi}$ are the same as the energy estimates derived in [54]. Proposition 9.8 provides, in particular, initial conditions for various tensorfields constructed out of the eikonal function that are relevant for the study of $\widetilde{\mathcal{M}}^{(Ext)}$. We emphasize how important the proposition is for the viability of our approach. For example, if we had instead chosen the "simpler" initial condition $u|_{\Sigma_0} := -r$, where r is the standard Euclidean radial variable, then given the limited regularity of the fluid solution, the null mean curvature of the spheres $\{r = \text{const}\}$ with respect to the metric induced on them by the acoustical metric $\mathbf{g}(\vec{\Psi})$ *would not generally have enjoyed any useful quantitative pointwise boundedness properties*. This could have led to the instantaneous formation of null focal points[53] and the breakdown of our geometric coordinate system. In contrast, (281) and the estimates of the proposition imply, for example, that $\|\tilde{r}^{1/2}\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L^\infty(\Sigma_0)} \lesssim \lambda^{-1/2}$. Initial condition bounds of this type play a crucial role in the proof of Proposition 10.1, which provides the main estimates for the acoustic geometry.

Proposition 9.8 (Existence and properties of the initial foliation). *On the hypersurface*[54] Σ_0, *there exists a function* $w = w(x)$ *on the domain implicitly defined by* $0 \leq w \leq w_{*;(\lambda)} := \frac{4}{5}T_{*;(\lambda)}$, *such that* $w(\mathbf{z}) = 0$ *(where* $\mathbf{z}$ *is the point in* Σ_0 *mentioned in Sect. 9.4), such that* w *is smooth away from* $\mathbf{z}$, *such that its levels sets* S_w *are diffeomorphic to* $\mathbb{S}^2$ *for* $0 < w \leq w_{*;(\lambda)}$, *such that* $\mathscr{O} := \cup_{0 \leq w < w_{*;(\lambda)}} S_w$ *is a neighborhood of* $\mathbf{z}$ *contained in the metric ball* $B_{T_{*;(\lambda)}}(\mathbf{z}, g)$ *(with respect to the rescaled first fundamental form* g *of* Σ_0*) of radius* $T_{*;(\lambda)}$ *centered at* $\mathbf{z}$, *and such that the following relations hold, where* $a = \dfrac{1}{\sqrt{(g^{-1})^{cd}\partial_c w \partial_d w}}$ *is the lapse,* $\mathrm{tr}_g k := (g^{-1})^{cd}k_{cd}$, *and* $\mathbf{\Gamma}_L := \mathbf{\Gamma}_\alpha L^\alpha$ *is a contracted (and lowered) Cartesian Christoffel symbol of the rescaled spacetime metric* $\mathbf{g}$*:*

$$\mathrm{tr}_{\not{g}}\theta + k_{NN} = \frac{2}{aw} + \mathrm{tr}_g k - \mathbf{\Gamma}_L, \qquad a(\mathbf{z}) = 1. \tag{280}$$

[52] As we highlighted in Remark 9.1, the hypersurface that we denote by "Σ_0" here corresponds to the hypersurface that we denoted by "Σ_{t_k}" in Sects. 3–8. Hence, to control the appropriate Sobolev norms of $\vec{\Psi}$ along these hypersurfaces, we need the energy estimates. We also point out that Propositions 4.1 and 5.1 yield energy estimates for the non-rescaled solution variables, while in the expression "$g_{ij}(\vec{\Psi})$" in the present section, $\vec{\Psi}$ denotes the rescaled solution (see Sect. 9.3). Hence, one needs to account for the rescaling when controlling the size of the L^2 norms of the derivatives of $g_{ij}(\vec{\Psi})$ (such bounds are needed to prove Proposition 9.8 using the arguments given in [51, Appendix C]).

[53] More precisely, this would have led to the possibility that $\|\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L^1([0,T])L^\infty_x}$ is infinite no matter how small T is; see, for example, the proofs of (351b) and (355) for clarification on the connection between having quantitative control of time integrals of $\|\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L^\infty_x(\Sigma_t)}$ and obtaining control over the local separation of the integral curves of L.

[54] As we highlighted in Remark 9.1, the hypersurface that we denote by "Σ_0" in this proposition corresponds to the hypersurface that we denoted by "Σ_{t_k}" in Sects. 3–8.

Note that by (200)*,* (207)*, and the relation* $\tilde{r}(0,-u) = w$ *(for* $-w_{*;(\lambda)} \leq u \leq 0$*), the first equation in* (280) *is equivalent to*

$$tr_{\widetilde{g}}\widetilde{\chi}^{(Small)}|_{\Sigma_0} = \frac{2(1-a)}{aw}, \qquad \textit{for } 0 \leq w \leq w_{*;(\lambda)}. \tag{281}$$

Let q_* *satisfy* $0 < 1 - \frac{2}{q_*} < N - 2$ *and let* $\not{e} = \not{e}(\omega)$ *be the standard round metric on the Euclidean unit sphere* $\mathbb{S}^2$*, where the angular coordinates* $\{\omega^A\}_{A=1,2}$ *are as in Sect.* 9.4.2*. Then if* q_* *is sufficiently close to* 2*, the following estimates hold*[55] *on* $\Sigma_0^{w_{*;(\lambda)}} := \cup_{0 \leq w \leq w_{*;(\lambda)}} S_w$*, where* ϵ_0 *is as in Sect.* 3.3*, where the role of* q *is played by* q_**:*

$$|a-1| \lesssim \lambda^{-4\epsilon_0} \leq \frac{1}{4}, \|w^{-1/2}(a-1)\|_{L_w^\infty C_\omega^{0,1-\frac{2}{q_*}}(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-1/2}, \upsilon(w,\omega) := \frac{\sqrt{det\not{g}}(w,\omega)}{\sqrt{det\not{e}}(\omega)} \approx w^2, \tag{282a}$$

$$\begin{aligned} \|w^{\frac{1}{2}-\frac{2}{q_*}}(\hat{\theta}, \nabla\!\!\!/ \ln a)\|_{L_w^\infty L_{\not{g}}^{q_*}(\Sigma_0^{w_{*;(\lambda)}})} &\lesssim \lambda^{-1/2}, \\ \|\nabla\!\!\!/ \ln a\|_{L_w^2 L_\omega^\infty(\Sigma_0^{w_{*;(\lambda)}})}, \ \|\hat{\chi}\|_{L_w^2 L_\omega^\infty(\Sigma_0^{w_{*;(\lambda)}})} &\lesssim \lambda^{-1/2}, \end{aligned} \tag{282b}$$

$$\max_{A,B=1,2} \left\| w^{-2}\not{g}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - \not{e}\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) \right\|_{L^\infty(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-4\epsilon_0}, \tag{282c}$$

$$\max_{A,B,C=1,2} \left\| \frac{\partial}{\partial\omega^A}\left\{ w^{-2}\not{g}\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right) - \not{e}\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right)\right\} \right\|_{L_w^\infty L_\omega^{q_*}(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-4\epsilon_0}, \tag{282d}$$

$$\|w^{\frac{1}{2}-\frac{2}{q_*}} \nabla\!\!\!/ \ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L_w^\infty L_{\not{g}}^{q_*}(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-1/2}. \tag{282e}$$

Finally, $\Sigma_0^{w_{*;(\lambda)}}$ *is contained in the Euclidean ball of radius* $T_{*;(\lambda)}$ *in* Σ_0 *centered at* $\mathbf{z}$*.*

Proof (Discussion of the proof) Based on the energy estimates we derived in Propositions 4.1 and 5.1 (which are estimates for the non-rescaled solution variables), the proof is the same as the proof of [54, Proposition 4.3], which is given in [51, Appendix C]. □

55 In [54, Proposition 4.3], the author stated the weaker estimate $\|w^{-1/2}(a-1)\|_{L^\infty(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-1/2}$ in place of the stronger estimate $\|w^{-1/2}(a-1)\|_{L_w^\infty C_\omega^{0,1-\frac{2}{q_*}}(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-1/2}$ appearing in (282a). However, the desired stronger estimate follows from the Morrey-type estimate (318) and the analysis given just above [51, Equation (10.113)].

9.14 Initial conditions on the cone-tip axis tied to the eikonal function

The next lemma complements Proposition 9.8 by providing the initial conditions on the cone-tip axis for various tensorfields tied to the eikonal function, i.e., initial conditions relevant for the study of $\widetilde{\mathcal{M}}^{(Int)}$.

Lemma 9.9 (Initial conditions on the cone-tip axis tied to the eikonal function) *The following estimates hold on any acoustic null cone $\mathcal{C}_u$ emanating from a point on the cone-tip axis with $0 \leq u = t \leq T_{*;(\lambda)}$, where "$\xi = \mathcal{O}(\tilde{r})$ as $t \downarrow u$" means that*[56] $|\xi|_{g\!\!/} \lesssim (t-u)$ *as $t \downarrow u$:*

$$tr_{g\!\!/}\chi - \frac{2}{\tilde{r}},\ \tilde{r}tr_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)},\ |\hat{\chi}|_{g\!\!/},\ |\tilde{r}\Pi\!\!\!/^{a}_{j}\partial_a L^i - \Pi\!\!\!/^{i}_{j}|,\ b-1,\ |\zeta|_{g\!\!/},\ \sigma, \tag{283a}$$

$$\begin{aligned}&\tilde{r}|\nabla\!\!\!/ tr_{g\!\!/}\chi|_{g\!\!/},\ \tilde{r}^2|\nabla\!\!\!/ tr_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}|_{g\!\!/},\ \tilde{r}|\nabla\!\!\!/\hat{\chi}|_{g\!\!/},\ \tilde{r}|\nabla\!\!\!/ b|_{g\!\!/},\ \tilde{r}|\nabla\!\!\!/\zeta|_{g\!\!/},\ \tilde{r}|\nabla\!\!\!/\sigma|_{g\!\!/},\\ &\tilde{r}^2\Delta\!\!\!/ b,\ \tilde{r}^2\Delta\!\!\!/\sigma,\ \tilde{r}^2\mu,\ \tilde{r}^2\check{\mu}\\ &= \mathcal{O}(\tilde{r}) \text{ as } t\downarrow u,\end{aligned}$$

$$\lim_{t\downarrow u}\|(\underline{\zeta}, k)\|_{L^\infty_\omega(S_{t,u})} < \infty. \tag{283b}$$

Moreover, with $e\!\!/$ denoting the standard round metric on the Euclidean unit sphere $\mathbb{S}^2$, we have

$$\lim_{t\downarrow u}\left\{\tilde{r}^{-2}(t,u)g\!\!/(t,u,\omega)\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right)\right\} = e\!\!/(\omega)\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right), \tag{284a}$$

$$\lim_{t\downarrow u}\left\{\tilde{r}^{-2}(t,u)\frac{\partial}{\partial\omega^C}g\!\!/(t,u,\omega)\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right)\right\} = \frac{\partial}{\partial\omega^C}\left\{e\!\!/(\omega)\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right)\right\}. \tag{284b}$$

Moreover, with $w_{*;(\lambda)} := \frac{4}{5}T_{*;(\lambda)}$ (as in Proposition 9.8), on $\Sigma_0^{w_{*;(\lambda)}} := \cup_{w\in(0,w_{*;(\lambda)}]}S_w$, we have (recalling that $w = -u|_{\Sigma_0} \geq 0$):

$$\begin{aligned}&\|w tr_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}\|_{L^\infty(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-4\epsilon_0},\\ &\|w^{3/2}\nabla\!\!\!/ tr_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}\|_{L^\infty_w L^p_\omega(\Sigma_0^{w_{*;(\lambda)}})},\ \|w^{1/2}tr_{\tilde{g\!\!/}}\widetilde{\chi}^{(Small)}\|_{L^\infty_w C^{0,1-\frac{2}{p}}_\omega(\Sigma_0^{w_{*;(\lambda)}})} \lesssim \lambda^{-1/2}.\end{aligned} \tag{285}$$

Finally, with N denoting the unit outward normal to S_w in Σ_0 and $\Pi\!\!\!/$ denoting the g-orthogonal projection tensorfield onto S_w (where g is the rescaled metric on Σ_0),

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[56] On RHS (283a), the implicit constants are allowed to depend on the L^∞ norm of the higher derivatives of the fluid solution. However, these constants never enter into our estimates since, in our subsequent analysis, (283a) will be used only to conclude that LHS (283a) is 0 along the cone-tip axis.

we have

$$\sum_{i,j=1,2,3} |w \not{\Pi}_j^c \partial_c N^i - \not{\Pi}_j^i| = \mathcal{O}(w) as\ w \downarrow 0. \tag{286}$$

Proof (Discussion of the proof) The lemma follows from the same arguments, based on Taylor expansions, that are found in [34,49], and we therefore omit the details. We refer to [54, Lemma 5.1] and [51, Appendix C] for the analogous results in the context of quasilinear wave equations. We also remark that there are simpler, alternative proofs available in [15, Appendix B] and [38, Sect. 3]. We further clarify that in [34,49], the expansions along null cones were derived not in terms of $\tilde{r}$, but rather in terms of the affine parameter $A = A(t, u, \omega)$ of the geodesic null vectorfield $b^{-1}L$ (i.e., $LA = b$, where b is defined in (178)), normalized by $A(u, u, \omega) = 0$. However, the same asymptotic expansions hold with $\tilde{r}$ in place of A, thanks in part to the asymptotic relation $\lim_{t\downarrow u} \frac{A(t,u,\omega)}{\tilde{r}(t,u)} = 1$, which follows from the identities $LA = b$ and $L\tilde{r} = Lt = 1$, and the following fact, which can be independently established with the help of (180): $\lim_{t\downarrow u} \{b(t, u, \omega) - 1\} = 0$. We also clarify that the estimate $\|w^{1/2}\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L_w^\infty C_\omega^{0,1-\frac{2}{p}}(\Sigma_0^{w_*;(\lambda)})} \lesssim \lambda^{-1/2}$ in (285) is stronger than the analogous estimate $\|w^{1/2}\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L^\infty(\Sigma_0^{w_*;(\lambda)})} \lesssim \lambda^{-1/2}$ stated [54, Lemma 5.1]; the desired stronger estimate is a simple consequence of (281) and the first and second estimates in (282a). □

10 Estimates for quantities constructed out of the eikonal function

Our main goal in this section is to prove Proposition 10.1, which provides estimates for the acoustic geometry. As we explain in Sect. 11, these estimates are the last new ingredient needed to prove the frequency-localized Strichartz estimate of Theorem 7.2. The proof of Proposition 10.1 is based on a bootstrap argument and is located in Sect. 10.9. Before proving the proposition, we first introduce the bootstrap assumptions (see Sect. 10.2) and provide a series of preliminary inequalities and estimates. Many of these preliminary results have been derived in prior works, and we typically do not repeat the proofs. In Lemma 10.5, we isolate the new estimates that are not found in earlier works; the results of Lemma 10.5 in particular quantify the effect of the high order derivatives of the vorticity and entropy on the evolution of the acoustic geometry; this will become clear during the proof of Proposition 10.1.

Remark 10.1 We remind the reader that in Sect. 10, we are operating under the conventions of Sect. 9.3.

10.1 The main estimates for the eikonal function quantities

Recall that $p > 2$ denotes the fixed number satisfying (276), where δ_0 is the parameter that we fixed in (35c). We now state the main result of Sect. 10; see Sect. 10.9 for the proof.

Proposition 10.1 (The main estimates for the eikonal function quantities). *Let p be as in (276), assume that $q > 2$ is sufficiently close to 2, and recall that we fixed several small parameters, including ϵ_0, in Sect. 3.3. There exists a large constant $\Lambda_0 > 0$ such that under the bootstrap assumptions of Sect. 10.2, if $\lambda \geq \Lambda_0$, then the following estimates hold on $\widetilde{\mathcal{M}} \subset [0, T_{*;(\lambda)}] \times \mathbb{R}^3$, where the norms referred to below are defined in Sect. 9.10, and the corresponding spacetime regions such as $\widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}$ are defined in Sect. 9.5.*

Estimates for connection coefficients*: The connection coefficients from Sects. 9.6.5, 9.7.1, and 9.7.3 verify the following estimates:*

$$\|(tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}, \ \|\tilde{r}\not{\mathbf{D}}_L(tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \quad (287a)$$

$$\|\tilde{r}^{1/2}(tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\|_{L_t^\infty L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \quad (287b)$$

$$\|\tilde{r}(tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\|_{L_t^\infty L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-4\epsilon_0}, \quad (287c)$$

$$\tilde{r}\, tr_{\tilde{g}}\widetilde{\chi} \approx 1, \quad (288a)$$

$$\|\tilde{r}^{1/2} tr_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2}, \quad (288b)$$

$$\|\tilde{r}^{3/2}\not{\nabla} tr_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2}, \quad (288c)$$

$$\|\tilde{r}(\not{\nabla} tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \not{\nabla}\hat{\chi})\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \quad (288d)$$

$$\|(tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\|_{L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}. \quad (288e)$$

In addition, the null lapse b defined in (178) verifies the following estimates:

$$\left\| \frac{b^{-1}-1}{\tilde{r}} \right\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}, \left\| \frac{b^{-1}-1}{\tilde{r}^{1/2}} \right\|_{L_t^\infty L_u^\infty L_\omega^{2p}(\widetilde{\mathcal{M}})}, \left\| \tilde{r}(\mathbf{D}\!\!\!/_L, \nabla\!\!\!\!/) \left(\frac{b^{-1}-1}{\tilde{r}} \right) \right\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}. \tag{289}$$

Furthermore, for any $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$, $t \in [[u]_+, T_{*;(\lambda)}]$, and $\omega \in \mathbb{S}^2$, the Cartesian spatial components L^i verify the following estimate:

$$|L^i(t,u,\omega) - L^i(0,0,\omega)| \lesssim \lambda^{-4\epsilon_0}. \tag{290}$$

Moreover, for any smooth scalar-valued function of the type described in Sect. 9.9.1, we have:

$$\|\mathrm{f}_{(\vec{L})}\|_{L_t^\infty L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} \lesssim 1. \tag{291}$$

Furthermore,

$$\left\| \left(tr_{\tilde{g}\!\!\!/}\widetilde{\chi}^{(Small)}, \hat{\chi}, tr_{g\!\!\!/}\chi - \frac{2}{\tilde{r}} \right) \right\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}, \|\zeta\|_{L_t^{\frac{q}{2}} L_x^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}. \tag{292}$$

Improved estimates in the interior region: *We have the following improved*[57] *estimates*[58] *in the interior region:*

$$\left\| \frac{b^{-1}-1}{\tilde{r}} \right\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-1/2-4\epsilon_0}, \tag{293}$$

[57] The most important improvement afforded by (295) is that on the LHSs of the estimates, the L_t^2 norms are taken *after* a spatial norm along constant-time hypersurfaces. This is crucial for the proof of Theorem 11.3 and contrasts with, for example, the estimate (288e), in which *only the angular* C_ω^{0,δ_0} norm is taken before the L_t^2 norm.

[58] Our estimate (295) involves Hölder norms in the angular variables, while the analogous estimates in [54] involved weaker L^∞-norms. The reason for the discrepancy is that $L_\omega^\infty(S_{t,u})$ bound for $\hat{\chi}$ proved just below [54, Equation (5.87)] relies on the invalid Calderon–Zygmund estimate $\|\xi\|_{L_\omega^\infty(S_{t,u})} \lesssim \sum_{i=1,2} \|\mathfrak{F}_{(i)}\|_{L_\omega^\infty(S_{t,u})} \ln\left(2 + \|\tilde{r}^{3/2}\nabla\!\!\!\!/\mathfrak{F}_{(i)}\|_{L_\omega^Q(S_{t,u})}\right) + \|\tilde{r}\mathfrak{G}\|_{L_\omega^Q(S_{t,u})}$ for solutions to the elliptic PDE (363). Unfortunately, this estimate cannot be correct because the power of $\tilde{r}^{3/2}$ on the RHS is not compatible with the natural scaling of (363) on Euclidean round spheres of radius $\tilde{r}$; the natural scaling coefficient would be $\tilde{r}$, not $\tilde{r}^{3/2}$, and the distinction is especially crucial near $\tilde{r} = 0$. In particular, since the correct power is $\tilde{r}$, one cannot combine the correct Calderon–Zygmund estimate with the $\tilde{r}^{3/2}$-involving bound (288c) to obtain the estimate for $\hat{\chi}$ stated in [54, Equation (5.11)]. For this reason, we use an alternate approach in deriving some of the estimates for $\hat{\chi}$, one that involves Hölder norms in the angular variables and the corresponding Calderon–Zygmund estimate (365).

$$\|\tilde{r}^{1/2}(tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\|_{L^{2p}_{\omega}L^{\infty}_t(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \qquad if\, \widetilde{\mathcal{C}}_u \in \widetilde{\mathcal{M}}^{(Int)}, \tag{294}$$

$$\|(tr_{\tilde{g}}\widetilde{\chi}^{(Small)}, tr_{g}\chi - \frac{2}{\tilde{r}}, \hat{\chi})\|_{L^2_t L^{\infty}_u C^{0,\delta_0}_{\omega}(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-1/2-3\epsilon_0}, \quad \|\zeta\|_{L^2_t L^{\infty}_x(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-1/2-3\epsilon_0}. \tag{295}$$

Estimates for the geometric angular coordinate components of g: *With* e *denoting the standard round metric on the Euclidean unit sphere* $\mathbb{S}^2$*, we have*

$$\max_{A,B=1,2} \left\| \left\{ \tilde{r}^{-2} g\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) - e\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) \right\} \right\|_{L^{\infty}(\widetilde{\mathcal{M}})} \lesssim \lambda^{-4\epsilon_0}, \tag{296a}$$

$$\max_{A,B,C=1,2} \left\| \frac{\partial}{\partial\omega^A} \left\{ \tilde{r}^{-2} g\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right) - e\left(\frac{\partial}{\partial\omega^B}, \frac{\partial}{\partial\omega^C}\right) \right\} \right\|_{L^p_{\omega}L^{\infty}_t(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-4\epsilon_0}. \tag{296b}$$

Estimates for υ ***and*** b: *The following estimates hold*[59] *for the volume form ratio* υ *defined in* (193) *and the null lapse* b *defined in* (178)*:*

$$\upsilon := \frac{\sqrt{det g}}{\sqrt{det e}} \approx \tilde{r}^2, \tag{297a}$$

$$\|b - 1\|_{L^{\infty}(\widetilde{\mathcal{M}})} \lesssim \lambda^{-4\epsilon_0} < \frac{1}{4}. \tag{297b}$$

Furthermore,

$$\|\tilde{r}^{\frac{1}{2}} \nabla \ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L^{\infty}_t L^{\infty}_u L^p_{\omega}(\widetilde{\mathcal{M}})}, \ \|\nabla \ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L^2_t L^p_{\omega}(\widetilde{\mathcal{C}}_u)}, \ \|\tilde{r} L \nabla \ln\left(\tilde{r}^{-2}\upsilon\right)\|_{L^2_t L^p_{\omega}(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}. \tag{298}$$

Estimates for μ ***and*** $\nabla\zeta$*: The torsion defined in* (196) *and the mass aspect function* μ *defined in* (208) *verify the following estimates:*

$$\|(\tilde{r}\mu, \tilde{r}\nabla\zeta)\|_{L^2_t L^p_{\omega}(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}. \tag{299}$$

Interior region estimates for σ*: The conformal factor* σ *from Definition* 9.3 *verifies the following estimates in the interior region:*

$$\|\tilde{r}^{\frac{1}{2}} L\sigma\|_{L^{\infty}_t L^{2p}_{\omega}(\widetilde{\mathcal{C}}_u)}, \ \|\tilde{r}^{\frac{1}{2}-\frac{2}{p}} \nabla\sigma\|_{L^p_{g} L^{\infty}_t(\widetilde{\mathcal{C}}_u)}, \ \|\tilde{r}^{\frac{1}{2}} \nabla\sigma\|_{L^p_{\omega}L^{\infty}_t(\widetilde{\mathcal{C}}_u)},$$
$$\|\nabla\sigma\|_{L^2_t L^p_{\omega}(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \quad if\, \widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}^{(Int)}, \tag{300a}$$

[59] We point out that we prove (297a)–(297b) independently in the proof of Proposition 10.2, which in turn plays a role in the proofs of the remaining estimates of Proposition 10.1. It is only for convenience that we have restated (297a)–(297b) in Proposition 10.1.

$$\|\sigma\|_{L^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-8\epsilon_0}, \tag{300b}$$

$$\|\tilde{r}^{-1/2}\sigma\|_{L^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}. \tag{300c}$$

Interior region estimates for σ, $\check{\mu}$, $\widetilde{\zeta}$, ***and*** $\not{\mu}$: *The conformal factor* σ *from Definition 9.3, the modified mass aspect function* $\check{\mu}$ *defined in (209), and the modified torsion* $\widetilde{\zeta}$ *defined in (211) verify the following estimates in the interior region:*

$$\|\not\nabla\sigma\|_{L_u^2 L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})}, \ \|(\tilde{r}\check{\mu}, \tilde{r}\not\nabla\widetilde{\zeta})\|_{L_u^2 L_t^2 L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}, \tag{301a}$$

$$\|\tilde{r}^{\frac{3}{2}}\check{\mu}\|_{L_u^2 L_t^\infty L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}. \tag{301b}$$

In addition, the one-form $\not{\mu}$, *which satisfies the Hodge system (210), verifies the following estimates:*

$$\|(\tilde{r}\not\nabla\not{\mu}, \not{\mu})\|_{L_t^2 L_u^2 L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})}, \ \|\not{\mu}\|_{L_t^2 L_u^2 L_\omega^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}. \tag{302}$$

Delicate decomposition of $\not\nabla\sigma$ ***and corresponding estimates in the interior region***: *Finally, in* $\widetilde{\mathcal{M}}^{(Int)}$, *we can decompose* $\not\nabla\sigma$ *into* $S_{t,u}$*-tangent one-forms as follows:*

$$\not\nabla\sigma = -\zeta + (\widetilde{\zeta} - \not{\mu}) + \not{\mu}_{(1)} + \not{\mu}_{(2)}. \tag{303}$$

In (303), ζ *is the torsion from (196),* $\widetilde{\zeta}$ *and* $\not{\mu}$ *are as in Definition 9.5, and* $\not{\mu}_{(1)}$ *and* $\not{\mu}_{(2)}$ *are as in (242) and are respectively solutions to the Hodge-transport systems (243a)–(243b) and (244a)–(244b) on* $S_{t,u}$ *that satisfy the following asymptotic conditions near the cone-tip axis:*

$$\tilde{r}\not{\mu}_{(1)}(t,u,\omega),\ \tilde{r}\not{\mu}_{(2)}(t,u,\omega) = \mathcal{O}(\tilde{r}) \text{as } t \downarrow u. \tag{304}$$

Moreover, the following bounds hold:

$$\|\widetilde{\zeta} - \not{\mu}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})}, \ \|\not{\mu}_{(1)}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}, \tag{305a}$$

$$\|\not{\mu}_{(2)}\|_{L_u^2 L_t^\infty L_\omega^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}. \tag{305b}$$

10.2 Assumptions, including bootstrap assumptions for the eikonal function quantities

In this subsection, we recall some important results proved in previous sections and state some bootstrap assumptions that will play a role in our proof of Proposition 10.1.

10.2.1 Restatement of assumptions and results from prior sections

From scaling considerations, it is straightforward to see that (107a)–(107b) imply that the *rescaled* solution variables (as defined in Sect. 9.1 and under the conventions of Sect. 9.3) verify the following bootstrap assumptions (where δ_0, ϵ_0, and the other parameters in our analysis are defined in Sect. 3.3):

$$\|\boldsymbol{\partial}\vec{\Psi}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})} + \lambda^{\delta_0}\sqrt{\sum_{\nu\geq 2}\nu^{2\delta_0}\|P_\nu\boldsymbol{\partial}\vec{\Psi}\|^2_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}} \leq \lambda^{-1/2-4\epsilon_0}, \tag{306a}$$

$$\|\partial(\vec{\Omega},\vec{S})\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})} + \lambda^{\delta_0}\sqrt{\sum_{\nu\geq 2}\nu^{2\delta_0}\|P_\nu\partial(\vec{\Omega},\vec{S})\|^2_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}} \leq \lambda^{-1/2-4\epsilon_0}. \tag{306b}$$

We will use (306a)–(306b) throughout the rest of Sect. 10. We will also use the bootstrap assumption (40). We clarify that, although the bootstrap assumption (40) refers to the non-rescaled solution, it also implies that the rescaled solution is contained in $\mathfrak{K}$ on the spacetime domain $\widetilde{\mathcal{M}}$. Moreover, we recall that we will assume that λ is sufficiently large; that is, there exists a (non-explicit) $\Lambda_0 > 0$ such that all of our estimates hold whenever $\lambda \geq \Lambda_0$. Moreover, throughout Sect. 10, we will use the top-order energy estimates of Proposition 5.1 along constant-time hypersurfaces and the energy estimates of Proposition 6.1 along acoustic null hypersurfaces (both of which concern estimates for the non-rescaled solution variables, from which estimates for the rescaled variables immediately follow via scaling considerations).

Next, for use throughout the rest of the article, we use (306a)–(306b), the product estimate (80), the energy estimates of Proposition 5.1, and the harmonic analysis results mentioned in the proof discussion of Corollary 7.1 to deduce the following estimates for the rescaled solution, valid for any smooth function f:

$$\|\boldsymbol{\partial}\mathbf{g}(\vec{\Psi})\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-4\epsilon_0}, \tag{307a}$$

$$\begin{aligned}&\|(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S},\vec{\mathcal{C}},\mathcal{D})\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}\\ &+\lambda^{\delta_0}\sqrt{\sum_{\nu\geq 2}\nu^{2\delta_0}\left\|P_\nu\left\{\mathrm{f}(\vec{\Psi},\vec{\Omega},\vec{S})(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S},\vec{\mathcal{C}},\mathcal{D})\right\}\right\|^2_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}} \lesssim \lambda^{-1/2-4\epsilon_0},\end{aligned} \tag{307b}$$

$$\left\|\mathrm{f}(\vec{\Psi},\vec{\Omega},\vec{S})(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S},\vec{\mathcal{C}},\mathcal{D})\right\|_{L_t^2 C_x^{0,\delta_0}(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-4\epsilon_0}. \tag{307c}$$

We clarify that to obtain the bounds in (307b) involving $\partial_t(\vec{\Omega},\vec{S})$, we use (157) to algebraically solve for $\partial_t(\vec{\Omega},\vec{S})$. Moreover, to obtain the bounds in (307b) involving

$\vec{\mathcal{C}}$ and $\mathcal{D}$, we use (153a)–(153b) to express $(\vec{\mathcal{C}}, \mathcal{D}) = \mathrm{f}(\vec{\Psi}, \vec{\Omega}, \vec{S}) \cdot \partial(\vec{\Psi}, \vec{\Omega}, \vec{S})$, where f is a schematically depicted smooth function.

10.2.2 Bootstrap assumptions for the eikonal function quantities

Recall that p denotes the number we fixed in (276). We assume that

$$\max_{A,B=1,2} \left\| \left\{ \tilde{r}^{-2} \not{g} \left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B} \right) - \not{e} \left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B} \right) \right\} \right\|_{L^\infty(\widetilde{\mathcal{M}})} \leq \lambda^{-\epsilon_0}, \tag{308a}$$

$$\max_{A,B,C=1,2} \left\| \frac{\partial}{\partial \omega^A} \left\{ \tilde{r}^{-2} \not{g} \left(\frac{\partial}{\partial \omega^B}, \frac{\partial}{\partial \omega^C} \right) - \not{e} \left(\frac{\partial}{\partial \omega^B}, \frac{\partial}{\partial \omega^C} \right) \right\} \right\|_{L_t^\infty L_\omega^p(\widetilde{\mathcal{C}}_u)} \leq \lambda^{-\epsilon_0}. \tag{308b}$$

We also assume that for any $\widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}$, we have

$$\|(\mathrm{tr}_{\tilde{\not{g}}} \widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\|_{L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)} \leq \lambda^{-1/2+2\epsilon_0}. \tag{309}$$

Moreover, we assume that for any $S_{t,u} \subset \widetilde{\mathcal{M}}$, we have

$$\|\tilde{r}(\hat{\chi}, \mathrm{tr}_{\tilde{\not{g}}} \widetilde{\chi}^{(Small)}, \zeta)\|_{L_\omega^p(S_{t,u})} \leq 1, \tag{310a}$$

$$\|b - 1\|_{L_\omega^\infty(S_{t,u})} \leq \frac{1}{2}. \tag{310b}$$

In addition, we assume that for every $u \in [-\frac{4}{5} T_{*;(\lambda)}, T_{*;(\lambda)}]$, $t \in [[u]_+, T_{*;(\lambda)}]$, and $\omega \in \mathbb{S}^2$, we have

$$|L^i(t, u, \omega) - L^i(0, 0, \omega)| \leq 1. \tag{311}$$

Finally, we assume that the following estimates hold in the interior region:

$$\|(\hat{\chi}, \mathrm{tr}_{\tilde{\not{g}}} \widetilde{\chi}^{(Small)})\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})} \leq \lambda^{-1/2}, \quad \|\zeta\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})} \leq \lambda^{-1/2}. \tag{312}$$

Remark 10.2 Our bootstrap assumptions are similar to the ones in [54, Section 5], except that for convenience, we have strengthened a few and included a few additional ones. We also note that we derive a strict improvement of (308a) in (296a), of (308b) in (296b), of (309) in (288e), of (310a) in (287c), of (310b) in (297b), of (311) in (290), and of (312) in (295).

10.3 Analytic tools

In this subsection, we record some inequalities that will play a role in the forthcoming analysis. All of the results are the same as or simple consequences of results from [54, Section 5].

10.3.1 Norm comparisons, trace inequalities, and Sobolev inequalities

Proposition 10.2 (Norm comparisons, trace inequalities, and Sobolev inequalities). *Under the assumptions of Sect. 10.2, the following estimates hold (see Sect. 9.10 for the definitions of the norms).*

Comparison of $S_{t,u}$ ***-norms with different volume forms***: *If* $1 \le Q < \infty$, *then for any* $S_{t,u}$*-tangent tensorfield* ξ, *we have*

$$\|\xi\|_{L^Q_{\not g}(S_{t,u})} \approx \|\tilde{r}^{\frac{2}{Q}}\xi\|_{L^Q_\omega(S_{t,u})}. \tag{313}$$

Trace inequalities: *For any* $S_{t,u}$*-tangent tensorfield* ξ, *we have*

$$\|\tilde{r}^{-1/2}\xi\|_{L^2_{\not g}(S_{t,u})} + \|\xi\|_{L^4_{\not g}(S_{t,u})} \lesssim \|\xi\|_{H^1(\widetilde{\Sigma}_t)}. \tag{314}$$

Sobolev and Morrey-type inequalities: *For any* $S_{t,u}$*-tangent tensorfield* ξ, *we have*

$$\|\xi\|_{L^2_u L^2_\omega(\widetilde{\Sigma}_t)} \lesssim \|\xi\|_{H^1(\widetilde{\Sigma}_t)}, \tag{315}$$

$$\|\tilde{r}^{1/2}\xi\|^2_{L^{2p}_\omega L^\infty_t(\widetilde{\mathcal{C}}_u)} \lesssim \left\{\|\tilde{r}\not{\mathbf{D}}_L\xi\|_{L^p_\omega L^2_t(\widetilde{\mathcal{C}}_u)} + \|\xi\|_{L^p_\omega L^2_t(\widetilde{\mathcal{C}}_u)}\right\}\|\xi\|_{L^\infty_\omega L^2_t(\widetilde{\mathcal{C}}_u)}. \tag{316}$$

Furthermore, if $2 < Q < \infty$, *then for any* $S_{t,u}$*-tangent tensorfield* ξ, *we have*

$$\|\xi\|_{L^Q_\omega(S_{t,u})} \lesssim \|\tilde{r}\not\nabla\xi\|^{1-\frac{2}{Q}}_{L^2_\omega(S_{t,u})}\|\xi\|^{\frac{2}{Q}}_{L^2_\omega(S_{t,u})} + \|\xi\|_{L^2_\omega(S_{t,u})}. \tag{317}$$

Moreover, if $2 < Q \le p$ *(where* p *is as in Sect. 10.1), then for any* $S_{t,u}$*-tangent tensorfield* ξ, *we have*

$$\|\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \lesssim \|\tilde{r}\not\nabla\xi\|_{L^Q_\omega(S_{t,u})} + \|\xi\|_{L^2_\omega(S_{t,u})}. \tag{318}$$

In addition, if $2 \le Q$, *then for any* $S_{t,u}$*-tangent tensorfield* ξ, *we have*

$$\|\tilde{r}^{\frac{1}{2}-\frac{1}{Q}}\xi\|^2_{L^{2Q}_{\not g}L^\infty_u(\widetilde{\Sigma}_t)} \lesssim \left\{\|\tilde{r}(\not{\mathbf{D}}_N, \not\nabla)\xi\|_{L^Q_\omega L^2_u(\widetilde{\Sigma}_t)} + \|\xi\|_{L^Q_\omega L^2_u(\widetilde{\Sigma}_t)}\right\}\|\xi\|_{L^\infty_\omega L^2_u(\widetilde{\Sigma}_t)}. \tag{319}$$

Finally, if $0 < 1 - \frac{2}{Q} < N - 2$, *then for any scalar function* f, *we have*

$$\|\tilde{r} f\|_{L^2_u L^Q_\omega(\widetilde{\Sigma}_t)} \lesssim \|f\|_{H^{N-2}(\widetilde{\Sigma}_t)}. \tag{320}$$

Remark 10.3 *(Silent use of* (313)*)* Following the proof of the proposition, in the rest of the article, we will often use the estimate (313) without explicitly mentioning it. For example, when deriving (387), we silently use (313) when controlling the term $\|\tilde{r}^{1-\frac{2}{Q}} \mathfrak{G}\|_{L^Q_{\not g}(S_{t,u})}$ on the right-hand side of the Calderon–Zygmund estimate (367).

Proof (Discussion of proof) To obtain the desired estimates, we first note that the following bounds hold: $\upsilon \approx \tilde{r}^2$ and $\|b - 1\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-4\epsilon_0} \leq \frac{1}{4}$. These bounds follow from the proof of [54, Lemma 5.4], based on the transport equations (233) and (235a), the initial conditions (282a), (283a), and (284a) (recall that $b|_{\Sigma_0} = a$ and that $u|_{\Sigma_0} = -w$), and the bootstrap assumptions. The estimates in the proposition can be proved using only on these estimates for υ and $b - 1$ and the bootstrap assumptions, especially (308a)–(308b), which capture the fact that $\tilde{r}^{-2}\not g$ is close, in appropriate norms, to the standard round Euclidean metric.

The desired bound (313) follows from the estimate $\upsilon \approx \tilde{r}^2$ and the definitions of the norms on the left- and right-hand sides. All of the remaining estimates follow from proofs given in other works, thanks to the bounds for υ and b mentioned in the previous paragraph and the bootstrap assumptions; for the reader's convenience, we now provide references. (314) follows from straightforward adaptations of the proofs of [50, Lemma 7.4] and [50, Equation (7.4)]. (315) follows from a standard adaptation of the proof of [50, Proposition 7.5], together with (313) and (314). The estimate (316) follows from a straightforward adaptation of the proof of [50, Equation (8.17)], where one uses $\tilde{r}^2$ in the role of υ; see also [52, Lemma 2.13], in which an estimate equivalent (taking into account (313)) to (316) is stated.

(317) and (318) can be proved by first noting that the same estimates hold for the round metric $\not e$ on the Euclidean-unit sphere (with $\tilde{r}$ replaced by unity and $\not\nabla$ replaced by the connection of $\not e$), and then using the bootstrap assumptions (308a)–(308b) to conclude the desired estimates as "perturbations" of the corresponding ones for the round metric. We will give the details for (318) and omit the argument for (317), which can be proved using similar arguments. Let $\not\nabla$ denote the Levi-Civita connection of $\not g$, and let ${}^{(0)}\not\nabla$ denote the Levi-Civita connection of $\not e$. Let Γ schematically denote the Christoffel symbols of $\not g$ relative to the geometric angular coordinates, and let ${}^{(0)}\Gamma$ schematically denote the corresponding Christoffel symbols of $\not e$, i.e., schematically, we have $\Gamma = (\not g^{-1})^{AB} \frac{\partial}{\partial \omega^C} \not g \left(\frac{\partial}{\partial \omega^D}, \frac{\partial}{\partial \omega^E} \right)$ and ${}^{(0)}\Gamma = (\not e^{-1})^{AB} \frac{\partial}{\partial \omega^C} \not e \left(\frac{\partial}{\partial \omega^D}, \frac{\partial}{\partial \omega^E} \right)$. Then schematically, relative to the geometric angular coordinates, we have $\not\nabla \xi = {}^{(0)}\not\nabla \xi + (\Gamma - {}^{(0)}\Gamma) \cdot \xi$. In view of Definition 9.7, we see that the standard Morrey inequality on the round sphere for type $\binom{m}{n}$ tensorfields ξ yields: $\|\tilde{r}^{n-m} \xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \lesssim \||{}^{(0)}\not\nabla \xi|_{\not e}\|_{L^Q_\omega(S_{t,u})} + \||\xi|_{\not e}\|_{L^2_\omega(S_{t,u})}$. Hence, multiplying both sides of this inequality by $\tilde{r}^{m-n}$ and using (308a), we find that

$\|\xi\|_{C_{\omega}^{0,1-\frac{2}{Q}}(S_{t,u})} \lesssim \|\tilde{r}^{(0)}\nabla\!\!\!/\xi\|_{L_{\omega}^{Q}(S_{t,u})} + \|\xi\|_{L_{\omega}^{2}(S_{t,u})}$ and thus

$$\|\xi\|_{C_{\omega}^{0,1-\frac{2}{Q}}(S_{t,u})} \lesssim \|\tilde{r}\nabla\!\!\!/\xi\|_{L_{\omega}^{Q}(S_{t,u})} + \|\xi\|_{L_{\omega}^{2}(S_{t,u})} + \|\tilde{r}^{m-n}|(\Gamma - {}^{(0)}\Gamma)\cdot\xi|_{e\!\!\!/}\|_{L_{\omega}^{Q}(S_{t,u})}. \tag{321}$$

The bootstrap assumptions (308a)–(308b) imply that the last term on RHS (321) satisfies the estimate

$$\begin{aligned}
&\|\tilde{r}^{m-n}|(\Gamma - {}^{(0)}\Gamma)\xi|_{e\!\!\!/}\|_{L_{\omega}^{Q}(S_{t,u})} \\
&\lesssim \sum_{A,B,C=1,2} \left\|\Gamma_{A\ B}^{\ C} - {}^{(0)}\Gamma_{A\ B}^{\ C}\right\|_{L_{\omega}^{Q}(S_{t,u})} \|\xi\|_{L_{\omega}^{\infty}(S_{t,u})} \\
&\lesssim \sum_{A,B,C,D,E=1,2} \left\|(g\!\!\!/^{-1})^{AB}\frac{\partial}{\partial\omega^C} g\!\!\!/\left(\frac{\partial}{\partial\omega^D},\frac{\partial}{\partial\omega^E}\right) - (e\!\!\!/^{-1})^{AB}\frac{\partial}{\partial\omega^C} e\!\!\!/\left(\frac{\partial}{\partial\omega^D},\frac{\partial}{\partial\omega^E}\right)\right\|_{L_{\omega}^{Q}(S_{t,u})} \|\xi\|_{L_{\omega}^{\infty}(S_{t,u})} \\
&\lesssim \lambda^{-\epsilon_0}\|\xi\|_{L_{\omega}^{\infty}(S_{t,u})}.
\end{aligned} \tag{322}$$

From (322), in view of Definition 9.7, we see that if λ is sufficiently large, then we can absorb the last term on RHS (321) back into the left, at the expense of doubling the (implicit) constants on the RHS. We have therefore proved (318).

The estimate (319) follows from a straightforward adaptation of the proof of [50, Equation (8.17)], where one uses the geometric coordinate partial derivative vectorfield $\frac{\partial}{\partial u}$ in the role of the vectorfield $\frac{\partial}{\partial t}$ and $\tilde{r}^2$ in the role of υ (note also that $|\mathbf{D}\!\!\!/_{\frac{\partial}{\partial u}}\xi|_{g\!\!\!/} \lesssim |(\mathbf{D}\!\!\!/_N, \nabla\!\!\!/)\xi|_{g\!\!\!/}$). Finally, we note that the estimate (320) is proved as [54, Equation (5.39)] as a consequence of (317)–(318).

10.3.2 Hardy–Littlewood maximal function

If $f = f(t)$ is a scalar function defined on the interval I, then we define the corresponding Hardy–Littlewood maximal function $\mathcal{M}(f) = \mathcal{M}(f)(t)$ to be the following scalar function on I:

$$\mathcal{M}(f)(t) := \sup_{t'\in I\cap(-\infty,t)} \frac{1}{|t-t'|}\int_{t'}^{t} f(\tau)\,d\tau. \tag{323}$$

We will use the following well-known estimate, valid for $1 < Q \leq \infty$:

$$\|\mathcal{M}(f)\|_{L^Q(I)} \lesssim \|f\|_{L^Q(I)}. \tag{324}$$

10.3.3 Transport lemma

Many of the geometric quantities that we must estimate satisfy transport equations along the integral curves of L. Our starting point for the analysis of such quantities will often be based on the following standard "transport lemma."

Lemma 10.3 (Transport lemma). *Let m be a constant, and let ξ and $\mathfrak{F}$ be $S_{t,u}$-tangent tensorfields such that the following transport equation holds along the null cone portion $\widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}$:*

$$\not{\mathbf{D}}_L \xi + m tr_{\not{g}} \chi \xi = \mathfrak{F}. \tag{325}$$

Then we have the following identities, where $\tilde{r}$ and υ are defined in Sect. 9.6.3, and we recall that $[u]_+ := \max\{u, 0\}$ (and thus $[u]_+$ denotes the minimum value of t along $\widetilde{\mathcal{C}}_u$):

$$\left[\upsilon^m \xi\right](t, u, \omega) = \lim_{\tau \downarrow [u]_+} \left[\upsilon^m \xi\right](\tau, u, \omega) + \int_{[u]_+}^{t} \left[\upsilon^m \mathfrak{F}\right](\tau, u, \omega)\, d\tau, \tag{326a}$$

$$\begin{aligned}\left[\tilde{r}^{2m} \xi\right](t, u, \omega) = \lim_{\tau \downarrow [u]_+} \left[\tilde{r}^{2m} \xi\right](\tau, u, \omega) + \int_{[u]_+}^{t} \Bigg\{ \left[\tilde{r}^{2m} \mathfrak{F}\right](\tau, u, \omega) \\ + m \left[\tilde{r}^{m} \left(\frac{2}{\tilde{r}} - tr_{\not{g}} \chi\right) \xi\right](\tau, u, \omega) \Bigg\}\, d\tau.\end{aligned} \tag{326b}$$

Similarly, if ξ, $\mathfrak{F}$, and $\mathfrak{G}$ are $S_{t,u}$-tangent tensorfields such that the following transport equation holds:

$$\not{\mathbf{D}}_L \xi + \frac{2m}{\tilde{r}} \xi = \mathfrak{G} \cdot \xi + \mathfrak{F}, \tag{327}$$

and if

$$\|\mathfrak{G}\|_{L^\infty_\omega L^1_t(\widetilde{\mathcal{C}}_u)} \leq C, \tag{328}$$

then under the assumptions of Sect. 10.2, the following estimate holds (where the implicit constants in (329) depend on the constant C on RHS (328)):

$$|\tilde{r}^{2m} \xi|_{\not{g}}(t, u, \omega) \lesssim \lim_{\tau \downarrow [u]_+} |\tilde{r}^{2m} \xi|_{\not{g}}(\tau, u, \omega) + \int_{[u]_+}^{t} |\tilde{r}^{2m} \mathfrak{F}|_{\not{g}}(\tau, u, \omega)\, d\tau. \tag{329}$$

Proof (Discussion of proof) The results are restatements of [54, Lemma 5.11] and can be proved using the same arguments, based on Eq. (212a) and the estimate $\upsilon \approx \tilde{r}^2$ noted in the proof of Proposition 10.2. ◻

10.4 Estimates for the fluid variables

Recall that Proposition 9.7 provides the PDEs verified by the geometric quantities under study and that some source terms in those PDEs depend on the fluid variables. In Proposition 10.4, we provide some estimates that are useful for controlling the fluid variable source terms. In particular, we use the estimates of Proposition 10.4 in our proof of Lemma 10.5, which provides the main new estimates needed to prove Proposition 10.1.

Proposition 10.4 (Estimates for the fluid variables) *Under the assumptions of Sect. 10.2, for any $2 \le Q \le p$ (where p is as in (276)), the following estimates hold on $\widetilde{\mathcal{M}}$:*

$$\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^2_u L^p_\omega(\widetilde{\Sigma}_t)}, \ \|\tilde{r}^{1/2}\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^\infty_u L^{2p}_\omega(\widetilde{\Sigma}_t)} \lesssim \lambda^{-1/2}, \tag{330a}$$

$$\|\tilde{r}^{1-\frac{2}{Q}}\boldsymbol{\partial}^2(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^2_u L^Q_{g\!\!/}(\widetilde{\Sigma}_t)} \lesssim \lambda^{-1/2}, \tag{330b}$$

$$\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^2_t L^\infty_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-4\epsilon_0}, \tag{330c}$$

$$\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^2_t L^p_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-4\epsilon_0}, \tag{330d}$$

$$\|\tilde{r}\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^2_t L^p_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{1/2-12\epsilon_0}, \tag{330e}$$

$$\|(\nabla\!\!\!/, \mathbf{D}\!\!\!\!/_L)\boldsymbol{\partial}\vec{\Psi}\|_{L^2(\widetilde{\mathcal{C}}_u)}, \ \|\tilde{r}^{1-\frac{2}{p}}(\nabla\!\!\!/, \mathbf{D}\!\!\!\!/_L)\boldsymbol{\partial}\vec{\Psi}\|_{L^2_t L^p_{g\!\!/}(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \tag{330f}$$

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_u L^p_\omega(\widetilde{\Sigma}_t)}, \ \|\tilde{r}^{1/2}(\vec{\mathcal{C}}, \mathcal{D})\|_{L^\infty_u L^{2p}_\omega(\widetilde{\Sigma}_t)} \lesssim \lambda^{-1/2}, \tag{331a}$$

$$\|\tilde{r}^{1-\frac{2}{Q}}\boldsymbol{\partial}(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_u L^Q_{g\!\!/}(\widetilde{\Sigma}_t)} \lesssim \lambda^{-1/2}, \tag{331b}$$

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^\infty_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-4\epsilon_0}, \tag{331c}$$

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^p_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-4\epsilon_0}, \tag{331d}$$

$$\|\tilde{r}(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^\infty_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{1/2-12\epsilon_0}, \tag{331e}$$

$$\|\boldsymbol{\partial}(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2(\widetilde{\mathcal{C}}_u)}, \ \|\tilde{r}^{1-\frac{2}{p}}(\nabla\!\!\!/, \mathbf{D}\!\!\!\!/_L)(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^p_{g\!\!/}(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}. \tag{331f}$$

Moreover, for any smooth function f*, we have*

$$\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^2_u L^Q_\omega(\widetilde{\Sigma}_t)}, \ \|\tilde{r}^{1/2}\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^\infty_t L^\infty_u L^{2Q}_\omega(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2}, \tag{332a}$$

$$\left\| \tilde{r}(\nabla\!\!\!/, \mathbf{D}\!\!\!/_L) \left\{ \mathrm{f}(\vec{\Psi}, \vec{\Omega}, \vec{S}, \vec{L}) \boldsymbol{\partial} \vec{\Psi} \right\} \right\|_{L_t^2 L_\omega^Q(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \tag{332b}$$

$$\left\| \tilde{r} \boldsymbol{\partial} \left\{ \mathrm{f}(\vec{\Psi}, \vec{\Omega}, \vec{S}) \boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S}) \right\} \right\|_{L_u^2 L_\omega^Q(\widetilde{\Sigma}_t)} \lesssim \lambda^{-1/2}, \tag{332c}$$

$$\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L_u^2 L_\omega^Q(\widetilde{\Sigma}_t)}, \ \|\tilde{r}^{1/2}(\vec{\mathcal{C}}, \mathcal{D})\|_{L_t^\infty L_u^\infty L_\omega^{2Q}(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2}, \tag{333a}$$

$$\left\| \tilde{r}(\nabla\!\!\!/, \mathbf{D}\!\!\!/_L) \left\{ \mathrm{f}(\vec{\Psi}, \vec{\Omega}, \vec{S}, \vec{L})(\vec{\mathcal{C}}, \mathcal{D}) \right\} \right\|_{L_t^2 L_\omega^Q(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \tag{333b}$$

$$\left\| \tilde{r} \boldsymbol{\partial} \left\{ \mathrm{f}(\vec{\Psi}, \vec{\Omega}, \vec{S})(\vec{\mathcal{C}}, \mathcal{D}) \right\} \right\|_{L_u^2 L_\omega^Q(\widetilde{\Sigma}_t)} \lesssim \lambda^{-1/2}, \tag{333c}$$

$$\left\| \tilde{r}^{1/2} \mathrm{f}(\vec{\Psi}, \vec{L}) \boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S}) \right\|_{L_u^2 L_t^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-4\epsilon_0}. \tag{334}$$

Proof (Discussion of the proof) Thanks to the assumptions of Sect. 10.2, the availability of the energy-elliptic estimates of Proposition 5.1, the estimates of Proposition 6.1 along null hypersurfaces (with $\widetilde{\mathcal{C}}_u$ in the role of $\mathcal{N}$ in Proposition 6.1), and Proposition 10.2, all estimates except for (334) follow from the same arguments given in [54, Lemma 5.5], [54, Proposition 5.6], and [54, Lemma 5.7]. We clarify that, in view of definition (271), (330c) follows from the bootstrap assumptions (306a)–(306b) and the bound $\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_\omega^\infty(S_{t,u})} \leq \|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L^\infty(\Sigma_t)}$, which implies that $\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_t^2 L_\omega^\infty(\widetilde{\mathcal{C}}_u)} \leq \|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}$. Similar remarks apply to (331c), where we take into account definitions (153a)–(153b) and the remarks of Sect. 9.3. Similar remarks apply (331e), where we take into account the bound (177) for $\tilde{r}$. We also refer the readers to the proof of [52, Proposition 2.6] for further details on the role that the energy-elliptic estimates and the estimates along null hypersurface play in the proof of Proposition 10.4. To prove the remaining estimate (334), we use (177) and (330a) to conclude that $\|\tilde{r}^{1/2}\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_u^2 L_t^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{1/2-4\epsilon_0}\|\tilde{r}^{1/2}\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_u^\infty L_t^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-4\epsilon_0}$ as desired. □

10.5 The new estimates needed to prove Proposition 10.1

The following lemma provides the main new estimates needed to prove Proposition 10.1; the other estimates needed to prove Proposition 10.1 were essentially derived in [54].

Lemma 10.5 (The new estimates needed to prove Proposition 10.1). *Under the assumptions of Proposition 10.1, the following estimates hold whenever*[60] $q > 2$ *is sufficiently close to* 2*, where* p *is defined in* (276)*, and we recall that* $[u]_+ := \max\{u, 0\}$ *(and thus* $[u]_+$ *denotes the minimum value of* t *along* $\widetilde{\mathcal{C}}_u$*).*
Estimates for time-integrated terms:

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}(t,u)}\int_{[u]_+}^{t}|\tilde{r}(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L_t^2L_x^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-12\epsilon_0}, \tag{335a}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^2(t,u)}\int_{[u]_+}^{t}|\tilde{r}^2(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L_t^2L_x^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-12\epsilon_0}, \tag{335b}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^2(t,u)}\int_{[u]_+}^{t}|\tilde{r}^2(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L_t^{\frac{q}{2}}L_x^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}+2)}, \tag{335c}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{1/2}(t,u,\omega)}\int_{[u]_+}^{t}|\tilde{r}(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-12\epsilon_0}, \tag{335d}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{3/2}(t,u,\omega)}\int_{[u]_+}^{t}|\tilde{r}^2(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-12\epsilon_0}, \tag{335e}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}(t,u)}\int_{[u]_+}^{t}|\tilde{r}^2(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-16\epsilon_0}, \tag{335f}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{3/2}(t,u)}\int_{[u]_+}^{t}|\tilde{r}^2(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-12\epsilon_0}, \tag{335g}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^2(t,u)}\int_{[u]_+}^{t}|\tilde{r}^2(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1-8\epsilon_0}, \tag{335h}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{3/2}(t,u)}\int_{[u]_+}^{t}|\tilde{r}^3\nabla\!\!\!/(\vec{\mathcal{C}},\mathcal{D})|_{g\!\!\!/}(\tau,u,\omega)\,d\tau\right\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-8\epsilon_0}, \tag{336a}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{2}(t,u)}\int_{[u]_+}^{t}|\tilde{r}^3\nabla\!\!\!/(\vec{\mathcal{C}},\mathcal{D})|_{g\!\!\!/}(\tau,u,\omega)\,d\tau\right\|_{L_t^2 L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-8\epsilon_0}, \tag{336b}$$

[60] The estimates (335c) and (340b) in fact hold whenever $q \geq 2$, but in proving Propositions 10.1 and 11.1, we need these estimates only when $q > 2$ is sufficiently close to 2.

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{3/2}(t,u)}\int_{[u]_+}^{t}\left|\tilde{r}^3(\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\not{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\right|_{\not{g}}(\tau,u,\omega)\,d\tau\right\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{-1/2-12\epsilon_0}, \tag{337a}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{2}(t,u)}\int_{[u]_+}^{t}\left|\tilde{r}^3(\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\not{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\right|_{\not{g}}(\tau,u,\omega)\,d\tau\right\|_{L_t^2 L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{-1/2-12\epsilon_0}, \tag{337b}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}(t,u)}\int_{[u]_+}^{t}\left|\tilde{r}^2(\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\not{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\right|_{\not{g}}(\tau,u,\omega)\,d\tau\right\|_{L_u^2 L_t^2 L_\omega^p(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{-16\epsilon_0}, \tag{338a}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{1/2}(t,u)}\int_{[u]_+}^{t}\left|\tilde{r}^2(\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\not{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\right|_{\not{g}}(\tau,u,\omega)\,d\tau\right\|_{L_u^2 L_t^\infty L_\omega^p(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{-16\epsilon_0}, \tag{338b}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}(t,u)}\int_{[u]_+}^{t}|\tilde{r}^2(\boldsymbol{\partial}\vec{\mathcal{C}},\boldsymbol{\partial}\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L_u^2 L_t^2 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-12\epsilon_0}, \tag{339a}$$

$$\lambda^{-1}\left\|\frac{1}{\tilde{r}^{1/2}(t,u)}\int_{[u]_+}^{t}|\tilde{r}^2(\boldsymbol{\partial}\vec{\mathcal{C}},\boldsymbol{\partial}\mathcal{D})|(\tau,u,\omega)\,d\tau\right\|_{L_u^2 L_t^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-12\epsilon_0}. \tag{339b}$$

Standard spacetime norm estimates:

$$\lambda^{-1}\|\tilde{r}(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^2 L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-8\epsilon_0}, \tag{340a}$$

$$\lambda^{-1}\|\tilde{r}(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^{\frac{q}{2}} L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}+1)}, \tag{340b}$$

$$\lambda^{-1}\|\tilde{r}(\vec{\mathcal{C}},\mathcal{D})\|_{L_u^2 L_t^2 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-12\epsilon_0}, \tag{340c}$$

$$\lambda^{-1}\|\tilde{r}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L_u^2 L_t^1 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-\frac{1}{2}-8\epsilon_0}, \tag{340d}$$

$$\lambda^{-1}\|\tilde{r}(\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\not{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\|_{L_u^2 L_t^1 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-\frac{1}{2}-10\epsilon_0}. \tag{340e}$$

Proof Throughout the proof, we silently use the simple bound $\tilde{r}(\tau,u)/\tilde{r}(t,u) \lesssim 1$ for $\tau \leq t$.

To prove (335a), we first use (151) and the bound $\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^\infty_\omega(S_{t,u})} \leq \|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^\infty(\Sigma_t)}$ to deduce that

$$\begin{aligned}\text{LHS (335}a) &\lesssim \lambda^{-1}\left\|\int_{[u]_+}^t |(\vec{\mathcal{C}}, \mathcal{D})|(\tau, u, \omega)\, d\tau\right\|_{L^2_t L^\infty_u L^\infty_\omega(\widetilde{\mathcal{M}})} \\ &\lesssim \lambda^{-1/2-4\epsilon_0}\left\|\int_{[u]_+}^t \|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^\infty_\omega(S_{t,u})}\, d\tau\right\|_{L^\infty_t L^\infty_u} \\ &\lesssim \lambda^{-1/2-4\epsilon_0}\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^1_t L^\infty_x(\widetilde{\mathcal{M}})}.\end{aligned}$$

Using (151) and (307b), we bound the RHS of the previous expression by

$$\lesssim \lambda^{-8\epsilon_0}\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^\infty_x(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-12\epsilon_0}$$

as desired.

The estimate (335b) can be proved using an argument that is nearly identical to the one we used to prove (335a), and we therefore omit the details.

To prove (335c), we argue as above to deduce that

$$\begin{aligned}\text{LHS (335}c) &\lesssim \lambda^{-1}\left\|\int_{[u]_+}^t |(\vec{\mathcal{C}}, \mathcal{D})|(\tau, u, \omega)\, d\tau\right\|_{L_t^{\frac{q}{2}} L^\infty_u L^\infty_\omega(\widetilde{\mathcal{M}})} \\ &\lesssim \lambda^{-1}(\lambda^{1-8\epsilon_0})^{\frac{2}{q}}\left\|\int_{[u]_+}^t \|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^\infty_\omega(S_{t,u})}\, d\tau\right\|_{L^\infty_t L^\infty_u} \\ &\lesssim \lambda^{-1+\frac{2}{q}-\frac{16}{q}\epsilon_0}\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^1_t L^\infty_x(\widetilde{\mathcal{M}})}.\end{aligned}$$

Using (151) and (307b), we bound the RHS of the previous expression by

$$\lesssim \lambda^{-1/2-4\epsilon_0+\frac{2}{q}-\frac{16}{q}\epsilon_0}\|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^\infty_x(\widetilde{\mathcal{M}})} \lesssim \lambda^{\frac{2}{q}-1-\frac{16}{q}\epsilon_0-8\epsilon_0} = \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}+2)}$$

as desired.

The estimates (335d)–(335h) can be proved using similar arguments that also take into account the bound (177) for $\tilde{r}$, and we therefore omit the straightforward details.

To prove (336a), we first observe (switching the order of L^∞_u and L^∞_t) that it suffices to prove that for each fixed $u \in [-\frac{4}{5}\lambda^{1-8\epsilon_0}T_*, \lambda^{1-8\epsilon_0}T_*]$, we have

$$\lambda^{-1}\left\|\int_{[u]_+}^t |\tilde{r}^{3/2}\nabla\!\!\!/(\vec{\mathcal{C}}, \mathcal{D})|_{g\!\!\!/}(\tau, u, \omega)\, d\tau\right\|_{L^\infty_t L^p_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-8\epsilon_0}.$$

Using (151), (177), and (333b), we conclude that the LHS of the previous expression is

$$\lesssim \lambda^{-1}\|\tilde{r}^{3/2}\nabla\!\!\!/(\vec{\mathcal{C}}, \mathcal{D})\|_{L^1_t L^p_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-8\epsilon_0}\|\tilde{r}\nabla\!\!\!/(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^p_\omega(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-8\epsilon_0}$$

as desired.

The estimate (336b) can be proved using a similar argument, and we omit the details.

To prove (337a), we first observe (switching the order of L_u^∞ and L_t^∞ and using that $|\vec{S}| \lesssim 1$) that it suffices to prove that for each fixed $u \in [-\frac{4}{5}\lambda^{1-8\epsilon_0}T_*, \lambda^{1-8\epsilon_0}T_*]$, we have

$$\lambda^{-1}\left\| \int_{[u]_+}^t \left| \tilde{r}^{3/2}\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S}) \cdot (\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \tilde{r}^{-1}) \right|_{\not g} (\tau, u, \omega)\, d\tau \right\|_{L_t^\infty L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-12\epsilon_0}.$$

Using (151), (177), (307b), and (309), we deduce that the LHS of the previous expression is

$$\lesssim \lambda^{-8\epsilon_0}\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_t^2 L_\omega^\infty(\widetilde{\mathcal{C}}_u)} + \lambda^{1/2-12\epsilon_0}\|\boldsymbol{\partial}(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_t^2 L_\omega^\infty(\widetilde{\mathcal{C}}_u)} \left\|(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta)\right\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2-12\epsilon_0} \tag{341}$$

as desired.

The estimates (337b), (338a), and (338b) can be proved using similar arguments, and we omit the details.

To prove (339a), we first use (151) to deduce (switching the order of L_u^2 and L_t^2) that

$$\text{LHS } (339a) \lesssim \lambda^{-1/2-4\epsilon_0}\left\| \int_0^t \|\tilde{r}(\boldsymbol{\partial}\vec{\mathcal{C}}, \boldsymbol{\partial}\mathcal{D})\|_{L_u^2 L_\omega^p(\widetilde{\Sigma}_\tau)}\, d\tau \right\|_{L_t^\infty}.$$

Using (151) and (333c) with $Q := p$, we deduce that the RHS of the previous expression is

$$\lesssim \lambda^{-1/2-4\epsilon_0}\|\tilde{r}(\boldsymbol{\partial}\vec{\mathcal{C}}, \boldsymbol{\partial}\mathcal{D})\|_{L_t^1 L_u^2 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{1/2-12\epsilon_0}\|\tilde{r}(\boldsymbol{\partial}\vec{\mathcal{C}}, \boldsymbol{\partial}\mathcal{D})\|_{L_t^\infty L_u^2 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-12\epsilon_0}$$

as desired.

The estimate (339b) can be proved using similar arguments that also take into account the bound (177) for $\tilde{r}$, and we therefore omit the straightforward details.

To prove (340a), we use (151), (177), and (331a) to conclude that

$$\text{LHS } (340a) \lesssim \lambda^{-1/2-4\epsilon_0}\|\tilde{r}^{1/2}\|_{L^\infty(\widetilde{\mathcal{M}})}\|\tilde{r}^{1/2}(\vec{\mathcal{C}}, \mathcal{D})\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-8\epsilon_0} \tag{342}$$

as desired.

To prove (340b), we use a similar argument to conclude that

$$\lambda^{-1}\|\tilde{r}(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^{\frac{q}{2}}L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1}\lambda^{(1-8\epsilon_0)\frac{2}{q}}\|\tilde{r}^{1/2}\|_{L^\infty(\widetilde{\mathcal{M}})}\|\tilde{r}^{1/2}(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0-\frac{16}{q}\epsilon_0} = \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}+1)} \tag{343}$$

as desired.

The estimate (340c) follows easily from (340a) and the bounds (177) for u.

To prove (340d), we use (151) and (333c) with $Q := p$ to deduce that

$$\lambda^{-1}\|\tilde{r}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L_u^2 L_t^1 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1}\|\tilde{r}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^1 L_u^2 L_\omega^p(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{-8\epsilon_0}\|\tilde{r}\boldsymbol{\partial}(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^\infty L_u^2 L_\omega^p(\widetilde{\mathcal{M}})} \lesssim \lambda^{-\frac{1}{2}-8\epsilon_0}$$

as desired.

To prove (340e), we first use (151), (177), and the fact that $|\vec{S}| \lesssim 1$ to deduce that

$$\text{LHS (340e)} \lesssim \lambda^{-1/2-4\epsilon_0}\left\|\tilde{r}(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta,\tilde{r}^{-1})\right\|_{L_u^\infty L_t^1 L_\omega^p(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{1/2-8\epsilon_0}\|(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\|_{L_u^\infty L_t^2 L_\omega^\infty(\widetilde{\mathcal{M}})}\|(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\not g}\widetilde{\chi}^{(Small)},\hat{\chi},\zeta)\|_{L_u^\infty L_t^2 L_\omega^p(\widetilde{\mathcal{M}})}$$
$$+\lambda^{-8\epsilon_0}\|(\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\|_{L_u^\infty L_t^2 L_\omega^p(\widetilde{\mathcal{M}})}. \tag{344}$$

Using (307b) and the bootstrap assumptions (309), we conclude that RHS (344) $\lesssim \lambda^{-\frac{1}{2}-10\epsilon_0}$ as desired. □

10.6 Control of the integral curves of *L*

The main results of this subsection are Proposition 10.7 and Corollary 10.8. The proposition yields quantitative estimates showing that at fixed u, the distinct integral curves of L remain separated (see Footnote 40). The corollary is a simple consequence of the proposition and the bootstrap assumptions. It provides $L_t^2 L_u^\infty C_\omega^{0,\delta_0}$ estimates for the fluid variables. Later, we will combine these estimates with the Schauder-type estimate (365) to obtain $L_t^q L_u^\infty C_\omega^{0,\delta_0}$-control of $\hat{\chi}$ for several values of q; see the proofs of (292) and (295) for $\hat{\chi}$.

We start with some preliminary estimates, provided by the following lemma.

Lemma 10.6 (Preliminary results for controlling the integral curves of L). *Under the assumptions of Sect. 10.2, if λ is sufficiently large, then the following results hold.*

Results along Σ_0*: For $A = 1,2$ and $i = 1,2,3$, let $\left(\frac{\partial}{\partial\omega^A}\right)^i$ denote the Cartesian components of $\frac{\partial}{\partial\omega^A}$, and let $\Theta_{(A)}$ be the $S_{t,u}$-tangent vectorfield with Cartesian components $\Theta^i_{(A)} := \frac{1}{\tilde{r}}\left(\frac{\partial}{\partial\omega^A}\right)^i$, as in (225). Along Σ_0 (where $\tilde{r} = w = -u$), for $0 < w \leq w_{*;(\lambda)} := \frac{4}{5}T_{*;(\lambda)}$ and $\omega \in \mathbb{S}^2$, we view $\Theta^i_{(A)} = \Theta^i_{(A)}(0,w,\omega)$, and similarly for the Cartesian spatial components N^i and L^i. Then for each $\omega \in \mathbb{S}^2$,*

$\lim_{w \downarrow 0} N^i(0, w, \omega)$, $\lim_{w \downarrow 0} L^i(0, w, \omega)$, and $\lim_{w \downarrow 0} \Theta^i_{(A)}(0, w, \omega)$ exist, and we respectively denote the limits by $N^i(0, 0, \omega)$, $L^i(0, 0, \omega)$, and $\Theta^i_{(A)}(0, 0, \omega)$. Furthermore, for each $\omega \in \mathbb{S}^2$, we have that

$$g_{cd}(0, 0, \omega)\Theta^c_{(A)}(0, 0, \omega)\Theta^d_{(B)}(0, 0, \omega) = \not{g}(\omega)\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) + \mathcal{O}(\lambda^{-4\epsilon_0}).$$

In addition, the following estimates hold for $(w, \omega) \in [0, \frac{4}{5}T_{*;(\lambda)}] \times \mathbb{S}^2$, where $x^i(0, w, \omega)$ are the Cartesian spatial coordinates viewed as a function of w, ω along Σ_0, and $\mathbf{z}^i$ are the Cartesian spatial coordinates of the point $\mathbf{z} \in \Sigma_0$ (see Sect. 9.4):

$$x^i(0, w, \omega) = \mathbf{z}^i + w\left\{N^i(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0})\right\}, \tag{345a}$$

$$N^i(0, w, \omega) = N^i(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0}), \tag{345b}$$

$$L^i(0, w, \omega) = L^i(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0}), \tag{345c}$$

$$\Theta^i_{(A)}(0, w, \omega) = \Theta^i_{(A)}(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0}). \tag{345d}$$

Moreover, the following identity holds:

$$\frac{\partial}{\partial \omega^A} N^i(0, 0, \omega) = \frac{\partial}{\partial \omega^A} L^i(0, 0, \omega) = \Theta^i_{(A)}(0, 0, \omega). \tag{346}$$

In addition, with $d_{\not{g}}(\omega_{(1)}, \omega_{(2)})$ denoting the distance between the points $\omega_{(1)}, \omega_{(2)} \in \mathbb{S}^2$ with respect to the standard Euclidean round metric $\not{g}$ on $\mathbb{S}^2$, we have the following estimate:

$$\sum_{i=1}^{3} |N^i(0, 0, \omega_{(1)}) - N^i(0, 0, \omega_{(2)})| = \sum_{i=1}^{3} |L^i(0, 0, \omega_{(1)}) - L^i(0, 0, \omega_{(2)})| \approx d_{\not{g}}(\omega_{(1)}, \omega_{(2)}). \tag{347}$$

Finally, we have the following estimate, ($\alpha = 0, 1, 2, 3$):

$$\|L^\alpha\|_{L^\infty_u C^{0,\delta_0}_\omega(\widetilde{\Sigma}_0)} \lesssim 1. \tag{348}$$

***Results along the cone-tip axis**: In $\widetilde{\mathcal{M}}^{(Int)}$, let us view $\Theta^i_{(A)} = \Theta^i_{(A)}(t, u, \omega)$, and similarly for the Cartesian spatial components N^i and L^i. Then for each $(u, \omega) \in [0, T_{*;(\lambda)}] \times \mathbb{S}^2$, $\lim_{t \downarrow u} \Theta^i_{(A)}(t, u, \omega)$ exists, and we denote the limit by $\Theta^i_{(A)}(u, u, \omega)$. Furthermore, the following estimate holds for $(t, \omega) \in [0, T_{*;(\lambda)}] \times \mathbb{S}^2$: $g_{ab}(t, t, \omega)\Theta^a_{(A)}(t, t, \omega)\Theta^b_{(B)}(t, t, \omega) = \not{g}(\omega)\left(\frac{\partial}{\partial \omega^A}, \frac{\partial}{\partial \omega^B}\right) + \mathcal{O}(\lambda^{-\epsilon_0})$, and within each coordinate chart on $\mathbb{S}^2$, for each ω in the domain of the chart, $\{\Theta_{(1)}(t, t, \omega), \Theta_{(2)}(t, t, \omega)\}$ is a linearly independent set of vectors in $\mathbb{R}^3$.*

Moreover, along the cone-tip axis, that is, for $t \in [0, T_{*;(\lambda)}]$ *we have:*

$$N^i(t, t, \omega) = N^i(0, 0, \omega) + \mathcal{O}(\lambda^{-8\epsilon_0}), \tag{349a}$$

$$L^i(t, t, \omega) = L^i(0, 0, \omega) + \mathcal{O}(\lambda^{-8\epsilon_0}), \tag{349b}$$

$$\Theta^i_{(A)}(t, t, \omega) = \Theta^i_{(A)}(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0}). \tag{349c}$$

In addition, for $(t, \omega) \in [0, T_{*;(\lambda)}] \times \mathbb{S}^2$*, the following relations hold along the cone-tip axis, that is, for* $t \in [0, T_{*;(\lambda)}]$*:*

$$\frac{\partial}{\partial \omega^A} N^i(t, t, \omega) = \frac{\partial}{\partial \omega^A} L^i(t, t, \omega) = \Theta^i_{(A)}(t, t, \omega). \tag{350}$$

Results in $\widetilde{\mathcal{M}}$*: For* $u \in [-\frac{4}{5} T_{*;(\lambda)}, T_{*;(\lambda)}]$*,* $t \in [[u]_+, T_{*;(\lambda)}]$*, and* $\omega \in \mathbb{S}^2$*, we have*

$$L^i(t, u, \omega) = L^i(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0}), \tag{351a}$$

$$\Theta^i_{(A)}(t, u, \omega) = \Theta^i_{(A)}(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0}). \tag{351b}$$

Proof **Proof of the results along** Σ_0. We start by showing that $\lim_{w \downarrow 0} \Theta^i_{(A)}(0, w, \omega) := \Theta^i_{(A)}(0, 0, \omega)$ exists, and we exhibit the desired properties of the limit. We will use the evolution equation (227).

From the bootstrap assumptions, the simple bound $\|\mathrm{f}_{(\vec{L})}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim 1$ implied by them, the estimates of Proposition 9.8, (318) with $Q := p$, (332a) and (332c) along Σ_0 with $Q := p$, the bound (177) for $\tilde{r}|_{\Sigma_0} = w$, and the estimate $\|\Theta^i_{(A)}\|_{L^\infty(\widetilde{\Sigma}_0)} \lesssim 1$ implied by (282c), we find that the first term on RHS (227) verifies

$$\begin{aligned}
&\|a \cdot \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}) \cdot \vec{\Theta}_{(A)}\|_{L^1_w L^\infty_\omega(\widetilde{\Sigma}_0)} \\
&\lesssim \lambda^{1/2-4\epsilon_0} \|\tilde{r} \nabla\!\!\!\!/ \boldsymbol{\partial}\vec{\Psi}\|_{L^2_w L^p_\omega(\widetilde{\Sigma}_0)} + \lambda^{1/2-4\epsilon_0} \|\boldsymbol{\partial}\vec{\Psi}\|_{L^2_w L^2_\omega(\widetilde{\Sigma}_0)} + \lambda^{1/2-4\epsilon_0} \|\hat{\chi}\|_{L^2_w L^\infty_\omega(\widetilde{\Sigma}_0)} \\
&\lesssim \lambda^{-4\epsilon_0},
\end{aligned}$$

and that the last term on RHS (227) verifies the same bound:

$$\|\mathrm{f}_{(\vec{L})} \cdot \nabla\!\!\!\!/ a \cdot \vec{\Theta}_{(A)}\|_{L^1_w L^\infty_\omega(\widetilde{\Sigma}_0)} \lesssim \lambda^{1/2-4\epsilon_0} \|\nabla\!\!\!\!/ a\|_{L^2_w L^\infty_\omega(\widetilde{\Sigma}_0)} \lesssim \lambda^{-4\epsilon_0}.$$

We now integrate equation (227) with respect to w and use these estimates and the initial condition for $\vec{\Theta}_{(A)}$ at the convenient value $w = 1$ (which, by (282c), is a value at which the vectors $\vec{\Theta}_{(1)}$ and $\vec{\Theta}_{(2)}$ are known to be finite and linearly independent) thereby concluding that if λ is sufficiently large, then $\lim_{w \downarrow 0} \Theta^i_{(A)}(0, w, \omega)$ exists, that for $0 \le w \le \frac{4}{5} T_{*;(\lambda)}$ and $\omega \in \mathbb{S}^2$ we have

$$\Theta^i_{(A)}(0, w, \omega) = \Theta^i_{(A)}(0, 1, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0}),$$

and that

$$g_{cd}(0,0,\omega)\Theta^c_{(A)}(0,0,\omega)\Theta^d_{(B)}(0,0,\omega) \approx \not{e}(\omega)\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right).$$

Except for (346), these arguments yield all desired results for $\Theta^i_{(A)}$ along Σ_0, including (345d). To prove (346), we contract the estimate (286) against $\Theta^j_{(A)}(0,w,\omega)$, use the identities $w\Theta^j_{(A)}\not{\Pi}^c_j\partial_c N^i = \frac{\partial}{\partial\omega^A}N^i$, $\Theta^j_{(A)}\not{\Pi}^i_j = \Theta^i_{(A)}$, and $L^i = \mathbf{B}^i + N^i$, use that $\mathbf{B}^i(0,0,\omega) = \mathbf{B}^i|_{\mathbf{z}}$ is independent of ω, and use the previous results proved in this paragraph.

The results for L^i and N^i along Σ_0 stated in the lemma, including (345b) and (345c), can be obtained from similar reasoning based on the evolution equations in (224), and we omit the details.

Next, we consider the map $\mathfrak{N}(\omega) := \big(N^1(0,0,\omega), N^2(0,0,\omega), N^3(0,0,\omega)\big)$ from the domain $\mathbb{S}^2$ to the target

$$UT_{\mathbf{z}}\Sigma_0 := \{V \in T_{\mathbf{z}}\Sigma_0 \mid g_{cd}|_{\mathbf{z}}V^cV^d = 1\} \simeq \mathbb{S}^2.$$

The results from the first paragraph of this proof, including (346), yield that the differential of $\mathfrak{N}$ with respect to ω is injective. Thus, $\mathfrak{N}$ is a differentiable open map from $\mathbb{S}^2$ to $\mathbb{S}^2$, and it is a standard result of differential topology that $\mathfrak{N}$ must be a covering map (in particular, it is onto). Thus, taking into account the quantitative bounds for the differential of $\mathfrak{N}$ with respect to ω proved above, we conclude that there exists a uniform constant $0 < \beta < \pi$ such that if λ is sufficiently large, then (347) holds (with bounded implicit constants) for all pairs $\omega_{(1)}, \omega_{(2)} \in \mathbb{S}^2$ such that $d_{\not{e}}(\omega_{(1)},\omega_{(2)}) < \beta$. Moreover, since the domain $\mathbb{S}^2$ is path-connected and the target $UT_{\mathbf{z}}\Sigma_0 \simeq \mathbb{S}^2$ is simply connected, it is a standard result in algebraic topology that $\mathfrak{N}$ is in fact a diffeomorphism (see [33, Theorem 54.4] and note that $UT_{\mathbf{z}}\Sigma_0 \simeq \mathbb{S}^2$ has a trivial fundamental group since it is simply connected). In particular, $\mathfrak{N}$ is globally injective. This fact yields (347) (again, with bounded implicit constants) for all $\omega_{(1)}, \omega_{(2)} \in \mathbb{S}^2$ with $\beta \leq d_{\not{e}}(\omega_{(1)},\omega_{(2)}) \leq \pi$.

To prove (345a), we first use (167) to deduce $\frac{\partial}{\partial w}x^i(0,w,\omega) = [aN^i](0,w,\omega)$. Also using (282a) and (345b), we see that $\frac{\partial}{\partial w}x^i(0,w,\omega) = N^i(0,0,\omega) + \mathcal{O}(\lambda^{-4\epsilon_0})$. Integrating this estimate with respect to w starting from the value $w = 0$, and using the initial condition $x^i(0,w,\omega) = \mathbf{z}^i$, we conclude (345a).

We now show that for each $(u,\omega) \in [-\frac{4}{5}T_{*;(\lambda)}, 0) \times \mathbb{S}^2$,

$$\lim_{t\downarrow 0}\Theta^i_{(A)}(t,u,\omega) = \Theta^i_{(A)}(0,u,\omega)$$

and

$$g_{cd}(0,u,\omega)\Theta^c_{(A)}(0,u,\omega)\Theta^d_{(B)}(0,u,\omega) = \not{e}(\omega)\left(\frac{\partial}{\partial\omega^A},\frac{\partial}{\partial\omega^B}\right) + \mathcal{O}(\lambda^{-\epsilon_0}).$$

The desired results can be obtained by using arguments similar to the ones given in the first paragraph of this proof, based on the evolution equation (226) and the bootstrap assumptions, including (151), (306a), (308a), and (312).

Finally, we prove (348). The result is trivial for L^0 since this component is constantly unity. Next, we note the schematic identity $\nabla\!\!\!/ L^i = \mathrm{f}_{(\vec{L})}\chi + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, where on the LHS, we are viewing $\nabla\!\!\!/ L^i$ to be the angular gradient of the scalar function L^i. Hence, applying (318) with $Q := p$ and with the scalar function L^i in the role of ξ, and using the simple bound $\|\mathrm{f}_{(\vec{L})}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim 1$ implied by the bootstrap assumptions, we find that for $u \in [-\frac{4}{5}T_{*;(\lambda)}, 0]$, we have

$$\|L^i\|_{C^{0,1-\frac{2}{p}}_\omega(S_{0,u})} \lesssim \|\tilde{r}\chi\|_{L^p_\omega(S_{0,u})} + \|\tilde{r}\boldsymbol{\partial}\vec{\Psi}\|_{L^p_\omega(S_{0,u})} + 1.$$

Also using the first identities in (200) and (204c), (207), the schematic identity $k_{AB} = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, and the parameter relation (276), we find that

$$\|L^i\|_{L^\infty_u C^{0,\delta_0}_\omega(\widetilde{\Sigma}_0)} \lesssim \|\tilde{r}\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi}^{(Small)}\|_{L^\infty_u L^p_\omega(\widetilde{\Sigma}_0)} + \|\tilde{r}\hat{\theta}\|_{L^\infty_u L^p_\omega(\widetilde{\Sigma}_0)} + \|\tilde{r}\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty_u L^p_\omega(\widetilde{\Sigma}_0)} + 1.$$

From (282b) with $q_* := p$, (285), (332a), and (177) for $u|_{\Sigma_0} = -w$, we conclude that the RHS of the previous estimate is $\lesssim 1$, which yields (348).

Proof of the results along the cone-tip axis. The ODE (161) can be expressed in the schematic form $\frac{d}{dt}\vec{N}_\omega = \mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\vec{\Psi} \cdot \vec{N}_\omega$, Here, $\vec{N}_\omega = \vec{N}_\omega(t)$ denotes the array of Cartesian spatial components of the unit outward normal vector N (corresponding to the parameter $\omega \in \mathbb{S}^2$) along the cone-tip axis $\gamma_{\mathbf{z}}(t)$. That is, if $\vec{N}(t, u, \omega)$ denotes the array of Cartesian spatial components of N viewed as a function of the geometric coordinates (t, u, ω), then $\vec{N}_\omega(t) := \vec{N}(t, t, \omega)$. Moreover, in the previous expressions, we have abbreviated $\vec{\Psi} = \vec{\Psi} \circ \gamma_{\mathbf{z}}(t)$ and $\boldsymbol{\partial}\vec{\Psi} = [\boldsymbol{\partial}\vec{\Psi}] \circ \gamma_{\mathbf{z}}(t)$. Integrating the ODE in time and using the bootstrap assumptions, we deduce that $|\vec{N}_\omega(t) - \vec{N}_\omega(0)| \lesssim \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\, d\tau$. From this estimate, (151), and (306a), we arrive at the desired bound (349a). The desired bound for (349b) follows from (349a), the identity $L = \mathbf{B} + N$, and the estimate $|\mathbf{B}^\alpha(t, t, \omega) - \mathbf{B}^\alpha(0, 0, \omega)| \lesssim \lambda^{-8\epsilon_0}$, which follows from integrating the estimate $|\mathbf{B}\mathbf{B}^\alpha|(\tau, \tau, \omega) \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}$ (valid since $\mathbf{B}^\alpha = \mathbf{B}^\alpha(\vec{\Psi})$) with respect to τ and using (151) and (306a).

We now show that for each $(u, \omega) \in [0, T_{*;(\lambda)}] \times \mathbb{S}^2$, $\lim_{t \downarrow u} \Theta^i_{(A)}(t, u, \omega) := \Theta^i_{(A)}(u, u, \omega)$ exists and that

$$g_{cd}(u, u, \omega)\Theta^c_{(A)}(u, u, \omega)\Theta^d_{(B)}(u, u, \omega) = \not{e}(\omega)\left(\frac{\partial}{\partial\omega^A}, \frac{\partial}{\partial\omega^B}\right) + \mathcal{O}(\lambda^{-\epsilon_0}).$$

The desired results can be obtained by using arguments similar to the ones given in the first paragraph of this proof, based on the evolution equation (226) and the bootstrap assumptions, including (151), (306a), (308a), and (312); we omit the details.

We now prove (350). From the identity $L = \mathbf{B} + N$ and the fact that $\frac{\partial}{\partial\omega^A}\mathbf{B}^\alpha(t, t, \omega) = \frac{\partial}{\partial\omega^A}[\mathbf{B}^\alpha \circ \vec{\Psi} \circ \gamma_{\mathbf{z}}(t)] = 0$, we find that $\frac{\partial}{\partial\omega^A}N^i(t, t, \omega) =$

$\frac{\partial}{\partial\omega^A}L^i(t,t,\omega)$, as is stated in (350). From the fact that $\lim_{t\downarrow u}\Theta^i_{(A)}(t,u,\omega) = \Theta^i_{(A)}(u,u,\omega)$ and the asymptotic initial condition (283a) for $|\tilde{r}\not{\Pi}^a_j\partial_a L^i - \not{\Pi}^i_j|$, we find that $\frac{\partial}{\partial\omega^A}N^i(t,t,\omega) = \Theta^i_{(A)}(t,t,\omega)$, which finishes the proof of (350).

We now prove (349c). We differentiate the ODE (161) with respect to the parameter ω^A (that is, with the operator $\frac{\partial}{\partial\omega^A}$), use the fact that $\frac{\partial}{\partial\omega^A}[\vec{\Psi}\circ\gamma_{\mathbf{z}}(t)] = \frac{\partial}{\partial\omega^A}([\boldsymbol{\partial}\Psi]\circ\gamma_{\mathbf{z}}(t)) = 0$, integrate the resulting ODE in time, and use the bootstrap assumptions, thereby deducing that

$$\left|\frac{\partial}{\partial\omega^A}\vec{N}_\omega(t) - \frac{\partial}{\partial\omega^A}\vec{N}_\omega(0)\right| \lesssim \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\left|\frac{\partial}{\partial\omega^A}\vec{N}_\omega(t) - \frac{\partial}{\partial\omega^A}\vec{N}_\omega(0)\right| d\tau + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\left|\frac{\partial}{\partial\omega^A}\vec{N}_\omega(0)\right| d\tau. \tag{352}$$

From (352), (151), (306a), (286) (which, in view of (282c), implies that $\left|\frac{\partial}{\partial\omega^A}\vec{N}_\omega(0)\right| \lesssim 1$), and Grönwall's inequality, we find that $\frac{\partial}{\partial\omega^A}N^i(t,t,\omega) = \frac{\partial}{\partial\omega^A}N^i(0,0,\omega) + \mathcal{O}(\lambda^{-8\epsilon_0})$. From this estimate and (346), we deduce that $\frac{\partial}{\partial\omega^A}N^i(t,t,\omega) = \Theta^i_{(A)}(0,0,\omega) + \mathcal{O}(\lambda^{-4\epsilon_0})$. Finally, from this bound and (350), we conclude (349c).

Proof of the results in $\widetilde{\mathcal{M}}$. We now show that (351b) holds. This estimate can be obtained by using arguments similar to the ones given in the first paragraph of this proof, based on the evolution equation (226) and the bootstrap assumptions, including (151), (306a), (308a), and (312). The initial conditions for $\Theta^i_{(A)}$ on Σ_0 (which are relevant for the region $\widetilde{\mathcal{M}}^{(Ext)}$) can be related back to $\Theta^i_{(A)}(0,0,\omega)$ via the already proven estimate (345d), while the initial conditions for $\Theta^i_{(A)}$ on the cone-tip axis (which are relevant for the region $\widetilde{\mathcal{M}}^{(Ext)}$) can be related back to $\Theta^i_{(A)}(0,0,\omega)$ via (349c); we omit the details.

The estimate (351a) can be obtained in a similar fashion based on the evolution equation for L^i stated in (223), the bootstrap assumptions, (151), (306a), and the already proven estimates (345c) and (349b); we omit the details. □

We now derive quantitative control of the integral curves of L in $\widetilde{\mathcal{M}}$.

Proposition 10.7 (Control of the integral curves of L in $\widetilde{\mathcal{M}}$). *Let $\Upsilon_{u;\omega}(t)$ be the family of null geodesic curves from Sects. 9.4.1 and 9.4.2, which depend on the parameters $(u,\omega) \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}] \times \mathbb{S}^2$ and are parameterized by $t \in [[u]_+, T_{*;(\lambda)}]$ and normalized by $\Upsilon^0_{u;\omega}(t) = t$. Let $\omega_{(1)}, \omega_{(2)} \in \mathbb{S}^2$, and let $d_{\not{e}}(\omega_{(1)},\omega_{(2)})$ denote their distance with respect to the standard Euclidean round metric $\not{e}$. Under the assumptions of Sect. 10.2, the following estimate for the Cartesian components $\Upsilon^\alpha_{u;\omega}(t)$ (which can be identified with the Cartesian coordinate functions x^α, viewed as a function of (t,u,ω)) holds for $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$ and $t \in [[u]_+, T_{*;(\lambda)}]$:*

$$\sum_{\alpha=0}^{3}|\Upsilon^\alpha_{u;\omega_{(1)}}(t) - \Upsilon^\alpha_{u;\omega_{(2)}}(t)| \approx \tilde{r}d_{\not{e}}(\omega_{(1)},\omega_{(2)}). \tag{353}$$

Proof At the end of the proof, we will show that the following two estimates hold for $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$, $t \in [[u]_+, T_{*;(\lambda)}]$, and $\omega \in \mathbb{S}^2$, ($A = 1, 2$ and $i = 1, 2, 3$):

$$\Upsilon^i_{u;\omega}(t) = \Upsilon^i_{u;\omega}([u]_+) + (t - [u]_+)\left\{L^i(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0})\right\}, \tag{354}$$

$$\frac{\partial}{\partial \omega^A}\Upsilon^i_{u;\omega}(t) = \tilde{r}\left\{\Theta^i_{(A)}(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0})\right\}. \tag{355}$$

From (355) and the properties of the (linearly independent) set $\{\Theta_{(1)}(0, 0, \omega), \Theta_{(2)}(0, 0, \omega)\}$ shown in Lemma 10.6, it follows that the map $\omega \to \left(\Upsilon^1_{u;\omega}(t), \Upsilon^2_{u;\omega}(t), \Upsilon^3_{u;\omega}(t)\right)$ has an injective differential and, in particular, there exists $0 < \beta < \pi$ such that if λ is sufficiently large, then (353) holds whenever $d_{\not{e}}(\omega_{(1)}, \omega_{(2)}) < \beta$. From (347), (354), and the fact that $\Upsilon^i_{u;\omega}(u)$ is independent of ω when $u \in [0, T_{*;(\lambda)}]$, it follows that for this fixed value of β, if $\lambda > 0$ is sufficiently large, then (353) holds whenever $\beta \leq d_{\not{e}}(\omega_{(1)}, \omega_{(2)}) \leq \pi$, $u \in [0, T_{*;(\lambda)}]$, and $t \in [u, T_{*;(\lambda)}]$. (353) can be proved in the remaining case, in which $\beta \leq d_{\not{e}}(\omega_{(1)}, \omega_{(2)}) \leq \pi$, $u \in [-\frac{4}{5}T_{*;(\lambda)}, 0]$, and $t \in [0, T_{*;(\lambda)}]$, via a similar argument that also takes into account the estimate (345a), as we now explain. (345a) is relevant in that the identity $L^i = \mathbf{B}^i + N^i$, the fact that $\mathbf{B}^i(0, 0, \omega)$ is independent of ω, and the estimates (345a) and (354) collectively imply that for $u \in [-\frac{4}{5}T_{*;(\lambda)}, 0]$, $t \in [0, T_{*;(\lambda)}]$, and $\omega_{(1)}, \omega_{(2)} \in \mathbb{S}^2$, we have

$$\begin{aligned}&\sum_{\alpha=0}^{3}|\Upsilon^\alpha_{u;\omega_{(1)}}(t) - \Upsilon^\alpha_{u;\omega_{(2)}}(t)|\\ &\quad= (|u| + t)\left\{\sum_{i=1}^{3}|L^i(0, 0, \omega_{(1)}) - L^i(0, 0, \omega_{(2)})| + \mathcal{O}(\lambda^{-4\epsilon_0})\right\}.\end{aligned}$$

In view of (347) and the assumption $\beta \leq d_{\not{e}}(\omega_{(1)}, \omega_{(2)})$, we see that for λ sufficiently large, the $\mathcal{O}(\lambda^{-4\epsilon_0})$ term is negligible. Since $\tilde{r} = |u| + t$ when $u \leq 0$, we have completed the proof of (353).

It remains for us to prove (354)–(355). The estimate (355) follows directly from multiplying (351b) by $\tilde{r}$ and considering the definitions of $\Theta^i_{(A)}$ and $\Upsilon^i_{u;\omega}(t)$. To derive (354), we first use (351a) to deduce that $\frac{\partial}{\partial t}\Upsilon^i_{u;\omega}(t) = L^i(t, u, \omega) = L^i(0, 0, \omega) + \mathcal{O}(\lambda^{-4\epsilon_0})$. Integrating this estimate with respect to time starting from the time value $[u]_+$, we conclude (354). □

We now derive the main consequence of Proposition 10.7: a corollary that yields $L^2_t L^\infty_u C^{0,\delta_0}_\omega(\widetilde{\mathcal{M}})$ estimates for various fluid variables.

Corollary 10.8 ($L^2_t L^\infty_u C^{0,\delta_0}_\omega(\widetilde{\mathcal{M}})$ estimates). *Under the assumptions of Sect. 10.2, we have the following estimates:*

$$\|\boldsymbol{\partial}\vec{\Psi}\|_{L^2_t L^\infty_u C^{0,\delta_0}_\omega(\widetilde{\mathcal{M}})}, \|(\partial\vec{\Omega}, \partial\vec{S})\|_{L^2_t L^\infty_u C^{0,\delta_0}_\omega(\widetilde{\mathcal{M}})}, \|(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^\infty_u C^{0,\delta_0}_\omega(\widetilde{\mathcal{M}})} \lesssim \lambda^{-1/2-3\epsilon_0}. \tag{356}$$

Moreover,

$$\|(\vec{\Psi}, \vec{\Omega}, \vec{S})\|_{L_t^\infty L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} \lesssim 1. \tag{357}$$

Proof We prove (356) only for the first term on the LHS; the remaining terms on LHS (356) can be bounded using the same arguments. To proceed, we first use (353) to deduce that

$$\frac{|\boldsymbol{\partial}\vec{\Psi}(t,u,\omega_{(1)}) - \boldsymbol{\partial}\vec{\Psi}(t,u,\omega_{(2)})|}{[\tilde{r} d_{\not{g}}(\omega_{(1)}, \omega_{(2)})]^{\delta_0}} \lesssim \|\boldsymbol{\partial}\vec{\Psi}\|_{C_x^{0,\delta_0}(\widetilde{\Sigma}_t)}.$$

From this bound, the estimate (177) for $\tilde{r}$, and the inequality $\lambda^{(1-8\epsilon_0)\delta_0} \leq \lambda^{\delta_0} \leq \lambda^{\epsilon_0}$ (see (35b)–(35c)), we find, considering separately the cases $0 \leq \tilde{r} \leq 1$ and $1 \leq \tilde{r}$, that $\|\boldsymbol{\partial}\vec{\Psi}\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim \lambda^{\epsilon_0} \|\boldsymbol{\partial}\vec{\Psi}\|_{C_x^{0,\delta_0}(\widetilde{\Sigma}_t)}$. From this bound and (307c), we conclude the desired estimate (356).

To prove (357), we note that Proposition 5.1 and Sobolev embedding $H^N(\Sigma_t) \hookrightarrow C_x^{0,\delta_0}(\widetilde{\Sigma}_t)$ imply that the *non-rescaled* solution variables $(\vec{\Psi}, \vec{\Omega}, \vec{S})$ are bounded in the norm $\|\cdot\|_{L_t^\infty C_x^{0,\delta_0}(\widetilde{\mathcal{M}})}$ by $\lesssim 1$. It follows that the rescaled solution variables on LHS (357) (as defined in Sect. 9.1 and under the conventions of Sect. 9.3) are bounded in the norm $\|\cdot\|_{L^\infty(\widetilde{\mathcal{M}})}$ by $\lesssim 1$ and in the norm $\|\cdot\|_{L_t^\infty \dot{C}_x^{0,\delta_0}(\widetilde{\mathcal{M}})}$ by $\lesssim \lambda^{-\delta_0}$. From these estimates and arguments similar to the ones given in the previous paragraph, we conclude (357). □

10.7 Estimates for transport equations along the integral curves of *L* in Hölder spaces in the angular variables ω

We now derive estimates for transport equations along the integral curves of L with initial data and source terms that are Hölder-class in the geometric angular variables ω.

Lemma 10.9 (Estimates for transport equations along the integral curves of L in Hölder spaces with respect to ω). *Let $\widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}$. Let $\mathfrak{F}$ be a smooth scalar-valued function on $\widetilde{\mathcal{C}}_u$ and let $\mathring{\varphi}$ be a smooth scalar-valued function on $S_{[u]_+,u}$. For $(t,\omega) \in [[u]_+, T_{*;(\lambda)}] \times \mathbb{S}^2$, let the scalar-valued function φ be a smooth solution to the following inhomogeneous transport equation with data given on $S_{[u]_+,u}$:*

$$L\varphi(t,u,\omega) = \mathfrak{F}(t,u,\omega), \tag{358a}$$

$$\varphi([u]_+,u,\omega) = \mathring{\varphi}(\omega). \tag{358b}$$

Under the assumptions of Sect. 10.2, the following estimate holds for $t \in [[u]_+, T_{*;(\lambda)}]$:

$$\|\varphi\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim \|\mathring{\varphi}\|_{C_\omega^{0,\delta_0}(S_{[u]_+,u})} + \int_{[u]_+}^t \|\mathfrak{F}\|_{C_\omega^{0,\delta_0}(S_{\tau,u})}\, d\tau. \tag{359}$$

Moreover,

$$\left\| \int_{[u]_+}^{t} \mathfrak{F}(\tau, u, \omega)\, d\tau \right\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim \int_{[u]_+}^{t} \|\mathfrak{F}\|_{C_\omega^{0,\delta_0}(S_{\tau,u})}\, d\tau. \tag{360}$$

Proof The lemma is a straightforward consequence of the fundamental theorem of calculus and the fact that the angular geometric coordinate functions $\{\omega^A\}_{A=1,2}$ are constant along the integral curves of $L = \frac{\partial}{\partial t}$.

10.8 Calderon–Zygmund- and Schauder-type Hodge estimates on $S_{t,u}$

Some of the tensorfields under study are solutions to Hodge systems on $S_{t,u}$. To control them, we will use the Calderon–Zygmund and Schauder-type estimates provided by the following lemma.

Lemma 10.10 (Calderon–Zygmund- and Schauder-type Hodge estimates on $S_{t,u}$). *Under the assumptions of Sect. 10.2 and the estimates of Proposition 10.4, if ξ is an $S_{t,u}$-tangent one-form and $2 \leq Q \leq p$ (where p is as in (276)), then*

$$\|\nabla\!\!\!/ \xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \|\tilde{r}^{-1}\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} \lesssim \|\mathrm{d}\!\!\!/\mathrm{iv}\,\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \|\mathrm{cu}\!\!\!/\mathrm{rl}\,\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})}. \tag{361}$$

Similarly, if ξ is an $S_{t,u}$-tangent type $\binom{0}{2}$ symmetric trace-free tensorfield, then

$$\|\nabla\!\!\!/ \xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \|\tilde{r}^{-1}\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} \lesssim \|\mathrm{d}\!\!\!/\mathrm{iv}\,\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})}. \tag{362}$$

Moreover, let ξ be an $S_{t,u}$-tangent type $\binom{0}{2}$ symmetric trace-free tensorfield, let $\mathfrak{F}_{(1)}$ be a scalar function, let $\mathfrak{F}_{(2)}$ be an $S_{t,u}$-tangent type $\binom{0}{2}$ symmetric tensorfield, and let $\mathfrak{G}$ be an $S_{t,u}$-tangent one-form. Assume that[61]

$$\mathrm{d}\!\!\!/\mathrm{iv}\,\xi = \nabla\!\!\!/ \mathfrak{F}_{(1)} + \mathrm{d}\!\!\!/\mathrm{iv}\,\mathfrak{F}_{(2)} + \mathfrak{G}. \tag{363}$$

Let $2 < Q < \infty$, and let Q' be defined by $\frac{1}{2} + \frac{1}{Q} = \frac{1}{Q'}$. Then the following estimate holds:

$$\|\xi\|_{L^Q_{g\!\!\!/}(S_{t,u})} \lesssim \sum_{i=1,2} \|\mathfrak{F}_{(i)}\|_{L^Q_{g\!\!\!/}(S_{t,u})} + \|\mathfrak{G}\|_{L^{Q'}_{g\!\!\!/}(S_{t,u})}. \tag{364}$$

[61] On RHS (363), we made a minor change compared to [54]*Proposition 5.9: we allowed for the presence of the $\mathfrak{F}_{(2)}$ term, in particular so that we can handle the second term on RHS (230). We will now explain why the estimate (364) holds in the presence of this new term. First, we can split the $S_{t,u}$-tangent type $\binom{0}{2}$ symmetric tensorfield $\mathfrak{F}_{(2)}$ into its trace-free and pure-trace parts. We then bring the trace-free part over to the left-hand side of the equation (so that the new LHS is of the form $\mathrm{d}\!\!\!/\mathrm{iv}(\xi - \hat{\mathfrak{F}}_{(2)})$), while we absorb the pure-trace part of $\mathfrak{F}_{(2)}$ into the $\mathfrak{F}_{(1)}$ term. This allows one to reduce the proof of (364) to the case in which the $\mathfrak{F}_{(2)}$ term on RHS (363) is absent, as was assumed in [54, Proposition 5.9].

In addition, if $2 < Q \leq p$ *(where* p *is as in Sect. 10.1), then*

$$\|\xi\|_{C_{\omega}^{0,1-\frac{2}{Q}}(S_{t,u})} \lesssim \sum_{i=1,2} \|\mathfrak{F}_{(i)}\|_{C_{\omega}^{0,1-\frac{2}{Q}}} + \|\tilde{r}\mathfrak{G}\|_{L_{\omega}^{Q}(S_{t,u})}. \tag{365}$$

Similarly, assume that ξ*,* $\mathfrak{F}_{(1)}$*, and* $\mathfrak{F}_{(2)}$ *are* $S_{t,u}$*-tangent one-forms and* $\mathfrak{G}_{(1)}$*, and* $\mathfrak{G}_{(2)}$ *are scalar functions such that* ξ *satisfies the following Hodge system:*

$$\mathrm{d}\!\!/\mathrm{iv}\,\xi = \mathrm{d}\!\!/\mathrm{iv}\,\mathfrak{F}_{(1)} + \mathfrak{G}_{(1)}, \tag{366a}$$

$$\mathrm{cu}\!\!/\mathrm{rl}\,\xi = \mathrm{cu}\!\!/\mathrm{rl}\,\mathfrak{F}_{(2)} + \mathfrak{G}_{(2)}. \tag{366b}$$

Then under the same assumptions on Q *and* Q' *stated in the previous paragraph,* ξ *satisfies the estimates (364)–(365) with* $\mathfrak{G} := (\mathfrak{G}_{(1)}, \mathfrak{G}_{(2)})$.

Finally, assume that ξ*,* $\mathfrak{F} = (\mathfrak{F}_{(1)}, \mathfrak{F}_{(2)})$*, and* $\mathfrak{G}$ *are* $S_{t,u}$ *tensorfields of the type from the previous two paragraphs (in particular satisfying (363) or (366a)–(366b)). Assume that* $\mathfrak{F}$ *is the* $S_{t,u}$*-projection of a spacetime tensorfield* $\widetilde{\mathfrak{F}}$ *or is a contraction of a spacetime tensorfield* $\widetilde{\mathfrak{F}}$ *against* L*,* $\underline{L}$*, or* N*. If* $Q > 2$*,* $1 \leq m < \infty$*, and* $\delta' > 0$ *is sufficiently small, then the following estimates hold, where* $\vec{\widetilde{\mathfrak{F}}}$ *denotes the array of (scalar) Cartesian component functions of* $\widetilde{\mathfrak{F}}$*:*

$$\|\xi\|_{L_{\omega}^{\infty}(S_{t,u})} \lesssim \left\|\nu^{\delta'} P_{\nu}\vec{\widetilde{\mathfrak{F}}}\right\|_{\ell_{\nu}^{m} L_{\omega}^{\infty}(S_{t,u})} + \left\|\vec{\widetilde{\mathfrak{F}}}\right\|_{L_{\omega}^{\infty}(S_{t,u})} + \|\tilde{r}^{1-\frac{2}{Q}}\mathfrak{G}\|_{L_{g\!\!/}^{Q}(S_{t,u})}. \tag{367}$$

Proof (Discussion of proof) Aside from (365), these estimates are a restatement of [54, Lemma 5.8], [54, Proposition 5.9], and [54, Proposition 5.10]. Thanks to the bootstrap assumptions and the estimates of Proposition 10.4, the estimates can be proved using the same arguments given in [52, Lemma 2.18], [20, Proposition 6.20], and [52, Proposition 3.5].

The elliptic Schauder-type estimate (365) for Hodge systems can be proved using a perturbative argument, that is, using the (standard) fact that it holds on $\mathbb{S}^2$ equipped with the standard round metric $e\!\!/$, and then obtaining the desired estimate perturbatively, with the help of the bootstrap assumptions (308a)–(308b) (which imply that $\tilde{r}^{-2}g\!\!/$ is close to $e\!\!/$) and the Morrey-type estimate (318). Here we will give a detailed proof of (365) for one-forms ξ that solve the system (366a)–(366b). The estimate (365) for $S_{t,u}$-tangent type $\binom{0}{2}$ symmetric trace-free tensorfields ξ that solve (363) can be proved using similar arguments, and we omit those details.

To proceed, we let $\nabla\!\!\!/$, $g\!\!/$, Γ, ${}^{(0)}\nabla\!\!\!/$, $e\!\!/$, and ${}^{(0)}\Gamma$ be as in our proof of (318). Let $\mathrm{d}\!\!/\mathrm{iv}\,\xi$ denote the divergence of ξ with respect to $g\!\!/$, and let ${}^{(0)}\mathrm{d}\!\!/\mathrm{iv}\,\xi$ denote the divergence of ξ with respect to $e\!\!/$. Let $\epsilon\!\!/$ denote the type $\binom{0}{2}$ volume form of $g\!\!/$, let ${}^{(0)}\epsilon\!\!/$ denote the type $\binom{0}{2}$ volume form of $e\!\!/$, and let $\epsilon\!\!/^{\#\#}$ denote the type $\binom{2}{0}$ volume form of $g\!\!/$, i.e., the dual of $\epsilon\!\!/$ with respect to $g\!\!/$. Then by (366a)–(366b), ξ satisfies the following equations, schematically depicted relative to the geometric angular coordinates, where Id denotes the type $\binom{1}{1}$ identity tensorfield, $[e\!\!/ \cdot (\tilde{r}^2 g\!\!/^{-1})]^A_{\ B} := e\!\!/_{BC}(\tilde{r}^2 g\!\!/^{-1})^{AC}$, and

$[{}^{(0)}\not{g} \cdot (\tilde{r}^2 \not{g}^{\#\#})]^A_{\ B} := {}^{(0)}\not{g}_{BC}(\tilde{r}^2 \not{g}^{\#\#})^{AC}$ (and note that if $\not{g} = \tilde{r}^2 \not{e}$, then $\not{e} \cdot (\tilde{r}^2 \not{g}^{-1}) = \mathrm{Id}$ and ${}^{(0)}\not{g} \cdot (\tilde{r}^2 \not{g}^{\#\#}) = -\mathrm{Id}$):

$$
\begin{aligned}
{}^{(0)}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi &= {}^{(0)}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\left\{[\mathrm{Id} - \not{e} \cdot (\tilde{r}^2 \not{g}^{-1})] \cdot \xi\right\} + {}^{(0)}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\left\{\not{e} \cdot (\tilde{r}^2 \not{g}^{-1}) \cdot \mathfrak{F}_{(1)}\right\} \\
&\quad + \tilde{r}^2 \mathfrak{G}_{(1)} + ({}^{(0)}\Gamma - \Gamma) \cdot (\tilde{r}^2 \not{g}^{-1}) \cdot \mathfrak{F}_{(1)} + ({}^{(0)}\Gamma - \Gamma) \cdot (\tilde{r}^2 \not{g}^{-1}) \cdot \xi, \quad (368)
\end{aligned}
$$

$$
\begin{aligned}
{}^{(0)}\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\xi &= {}^{(0)}\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\left\{[\mathrm{Id} + {}^{(0)}\not{g} \cdot (\tilde{r}^2 \not{g}^{\#\#})] \cdot \xi\right\} - {}^{(0)}\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\left\{{}^{(0)}\not{g} \cdot (\tilde{r}^2 \not{g}^{\#\#}) \cdot \mathfrak{F}_{(2)}\right\} \\
&\quad + \tilde{r}^2 \mathfrak{G}_{(2)} + ({}^{(0)}\Gamma - \Gamma) \cdot (\tilde{r}^2 \not{g}^{\#\#}) \cdot \mathfrak{F}_{(2)} + ({}^{(0)}\Gamma - \Gamma) \cdot (\tilde{r}^2 \not{g}^{\#\#}) \cdot \xi. \quad (369)
\end{aligned}
$$

We view (368)–(369) as a div-curl system on the standard round sphere. To control the solutions, we will use the following simple product-type estimate, which can easily be seen to be valid for $S_{t,u}$-tangent tensorfields $\xi_{(1)}$ and $\xi_{(2)}$, where "$\cdot$" schematically denotes tensor products and natural contractions:

$$
\|\xi_{(1)} \cdot \xi_{(2)}\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim \|\xi_{(1)}\|_{L^\infty_\omega(S_{t,u})}\|\xi_{(2)}\|_{\dot{C}^{0,\delta_0}_\omega(S_{t,u})} + \|\xi_{(2)}\|_{L^\infty_\omega(S_{t,u})}\|\xi_{(1)}\|_{\dot{C}^{0,\delta_0}_\omega(S_{t,u})}. \quad (370)
$$

From (368)–(369) and the fact that the analog of (365) holds on the standard round sphere, we have, in view of Definition 9.7 and (278), (370), the Morrey estimate (318) for the standard round sphere, and (308a), the following estimate:

$$
\begin{aligned}
\|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} &\lesssim \sum_{A,B,C=1,2} \left\| \frac{\partial}{\partial\omega^C}\left[\mathrm{Id}^A_{\ B} - \not{e}_{BD} \cdot (\tilde{r}^2 \not{g}^{-1})^{AD}\right]\right\|_{L^Q_\omega(S_{t,u})} \|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \sum_{A,B=1,2} \left\|\mathrm{Id}^A_{\ B} - \not{e}_{BC} \cdot (\tilde{r}^2 \not{g}^{-1})^{AC}\right\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \sum_{A,B,C=1,2} \left\| \frac{\partial}{\partial\omega^C}\left[\mathrm{Id}^A_{\ B} + {}^{(0)}\not{g}_{BD}(\tilde{r}^2 \not{g}^{\#\#})^{AD}\right]\right\|_{L^Q_\omega(S_{t,u})} \|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \sum_{A,B=1,2} \left\|\mathrm{Id}^A_{\ B} + {}^{(0)}\not{g}_{BC}(\tilde{r}^2 \not{g}^{\#\#})^{AC}\right\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \sum_{A,B,C=1,2} \left\|{}^{(0)}\Gamma_{A\ B}^{\ C} - \Gamma_{A\ B}^{\ C}\right\|_{L^Q_\omega(S_{t,u})} \||\tilde{r}^2 \not{g}^{-1}|_{\not{e}}\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \sum_{A,B,C=1,2} \left\|{}^{(0)}\Gamma_{A\ B}^{\ C} - \Gamma_{A\ B}^{\ C}\right\|_{L^Q_\omega(S_{t,u})} \||\tilde{r}^2 \not{g}^{\#\#}|_{\not{e}}\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \||\tilde{r}^2 \not{g}^{-1}|_{\not{e}}\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\mathfrak{F}_{(1)}\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} + \||\tilde{r}^2 \not{g}^{\#\#}|_{\not{e}}\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\mathfrak{F}_{(2)}\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \sum_{A,B,C=1,2} \left\|{}^{(0)}\Gamma_{A\ B}^{\ C} - \Gamma_{A\ B}^{\ C}\right\|_{L^Q_\omega(S_{t,u})} \||\tilde{r}^2 \not{g}^{-1}|_{\not{e}}\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\mathfrak{F}_{(1)}\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \\
&+ \sum_{A,B,C=1,2} \left\|{}^{(0)}\Gamma_{A\ B}^{\ C} - \Gamma_{A\ B}^{\ C}\right\|_{L^Q_\omega(S_{t,u})} \||\tilde{r}^2 \not{g}^{\#\#}|_{\not{e}}\|_{L^\infty_\omega(S_{t,u})} \|\tilde{r}\mathfrak{F}_{(2)}\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})}
\end{aligned}
$$

$$+ \sum_{i=1,2} \|\tilde{r}^2 \mathfrak{G}_{(i)}\|_{L^Q_\omega(S_{t,u})}. \tag{371}$$

Using (371) and (308a)–(308b), we deduce that

$$\|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} \lesssim \lambda^{-\epsilon_0}\|\tilde{r}\xi\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} + \sum_{i=1,2}\|\tilde{r}\mathfrak{F}_{(i)}\|_{C^{0,1-\frac{2}{Q}}_\omega(S_{t,u})} + \sum_{i=1,2}\|\tilde{r}^2\mathfrak{G}_{(i)}\|_{L^Q_\omega(S_{t,u})}. \tag{372}$$

From (372), we see that if λ is sufficiently large, then we can absorb the first term on RHS (372) back into the left, at the expense of doubling the (implicit) constants on the RHS. We have therefore proved (365) for one-forms ξ that solve the system (366a)–(366b). □

10.9 Proof of Proposition 10.1

Armed with the previous results of Sect. 10, we are now ready to prove Proposition 10.1. Let us make some preliminary remarks. We mainly focus on estimating the terms that are new compared to [54], typically referring the reader to the relevant spots in [54] for terms that have already been handled. When we refer to [54] for proof details, we implicitly mean that those details can involve the results of Proposition 9.8, Lemma 9.9, the inequalities proved in Sect. 10.3, and Proposition 10.4, which subsume results derived in [54]. The arguments given in [54] often also involve the bootstrap assumptions of Sect. 10.2, which subsume the bootstrap assumptions made in [54]. We sometimes silently use the results of Proposition 9.8 and Lemma 9.9, which concern estimates for the initial data of various quantities. We also stress that the order in which we derive the estimates is important, though we do not always make this explicit. Moreover, throughout the proof, we silently use the simple bound $\tilde{r}(\tau,u)/\tilde{r}(t,u) \lesssim 1$ for $\tau \leq t$. Finally, we highlight that the factors of $\mathrm{f}_{(\vec{L})}$ appearing on the RHSs of the equations of Proposition 9.7 are, by virtue of the bootstrap assumptions, bounded in magnitude by $\lesssim 1$. Therefore, these factors of $\mathrm{f}_{(\vec{L})}$ are not important for the overwhelming majority of our estimates, and we typically do not even mention them in our discussion below.

Remark 10.4 In the PDEs that we estimate below, all of the terms that are new compared to [54] are easy to identify: they all are multiplied by λ^{-1}.

10.9.1 Proof of (297a)–(297b)

Based on the transport equations (233) and (235a) and Lemma 10.3, the proof of [54]*Lemma 5.4 goes through verbatim.

10.9.2 Proof of (290) and (291)

Throughout, we will use the simple product-type estimate (370). We will also use the simple estimate $\|\mathrm{f}\circ\vec{\varphi}\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim 1 + \|\vec{\varphi}\|_{C^{0,\delta_0}_\omega(S_{t,u})}$, which is valid for scalar functions f of array-valued functions $\vec{\varphi}$ on $S_{t,u}$ whenever f is smooth on an open set containing the image set $\vec{\varphi}(S_{t,u})$.

We first prove (291). To proceed, we note that from the bootstrap assumptions, it easily follows that $\|\mathrm{f}_{(\vec{L})}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim 1$, $\|\vec{\Psi}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim 1$, and $\|\vec{L}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim 1$. From these bounds, the estimates mentioned in the previous paragraph, and (357), we see that $\|\mathrm{f}_{(\vec{L})}\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim 1+\|\vec{\Psi}\|_{C^{0,\delta_0}_\omega(S_{t,u})}+\|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim 1+\|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{t,u})}$. Thus, to prove (291), it suffices to show that for $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$ and $t \in [[u]_+, T_{*;(\lambda)}]$, we have $\|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim 1$. To this end, we first note that Lemma 10.6 and the bootstrap assumptions imply that for $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$, we have the following estimate: $\|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{[u]_+,u})} \lesssim 1$ (in fact, (349c)–(350) imply the stronger bound $\|\vec{L}\|_{C^{0,1}_\omega(S_{u,u})} \lesssim 1$ for $u \in [0, T_{*;(\lambda)}]$, whose full strength we do not need here). From this "initial data bound," the first transport equation in (223), the estimates mentioned in the previous paragraph, the estimate $\|\mathrm{f}_{(\vec{L})}\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim 1 + \|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{t,u})}$, the estimate $\int_{[u]_+}^t \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}\, d\tau \lesssim \lambda^{-7\epsilon_0}$ (which follows from (151) and (356)), and inequality (359), we deduce that the following bound holds for $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$ and $t \in [[u]_+, T_{*;(\lambda)}]$, where we recall that $[u]_+ = \max\{0, u\}$ is the minimum value of t along $\widetilde{\mathcal{C}}_u$:

$$
\begin{aligned}
\|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{t,u})} &\lesssim \|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{[u]_+,u})} + \int_{[u]_+}^t \|\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}\, d\tau \\
&\lesssim 1 + \|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{[u]_+,u})} + \int_{[u]_+}^t \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}\, d\tau \\
&\quad + \int_{[u]_+}^t \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})} \|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}\, d\tau \\
&\lesssim 1 + \int_{[u]_+}^t \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})} \|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}\, d\tau.
\end{aligned}
\tag{373}
$$

From (373), the estimate $\int_{[u]_+}^t \|\boldsymbol{\partial}\vec{\Psi}\|_{C^{0,\delta_0}_\omega(S_{\tau,u})}\, d\tau \lesssim \lambda^{-7\epsilon_0}$ noted above, and Grönwall's inequality, we deduce that $\|\vec{L}\|_{C^{0,\delta_0}_\omega(S_{t,u})} \lesssim 1$, thereby completing the proof of (291).

We will now prove (290). To proceed, we again use the first transport equation in (223), the fundamental theorem of calculus, and the bootstrap assumptions and argue as above to deduce $|L^i(t,u,\omega) - L^i([u]_+,u,\omega)| \lesssim \int_{[u]_+}^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\widetilde{\Sigma}_\tau)}\, d\tau \lesssim \lambda^{-7\epsilon_0}$. From this estimate and the data bounds (345c) and (349b), we conclude the desired estimate (290).

10.9.3 Proof of (287a)–(287b) for $\hat{\chi}$, $\not{\mathbf{D}}_L\hat{\chi}$, ζ, and $\not{\mathbf{D}}_L\zeta$

We first prove (287a) for $\|\hat{\chi}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$. From the transport equation (231), (326a), and (297a), we deduce

$$
\begin{aligned}
|\tilde{r}^2\hat{\chi}|_{\not{g}}(t,u,\omega) &\lesssim \lim_{\tau\downarrow[u]_+} |\tilde{r}^2\hat{\chi}|_{\not{g}}(\tau,u,\omega) + \lambda^{-1}\int_{[u]_+}^t |\tilde{r}^2(\vec{\mathcal{C}},\mathcal{D})|(\tau,u,\omega)\,d\tau \\
&+ \int_{[u]_+}^t |\tilde{r}^2(\not{\nabla},\not{\mathbf{D}}_L)\xi|_{\not{g}}(\tau,u,\omega)\,d\tau + \int_{[u]_+}^t \left|\tilde{r}^2(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{\not{g}}}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{\not{g}}(\tau,u,\omega)\,d\tau,
\end{aligned} \tag{374}
$$

where the correction mentioned in Footnote 48 leads to $m=1$ in (326a), thus correcting the value $m=\frac{1}{2}$ appearing [54, Equation (5.69)]. We now divide (374) by $\tilde{r}^2(t,u)$ and take the norm $\|\cdot\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$. The arguments given just below [54, Equation (5.68)] yield that the norms of all terms on RHS (374) are $\lesssim \lambda^{-1/2}$ (the correction of the value of m mentioned above does not substantially affect the arguments given there), except the term multiplied by λ^{-1} was not present in [54]. To handle the remaining term, we use (335b). We clarify that to handle the case in which $u \leq 0$, this argument relies on the initial data bound $\|w^{1/2}\hat{\chi}\|_{L_w^\infty L_\omega^p(\Sigma_0^{w_*;(\lambda)})} \lesssim \lambda^{-1/2}$, which follows from **i)** using the first equation in (200) to express $\hat{\chi}$ in terms of $\hat{\theta}$ and $\hat{k}$; **ii)** bounding $\hat{\theta}$ in the norm $\|w^{1/2}\cdot\|_{L_w^\infty L_\omega^p(\Sigma_0^{w_*;(\lambda)})}$ by using the estimate (282b); and **iii)** bounding $\hat{k}$ in the norm $\|w^{1/2}\cdot\|_{L_w^\infty L_\omega^p(\Sigma_0^{w_*;(\lambda)})}$ by using the schematic identity $\hat{k}_{AB} = \mathrm{f}_{(\vec{L})}\cdot\boldsymbol{\partial}\vec{\Psi}$, the estimate (291), and the estimate (330a) for $\tilde{r}^{1/2}\boldsymbol{\partial}\vec{\Psi}$; in total, this allows one to deduce (recalling that $w=-u|_{\Sigma_0}\geq 0$ and that $\tilde{r}(\tau,u)=\tau-u$) the estimate $\|\frac{u^2}{\tilde{r}^2}\hat{\chi}(0,u,\omega)\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \||u|^{1/2}\hat{\chi}(0,u,\omega)\|_{L_u^\infty L_\omega^p} = \|w^{1/2}\hat{\chi}\|_{L_w^\infty L_\omega^p(\Sigma_0^{w_*;(\lambda)})} \lesssim \lambda^{-1/2}$, which is needed to control the term generated by the first term on RHS (374) when $u\leq 0$.

To prove the estimate for (287b) for $\|\tilde{r}^{1/2}\hat{\chi}\|_{L_t^\infty L_\omega^p(\widetilde{\mathcal{C}}_u)}$, we note that all terms on RHS (374) can, after being divided by $\tilde{r}^{3/2}$, be handled using similar arguments (see just below [54, Equation (5.73)], where we again note that the correction of the powers of $\tilde{r}$ mentioned above does not substantially affect the arguments), but the term multiplied by λ^{-1} was not present in [54]. To handle this remaining term, we use (335e).

We now prove (287a) for $\|\tilde{r}\not{\mathbf{D}}_L\hat{\chi}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$. We use the transport equation (231) to solve for $\not{\mathbf{D}}_L\hat{\chi}$, multiply the resulting identity by $\tilde{r}$, and then take the norm $\|\cdot\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$. Thanks to the already proven bound (287a) for $\|\hat{\chi}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$, the same arguments given in the paragraph below [54, Equation (5.73)] imply that all terms satisfy the desired estimate (where the correction mentioned in Footnote 48 is not important for this argument), except the following term was not present there: $\lambda^{-1}\tilde{r}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})$. To handle this remaining term, we use (340a).

The estimates (287a) and (287b) for ζ and $\not{\mathbf{D}}_L\zeta$ follow from a similar argument since, by (232), ζ satisfies a transport equation that is schematically similar to the one that $\hat{\chi}$ satisfies, except it features the additional source term $\zeta\cdot\hat{\chi}$, which can be handled with the bootstrap assumptions (309); we omit the details. We clarify that, in view of the second term on LHS (232), the correct power of $\tilde{r}$ in the analog of inequality (374) for ζ is $\tilde{r}$. Thus, to handle the λ^{-1}-multiplied terms, we use the estimates (335a) and (335d) in place of the estimates (335b) and (335e) we used to handle $\hat{\chi}$.

10.9.4 Proof of (294) for $\hat{\chi}$ and ζ

These estimates follow from (316) with $(\hat{\chi}, \zeta)$ in the role of ξ, the already proven estimates (287a) for $(\hat{\chi}, \zeta)$ and $(\not{\mathbf{D}}_L \hat{\chi}, \not{\mathbf{D}}_L \zeta)$, and the bootstrap assumptions (312) for $(\hat{\chi}, \zeta)$.

10.9.5 Proof of (287a), (287b), (294), (288a), (288b), and (288e) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}$, $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$, and $\not{\mathbf{D}}_L \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$

To prove (288a), we note that the definition (207) of $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$ implies that it suffices to prove the pointwise bound $|\tilde{r}\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}| \lesssim \lambda^{-4\epsilon_0}$. To this end, we first use the transport equation (228a) and (329) with $\mathfrak{G} := 0$ to deduce

$$
\begin{aligned}
&|\tilde{r}^2 \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}|(t,u,\omega) \\
&\lesssim \lim_{\tau \downarrow [u]_+} |\tilde{r}^2 \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}|(\tau,u,\omega) + \lambda^{-1} \int_{[u]_+}^t |\tilde{r}^2 (\vec{\mathcal{C}}, \mathcal{D})|(\tau,u,\omega)\, d\tau \\
&\quad + \int_{[u]_+}^t \left| \tilde{r}^2 (\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi} \right|_{\not{g}} (\tau,u,\omega)\, d\tau + \int_{[u]_+}^t |\tilde{r}^2 \hat{\chi} \cdot \hat{\chi}|_{\not{g}} (\tau,u,\omega)\, d\tau \\
&\quad + \int_{[u]_+}^t |\tilde{r}^2 \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)} \cdot \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}|(\tau,u,\omega)\, d\tau. \qquad (375)
\end{aligned}
$$

We now divide (375) by $\tilde{r}(t,u)$. To handle the term on RHS (375) that is multiplied by λ^{-1}, we use (335f). The remaining terms were suitably bounded in the arguments given just below [54, Equation (5.78)]. We have thus proved (288a). The estimate (288b) follows from nearly identical arguments, where one uses (335g) to handle the λ^{-1}-multiplied term; we omit the details.

The estimates (287b) and (294) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$ then follow as straightforward consequences of (288b).

We now prove the estimate (288e) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$. First, using the transport equation (228a), we deduce that

$$
\begin{aligned}
L(\tilde{r}^2 \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}) = \mathfrak{F} &:= \lambda^{-1}\tilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot (\vec{\mathcal{C}}, \mathcal{D}) + \tilde{r}^2 \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}, \tilde{r}^{-1}) \cdot \boldsymbol{\partial}\vec{\Psi} \\
&\quad + \tilde{r}^2 \mathrm{f}_{(\vec{L})} \hat{\chi} \cdot \hat{\chi} + \tilde{r}^2 \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)} \cdot \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}. \qquad (376)
\end{aligned}
$$

From (376) and the vanishing initial condition (along the cone-tip axis) for $\tilde{r}^2 \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$ guaranteed by (283a), we find, with $[u]_- := |\min\{u, 0\}|$ and $[u]_+ :=$

$\max\{u, 0\}$, that

$$\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}(t,u,\omega) = \frac{[u]_-^2}{(t+[u]_-)^2}\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}([u]_+,u,\omega) + \frac{1}{\tilde{r}^2(t,u)}\int_{[u]_+}^{t}\mathfrak{F}(\tau,u,\omega)\,d\tau. \tag{377}$$

Using (377), applying the product-type estimate (370) to the term $\mathfrak{F}$ in (376), using the already proven estimate (291) for $\mathrm{f}_{(\vec{L})}$, and using (360), we find, in view of the definition (323) of the Hardy–Littlewood maximal function, that for $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$ and $t \in [[u]_+, T_{*;(\lambda)}]$, we have

$$\begin{aligned}\|\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim{}& \frac{[u]_-^{3/2}}{(t+[u]_-)^2}\left\||u|^{1/2}\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\right\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_0)} \\ &+\lambda^{-1}\|(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^1C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)} + \|(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi})\|^2_{L_t^2C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)} \\ &+\mathcal{M}\left(\|\boldsymbol{\partial}\vec{\Psi}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t)}\right).\end{aligned} \tag{378}$$

From (378), the last estimate in (285), the parameter relation (276), (151), (309), and (356), we find that $\|\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{C_\omega^{0,\delta_0}(S_{t,u})} \lesssim \frac{[u]_-^{3/2}}{(t+[u]_-)^2}\lambda^{-1/2} + \lambda^{-1+4\epsilon_0} + \mathcal{M}\left(\|\boldsymbol{\partial}\vec{\Psi}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t)}\right)$. Taking the norm $\|\cdot\|_{L_t^2([[u]_+,T_{*;(\lambda)}])}$ of this inequality and using (151), (324) with $Q := 2$, and (356), we conclude the desired bound (288e) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$.

The estimate (287a) for $\|\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^2L_\omega^p(\widetilde{\mathcal{C}}_u)}$ then follows as a straightforward consequence of the estimate (288e) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$.

We now prove the estimate (287a) for $\|\tilde{r}\mathbf{D}\!\!\!\!/_L\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^2L_\omega^p(\widetilde{\mathcal{C}}_u)}$ by using the transport equation (228a) to algebraically solve for $\tilde{r}\mathbf{D}\!\!\!\!/_L\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$. Thanks to the bound (177) for $\tilde{r}$, the bootstrap assumptions, and the already proven bounds (287a) and (287b) for $\hat{\chi}$ and $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$, the same arguments given just below [54, Equation (5.80)] imply that all terms on RHS (228a) and the term $\frac{2}{\tilde{r}}\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$ on LHS (228a) satisfy (upon being multiplied by $\tilde{r}$) the desired estimate, except the following term was not present in [54]: $\lambda^{-1}\tilde{r}\mathrm{f}_{(\vec{L})}\cdot(\vec{\mathcal{C}},\mathcal{D})$. To bound this remaining term, we use (340a).

10.9.6 Proof of (287c)

(287c) follows from the already proven estimate (287b) and the bound (177) for $\tilde{r}$.

10.9.7 Proof of (288c) and (288d)

To prove (288d), we first note the following bound for some factors in the next-to-last product on RHS (228b), which follows from (151), (306a), and (309): $\|\mathrm{f}_{(\vec{L})}\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi})\|_{L_\omega^\infty L_t^1(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-2\epsilon_0} \leq 1$. From this bound, the transport equation

(228b), and (329) with $\mathfrak{G} := \mathrm{f}_{(\vec{L})} \cdot (\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi})$, we deduce

$$
\begin{aligned}
&|\tilde{r}^3 \nabla\!\!\!/ \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}|_{g\!\!\!/}(t,u,\omega) \\
&\lesssim \lim_{\tau\downarrow[u]_+} |\tilde{r}^3 \nabla\!\!\!/ \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}|_{g\!\!\!/}(\tau,u,\omega) + \lambda^{-1}\int_{[u]_+}^t |\tilde{r}^3 \nabla\!\!\!/ (\vec{\mathcal{C}},\mathcal{D})|_{g\!\!\!/}(\tau,u,\omega)\,d\tau \\
&\quad + \lambda^{-1}\int_{[u]_+}^t \left|\tilde{r}^3(\vec{S}\cdot\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Psi},\boldsymbol{\partial}\vec{\Omega},\boldsymbol{\partial}\vec{S})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\right|_{g\!\!\!/}(\tau,u,\omega)\,d\tau \\
&\quad + \int_{[u]_+}^t \left|\tilde{r}^3 \nabla\!\!\!/ \boldsymbol{\partial}\vec{\Psi}\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\tilde{r}^{-1})\right|_{g\!\!\!/}(\tau,u,\omega)\,d\tau \\
&\quad + \int_{[u]_+}^t |\tilde{r}^3 \nabla\!\!\!/ \hat{\chi}\cdot\hat{\chi}|_{g\!\!\!/}(\tau,u,\omega)\,d\tau \\
&\quad + \int_{[u]_+}^t \left|\tilde{r}^3(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi},\tilde{r}^{-1})\cdot(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\tilde{r}^{-1})\cdot\boldsymbol{\partial}\vec{\Psi}\right|_{g\!\!\!/}(\tau,u,\omega)\,d\tau.
\end{aligned}
\tag{379}
$$

In the arguments given in the paragraph below [54, Equation (5.81)], with the help of the bootstrap assumptions, all terms on RHS (379) were shown, after dividing by $\tilde{r}^2(t,u)$, to be bounded in the norm $\|\cdot\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$ by $\lesssim \lambda^{-1/2} + \lambda^{-2\epsilon_0}\|\tilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$, except that the two terms multiplied by λ^{-1} were not present there. To handle these remaining terms, we use (336b) and (337b), which in total yields

$$
\|\tilde{r}\nabla\!\!\!/ \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2} + \lambda^{-2\epsilon_0}\|\tilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}. \tag{380}
$$

Next, we note that the divergence equation (230), the Hodge estimate (362) with $Q := p$, and the same arguments given in the paragraph below [54, Equation (5.82)] yield

$$
\|\tilde{r}\nabla\!\!\!/\hat{\chi}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \|\tilde{r}\nabla\!\!\!/ \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} + \lambda^{-1/2}. \tag{381}
$$

From (380) and (381), we conclude (when λ is sufficiently large) the desired bounds in (288d).

As is noted just below [54, Equation (5.84)], the estimate (288c) can be proved using a similar argument, based on dividing (379) by $\tilde{r}^{3/2}(t,u)$, where we use (336a) and (337a) to handle the two λ^{-1}-multiplied terms on RHS (379).

10.9.8 Proof of (295) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$ and $\mathrm{tr}_{g\!\!\!/}\chi - \frac{2}{\tilde{r}}$

We first prove (295) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$. A slight modification of the proof of (378) yields the following bound:

$$
\begin{aligned}
\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t^{(Int)})} &\lesssim \lambda^{-1}\|(\vec{\mathcal{C}},\mathcal{D})\|_{L_t^1 L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})} + \mathcal{M}\left(\|\boldsymbol{\partial}\vec{\Psi}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t^{(Int)})}\right) \\
&\quad + \|(\boldsymbol{\partial}\vec{\Psi},\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)},\hat{\chi})\|^2_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})}.
\end{aligned}
\tag{382}
$$

From (382), (151), (312), and (356), we deduce

$$\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t^{(Int)})} \lesssim \lambda^{-1} + \mathcal{M}\left(\|\boldsymbol{\partial}\vec{\Psi}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t^{(Int)})}\right). \tag{383}$$

Taking the norm $\|\cdot\|_{L_t^2([0,T_{*;(\lambda)}])}$ of (383) and using (151), (324) with $Q := 2$, and (356), we conclude the desired bound (295) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$.

The estimate (295) for $\mathrm{tr}_{\not{g}}\chi - \frac{2}{\tilde{r}}$ then follows from the identity $\mathrm{tr}_{\not{g}}\chi - \frac{2}{\tilde{r}} = \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)} - \boldsymbol{\Gamma}_L$, the schematic relation $\boldsymbol{\Gamma}_L = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, the already proven estimate (295) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$, the product-type estimate (370), (291), and (356).

10.9.9 Proof of (295) for $\hat{\chi}$

We first use equation (230), the estimate (365) with $Q := p$, the parameter relation (276), the product-type estimate (370), (291), and Hölder's inequality to deduce that

$$\begin{aligned}\|\hat{\chi}\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})} &\lesssim \|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})} \\ &\quad + \|\tilde{r}^{1/2}\|_{L^\infty(\widetilde{\mathcal{M}})}\|\boldsymbol{\partial}\vec{\Psi}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}\left\|\tilde{r}^{1/2}(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi})\right\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})}.\end{aligned} \tag{384}$$

Using the bound (177) for $\tilde{r}$, the estimate (287b) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$ and $\hat{\chi}$, the estimate (330a) for $\boldsymbol{\partial}\vec{\Psi}$, the already proven estimate (295) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$, (306a) for $\boldsymbol{\partial}\vec{\Psi}$, and (356) for $\boldsymbol{\partial}\vec{\Psi}$, we conclude that RHS (384) $\lesssim \lambda^{-1/2-3\epsilon_0}$ as desired.

10.9.10 Proof of (292) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$ and $\mathrm{tr}_{\not{g}}\chi - \frac{2}{\tilde{r}}$

We first bound $\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})}$. We start by noting the following estimate, which is a simple consequence of the estimate proved just below (378), and which holds for $t \in [0, T_{*;(\lambda)}]$:

$$\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t)} \lesssim t^{-1/2}\lambda^{-1/2} + \lambda^{-1+4\epsilon_0} + \mathcal{M}\left(\|\boldsymbol{\partial}\vec{\Psi}\|_{L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\Sigma}_t)}\right). \tag{385}$$

Taking the norm $\|\cdot\|_{L_t^{\frac{q}{2}}([0,T_{*;(\lambda)}])}$ of (385) and using (151), (324) with $Q := \frac{q}{2}$, and (356), we conclude that if $q > 2$ is sufficiently close to 2, then the desired estimate (292) for $\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}$ holds.

To prove (292) for $\mathrm{tr}_{\not{g}}\chi - \frac{2}{\tilde{r}}$, we use (207), the schematic relation $\boldsymbol{\Gamma}_L = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, the product-type estimate (370), (151), (356), (291), the already proven estimate (292) for $\|\mathrm{tr}_{\widetilde{g}}\widetilde{\chi}^{(Small)}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})}$, to conclude that if $q > 2$ is sufficiently close to 2, then

$$\|\mathrm{tr}_{\not g}\chi - \frac{2}{\tilde r}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} \lesssim \|\mathrm{tr}_{\tilde{\not g}}\widetilde{\chi}^{(Small)}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})}$$
$$\lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)} + (\lambda^{1-8\epsilon_0})^{(\frac{2}{q}-\frac{1}{2})} \cdot \lambda^{-1/2-3\epsilon_0}$$
$$\lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}$$

as desired.

10.9.11 Proof of (292) for $\hat\chi$

A slight modification of the proof of (384) yields that

$$\|\hat\chi\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} \lesssim \|\mathrm{tr}_{\tilde{\not g}}\widetilde{\chi}^{(Small)}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})}$$
$$+ \|\tilde r^{1/2}\|_{L^\infty(\widetilde{\mathcal{M}})} \|\boldsymbol{\partial}\vec{\Psi}\|_{L_t^{\frac{q}{2}} L_x^\infty(\widetilde{\mathcal{M}})} \left\| \tilde r^{1/2}(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{\not g}}\widetilde{\chi}^{(Small)}, \hat\chi)\right\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})}. \tag{386}$$

From (386), (151), (177), (287b), the already proven bound (292) for $\mathrm{tr}_{\tilde{\not g}}\widetilde{\chi}^{(Small)}$, (332a), and (356), we conclude that if $q > 2$ is sufficiently close to 2, then $\|\hat\chi\|_{L_t^{\frac{q}{2}} L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}})} \lesssim \lambda^{\frac{2}{q}-1-4\epsilon_0(\frac{4}{q}-1)}$ as desired.

10.9.12 Proof of (288e) for $\hat\chi$

Using (318) with $Q := p$ and taking into account (276), we find that $\|\hat\chi\|_{L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)} \lesssim \|\tilde r \not\nabla \hat\chi\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} + \|\chi\|_{L_t^2 L_\omega^2(\widetilde{\mathcal{C}}_u)}$. Using the already proven estimates (288d) for $\|\tilde r \not\nabla \hat\chi\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$ and (287a) for $\|\chi\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$, we conclude that the RHS of the previous expression is $\lesssim \lambda^{-\frac{1}{2}}$ as desired.

10.9.13 Proof of (296a)–(296b)

Based on the transport equation (234), Lemma 10.3, (284a)–(284b), (282c)–(282d), (151), the bootstrap assumptions, Proposition 10.4, and the previously proven estimates (287a), (288d), (288e), and (295), the proof given in [54, Subsubsection 5.2.2] goes through verbatim.

10.9.14 Proof of (289) and (293)

Based on the transport equation (233), the bootstrap assumptions, and the previously proven estimates (297b) and (287a), the arguments given in the discussion surrounding [54, Equation (5.90)] go through verbatim.

10.9.15 Proof of (298)

Based on the evolution equations (235a)–(235b), Lemma 10.3, the bootstrap assumptions, and the previously proven estimate (288d), the proof of [54] Lemma 5.15 (in which $\ln\left(\tilde{r}^{-2}\upsilon\right)$ was denoted by "φ") goes through verbatim.

10.9.16 Proof of (288e) for $\|\zeta\|_{L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)}$ and (299)

We first simultaneously prove (288e) for $\|\zeta\|_{L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)}$ and (299) for $\|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$ with the help of the Hodge system (239a)–(239b). We now define the following two scalar functions: $\mathfrak{F} :=$ RHS (239a), $\mathfrak{G} :=$ RHS (239b). From the Calderon–Zygmund estimate (361) with $Q := p$, we deduce that for each fixed $u \in [-\frac{4}{5}T_{*;(\lambda)}, T_{*;(\lambda)}]$, we have $\|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \|\tilde{r}(\mathfrak{F}, \mathfrak{G})\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$. In the arguments given just below [54, Equation (5.97)], based on the bootstrap assumptions, (151), (177), and the previously proven estimates (287b), (288e) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$ and $\hat{\chi}$, and (298), all terms on RHSs (239a)–(239b) were shown to be bounded in the norm $\|\tilde{r} \cdot \|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$ by $\lesssim \lambda^{-1/2} + \lambda^{-4\epsilon_0}\|\zeta\|_{L_t^2 L_\omega^\infty(\widetilde{\mathcal{C}}_u)}$, except that the terms on RHS (239a) that are multiplied by λ^{-1} were not present there. To handle these remaining terms, we use (340a). We have thus shown that $\|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2} + \lambda^{-4\epsilon_0}\|\zeta\|_{L_t^2 L_\omega^\infty(\widetilde{\mathcal{C}}_u)}$. Moreover, using (318) with $Q := p$, the parameter relation (276), and (287a) (which implies that $\|\zeta\|_{L_t^2 L_\omega^2(\widetilde{\mathcal{C}}_u)} \lesssim \|\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}$), we find that $\|\zeta\|_{L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)} \lesssim \|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} + \|\zeta\|_{L_t^2 L_\omega^2(\widetilde{\mathcal{C}}_u)} \lesssim \|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} + \lambda^{-1/2}$. Combining the above estimates, we find that $\|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2} + \lambda^{-4\epsilon_0}\|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$, from which we readily conclude (when λ is sufficiently large) the desired bound (299) for $\|\tilde{r}\nabla\!\!\!/\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$ and the desired bound (288e) for $\|\zeta\|_{L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{C}}_u)}$.

To prove (299) for $\|\tilde{r}\mu\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$, we must show that $\|\tilde{r} \times \text{RHS (236)}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}$. In the arguments given just below [54, Equation (5.101)], based on the bootstrap assumptions, (151), (177), and the previously proven estimates (287b), (288e), and (298), all terms on RHS (236) were shown to satisfy the desired bound, except the term on RHS (236) that is multiplied by λ^{-1} was not present there. To handle this remaining term, we use (340a).

10.9.17 Proof of (295) for ζ and (292) for ζ

To prove (295) for ζ, we will use the Hodge system (239a)–(239b). From these equations and the Calderon–Zygmund estimate (367) with ζ in the role of ξ, with $Q := p$ and $m := 2$, with $\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$ in the role of $\mathfrak{F}$ (where $\mathfrak{F}$ represents the second terms on RHSs (239a) and (239b)), with $\mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\vec{\Psi}$ in the role of $\vec{\tilde{\mathfrak{F}}}$ and with $\delta' > 0$ chosen to be sufficiently small, we find (where the implicit constants can depend on δ') that

$$\|\zeta\|_{L^2_t L^\infty_x(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \left\| \tilde{r}(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \tilde{r}^{-1}) \cdot (\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}, \zeta) \right\|_{L^2_t L^\infty_u L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} + \lambda^{-1}\|\tilde{r}(\vec{\mathcal{C}}, \mathcal{D})\|_{L^2_t L^\infty_u L^p_\omega(\widetilde{\mathcal{M}})} + \left\| \tilde{r}\nabla\!\!\!/ \ln\left(\tilde{r}^{-2}\upsilon\right) \cdot (\boldsymbol{\partial}\vec{\Psi}, \zeta) \right\|_{L^2_t L^\infty_u L^p_\omega(\widetilde{\mathcal{M}})} + \left\| \nu^{\delta'} P_\nu \left(\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi} \right) \right\|_{L^2_t \ell^2_\nu L^\infty_x(\widetilde{\mathcal{M}})} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L^2_t L^\infty_x(\widetilde{\mathcal{M}})}. \tag{387}$$

Assuming that $\delta' > 0$ is chosen to be sufficiently small (in particular, at least as small as the parameter δ_0 in (306b)), the arguments given on [54, page 52] show that, thanks to the bootstrap assumptions, (151), (177), and the already proven estimates (287b), (288e), (295) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$ and $\hat{\chi}$, and (298), all terms on RHS (387) are $\lesssim \lambda^{-\frac{1}{2}-3\epsilon_0} + \lambda^{-4\epsilon_0}\|\zeta\|_{L^2_t L^\infty_x(\widetilde{\mathcal{M}}^{(Int)})}$, except that the term on RHS (387) that is multiplied by λ^{-1} was not present in [54]. To handle this remaining term, we use (340a). This shows that $\|\zeta\|_{L^2_t L^\infty_x(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}-3\epsilon_0} + \lambda^{-4\epsilon_0}\|\zeta\|_{L^2_t L^\infty_x(\widetilde{\mathcal{M}}^{(Int)})}$ which, when λ is sufficiently large, yields the desired bound (295) for ζ.

Similarly, based on the Hodge system (239a)–(239b), the Calderon–Zygmund estimate (367) with $Q := p$, the bootstrap assumptions, (151), (177), the already proven estimates (287b) and (298) and the already proven estimate (292) for $\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}$ and $\hat{\chi}$, the arguments given on [54]*page 52 yield the desired estimate (292) for ζ, where we use (340b) to handle the λ^{-1}-multiplied terms on RHSs (239a)–(239b).

10.9.18 Proof of (300a)–(300c)

Based on (201a)–(201b), Lemma 10.3, the bootstrap assumptions, and the previously proven estimate (295), the proof of these estimates given in [54, Lemma 6.1] goes through verbatim, except for the estimate (300a) for $\|\tilde{r}^{\frac{1}{2}}\nabla\!\!\!/\sigma\|_{L^p_\omega L^\infty_t(\widetilde{\mathcal{C}}_u)}$. To bound this remaining term, we first use (297a) to deduce (noting that $u \geq 0$ since, by assumption, we have $\widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}^{(Int)}$)

$$\begin{aligned}\|\tilde{r}^{\frac{1}{2}}\nabla\!\!\!/\sigma\|^p_{L^p_\omega L^\infty_t(\widetilde{\mathcal{C}}_u)} &\lesssim \int_{\mathbb{S}^2} \mathrm{ess\,sup}_{t\in[u,T_{*;(\lambda)}]}|\tilde{r}^{\frac{1}{2}}\nabla\!\!\!/\sigma|^p_{\tilde{g}}(t,u,\omega)\, d\varpi_{\not{e}(\omega)} \\ &\lesssim \int_{\mathbb{S}^2} \mathrm{ess\,sup}_{t\in[u,T_{*;(\lambda)}]}\left\{\upsilon(t,u,\omega)|\tilde{r}^{\frac{1}{2}-\frac{2}{p}}\nabla\!\!\!/\sigma|^p_{\tilde{g}}(t,u,\omega)\right\} d\varpi_{\not{e}(\omega)} \\ &:= \|\tilde{r}^{\frac{1}{2}-\frac{2}{p}}\nabla\!\!\!/\sigma\|^p_{L^p_{\tilde{g}} L^\infty_t(\widetilde{\mathcal{C}}_u)}.\end{aligned} \tag{388}$$

From (388) and the already proven bound (300a) for $\|\tilde{r}^{\frac{1}{2}-\frac{2}{p}}\nabla\!\!\!/\sigma\|_{L^p_{\tilde{g}} L^\infty_t(\widetilde{\mathcal{C}}_u)}$, we conclude that RHS (388) $\lesssim \lambda^{-p/2}$ as desired.

10.9.19 Proof of (301a)–(301b)

We make the bootstrap assumption $\|\nabla\!\!\!/\sigma\|_{L^2_u L^2_t L^\infty_\omega(\widetilde{\mathcal{M}}^{(Int)})} \leq 1$; this is viable because (301a) yields an improvement of this bootstrap assumption.

We start by deriving a preliminary estimate for $\|\tilde{r} \nabla\!\!\!/ \widetilde{\zeta}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$ using Hodge system (240a)–(240b). To proceed, we define the following two scalar functions: $\mathfrak{F} :=$ RHS (240a), $\mathfrak{G} :=$ RHS (240b). From these equations and the Calderon–Zygmund estimate (361), we deduce

$$\|\tilde{r} \nabla\!\!\!/ \widetilde{\zeta}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \|\tilde{r}\mathfrak{F}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} + \|\tilde{r}\mathfrak{G}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} + \|\tilde{r}\check{\mu}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}.$$

In the last two paragraphs of the proof of [54, Proposition 6.3], based on the bootstrap assumptions, (151), (177), and the previously proven estimates (287a) and (295), the author showed that all terms on RHSs (240a)–(240b) are bounded in the norm $\|\tilde{r} \cdot \|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$ by $\lesssim \lambda^{-4\epsilon_0}$, except that the terms on RHS (240a) that are multiplied by λ^{-1} were not present there. To handle these remaining terms, we use (340c), which in total yields the desired preliminary estimate $\|\tilde{r} \nabla\!\!\!/ \widetilde{\zeta}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0} + \|\tilde{r}\check{\mu}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$.

We now derive estimates for $\check{\mu}$. Using the transport equation (237), the identity (326a), the vanishing initial conditions for $\tilde{r}^2\check{\mu}$ along the cone-tip axis guaranteed by (283a), and (297a), we see that in $\widetilde{\mathcal{M}}^{(Int)}$, we have

$$|\tilde{r}^2\check{\mu}|(t, u, \omega) \lesssim \int_u^t \tilde{r}^2 \left\{|\mathfrak{I}_{(1)} + \mathfrak{I}_{(2)}|\right\}(\tau, u, \omega)\, d\tau, \tag{389}$$

where $\mathfrak{I}_{(1)}$ and $\mathfrak{I}_{(2)}$ are defined in (238a)–(238b). We now divide (389) by $\tilde{r}(t, u)$ and take the norm $\| \cdot \|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$. In the proof of [54, Proposition 6.3], the author derived estimates for the terms on RHS (389) that imply, based on the bootstrap assumptions, (151), (177), and the previously proven estimates (287a), (287c), (288c), (288d), and (295), that $\|\tilde{r}\check{\mu}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0} + \lambda^{-8\epsilon_0}\|\tilde{r}\check{\mu}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} + \lambda^{-4\epsilon_0}\|\tilde{r} \nabla\!\!\!/ \widetilde{\zeta}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$, except that the terms on RHS (238b) that are multiplied by λ^{-1} were not present there. To handle these remaining terms, we use (338a) and (339a). Considering also the preliminary estimate for $\|\tilde{r} \nabla\!\!\!/ \widetilde{\zeta}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$ derived in the previous paragraph, we deduce $\|\tilde{r}\check{\mu}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0} + \lambda^{-8\epsilon_0}\|\tilde{r}\check{\mu}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$. Thus, when λ is sufficiently large, we conclude the desired bound (301a) for $\|\tilde{r}\check{\mu}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$. Inserting this bound into the preliminary estimate for $\|\tilde{r} \nabla\!\!\!/ \widetilde{\zeta}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$ derived in the previous paragraph, we also conclude the desired bound (301a) for $\|\tilde{r} \nabla\!\!\!/ \widetilde{\zeta}\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})}$.

A similar argument yields (301b), where we divide (389) by $\tilde{r}^{1/2}(t, u)$, and to handle the terms on RHS (238b) that are multiplied by λ^{-1}, we use (338b) and (339b); we omit the details.

It remains for us to prove the estimate (301a) for $\|\nabla\!\!\!/ \sigma\|_{L^2_u L^2_t C^{0,\delta_0}_\omega(\widetilde{\mathcal{M}}^{(Int)})}$. First, using definition (211), (318) with $Q := p$, and the parameter relation (276), we see that

$$\|\nabla\!\!\!/ \sigma\|_{L^2_u L^2_t C^{0,\delta_0}_\omega(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \|\tilde{r} \nabla\!\!\!/ (\widetilde{\zeta}, \zeta)\|_{L^2_u L^2_t L^p_\omega(\widetilde{\mathcal{M}}^{(Int)})} + \|\nabla\!\!\!/ \sigma\|_{L^2_u L^2_t L^2_\omega(\widetilde{\mathcal{M}}^{(Int)})}.$$

We have already shown that $\|\tilde{r}\nabla\widetilde{\zeta}\|_{L_u^2 L_t^2 L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}$. To bound $\|\tilde{r}\nabla\zeta\|_{L_u^2 L_t^2 L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})}$ by $\lesssim \lambda^{-4\epsilon_0}$, we square the already proven estimate (299) for $\|\tilde{r}\nabla\zeta\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$, integrate with respect to u over $u \in [0, T_{*;(\lambda)}]$, and use the bound (177) for u. Finally, to obtain the bound $\|\nabla\sigma\|_{L_u^2 L_t^2 L_\omega^2(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}$, we square the already proven estimate (300a) for $\|\nabla\sigma\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)}$, integrate with respect to u over $u \in [0, T_{*;(\lambda)}]$, and use the bound (177) for u. Combining these estimates, we conclude that $\|\nabla\sigma\|_{L_u^2 L_t^2 C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}$ as desired.

Remark 10.5 Throughout the rest of the proof of Proposition 10.1, we silently use the following estimates, valid for $1 \leq Q \leq \infty$, which are simple consequences of (297a): $|\overline{f}(t, u)| \lesssim \|f\|_{L_\omega^Q(S_{t,u})}$ and $\|\overline{f}\|_{L_\omega^Q(S_{t,u})} \lesssim \|f\|_{L_\omega^Q(S_{t,u})}$ (see (205) regarding the "overline" notation).

10.9.20 Proof of (302)

Using the Hodge system (210), (318) with $Q := p$, and (361) with $Q := p$, we find that

$$\|(\tilde{r}\nabla\not{\mu}, \not{\mu})\|_{L_\omega^p(S_{t,u})}, \ \|\not{\mu}\|_{L_\omega^\infty(S_{t,u})} \lesssim \|\tilde{r}(\check{\mu} - \overline{\check{\mu}})\|_{L_\omega^p(S_{t,u})}. \tag{390}$$

Taking the norm $\|\cdot\|_{L_t^2 L_u^2}$ of (390) over the range of (t, u)-values corresponding to $\widetilde{\mathcal{M}}^{(Int)}$ and using the already proven estimate (301a) for $\|\tilde{r}\check{\mu}\|_{L_u^2 L_t^2 L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})}$, we arrive at the desired bound (302).

10.9.21 Proof of (303)–(305b)

We first note that the decomposition (303) follows from the definitions of the quantities involved.

Throughout the rest of proof, $\mathscr{D}^{-1}(\mathfrak{F}, \mathfrak{G})$ will denote the solution ξ the following Hodge system on $S_{t,u}$: $\mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi = \mathfrak{F}$, $\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\xi = \mathfrak{G}$. In our applications, ξ will be a one-form or a symmetric trace-free type $\binom{0}{2}$ tensor (where in the latter case, one can show that the one-forms $\mathfrak{F}$ and $\mathfrak{G}$ are constrained by the relation $\mathfrak{G}_A = \epsilon_{AB}\mathfrak{F}_A$, where ϵ_{AB} is the antisymmetric symbol with $\epsilon_{12} = 1$ relative to a $\not{g}$-orthonormal frame on $S_{t,u}$).

We start by proving (305a) for the term $\|\widetilde{\zeta} - \not{\mu}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})}$ on the LHS. We will use the Hodge system (241a)–(241b). Note that we can split $\widetilde{\zeta} - \not{\mu} = \mathscr{D}^{-1}(\mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi, \mathrm{cu}\not{\mathrm{r}}\mathrm{l}\xi) + \mathscr{D}^{-1}(\cdots, \cdots)$, where $\mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi$ is the first term on RHS (241a), $\mathrm{cu}\not{\mathrm{r}}\mathrm{l}\xi$ is the first term on RHS (241b), and $(\cdots, \cdots)$ denotes the remaining terms on RHSs (241a)–(241b). Recall that the $S_{t,u}$-tangent tensorfields denoted here by ξ have Cartesian component functions of the form $\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$ (and thus ξ satisfies the hypotheses needed to apply the estimate (367) with $\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$ in the role of $\mathfrak{F}$ and $\mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\vec{\Psi}$ in the role of $\vec{\vec{\mathfrak{F}}}$). Therefore, using (367) with $\delta' > 0$ chosen to be sufficiently small (at least as small as the parameter $\delta_0 > 0$ in (306b)) and $m := 2$ to handle the term $\mathscr{D}^{-1}(\mathrm{d}\not{\mathrm{i}}\mathrm{v}\xi, \mathrm{cu}\not{\mathrm{r}}\mathrm{l}\xi)$, and (318) and (361) with $Q := p$ to handle the term $\mathscr{D}^{-1}(\cdots, \cdots)$, we deduce that

$$\|\widetilde{\zeta} - \not\mu\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \|\tilde{r}(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\not g}\widetilde{\chi}^{(Small)}, \hat{\chi}, \zeta, \tilde{r}^{-1}) \cdot (\boldsymbol{\partial}\vec{\Psi}, \hat{\chi}, \zeta)\|_{L_t^2 L_u^\infty L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})}$$
$$+ \lambda^{-1}\|\tilde{r}(\vec{\mathcal{C}}, \mathcal{D})\|_{L_t^2 L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})}$$
$$+ \left\|\nu^{\delta'} P_\nu \left(\mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\vec{\Psi}\right)\right\|_{L_t^2 \ell_\nu^2 L_x^\infty(\widetilde{\mathcal{M}})} + \|\boldsymbol{\partial}\vec{\Psi}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}})}. \tag{391}$$

At the very end of the proof of [54, Proposition 6.4] (in which the author derived bounds for the second piece of a quantity denoted by "$\mathbf{A}^\dagger$," which was split into two pieces there), the author gave arguments showing that, thanks to the bootstrap assumptions, (177), and the previously proven estimates (287c) and (295), all terms on RHS (391) are $\lesssim \lambda^{-\frac{1}{2}-3\epsilon_0}$, except that the term $\lambda^{-1}\|\tilde{r}(\vec{\mathcal{C}}, \mathcal{D})\|_{L_t^2 L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})}$ was not present in [54]. To handle this remaining term, we use (340a). We have therefore proved (305a) for $\|\widetilde{\zeta} - \not\mu\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})}$.

To prove (304), we first note that in view of (245), it suffices to show that $\tilde{r}\not\mu(t, u, \omega) = \mathcal{O}(\tilde{r})$as $t \downarrow u$. The desired bound follows from applying the Calderon–Zygmund estimate (367) with $\mathfrak{F} = 0$ to the Hodge system (210) and using the asymptotic estimate (283a) for $\check{\mu}$.

We now prove the estimate (305a) for the remaining term $\|\not\mu_{(1)}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})}$ on the LHS. Note that $\not\mu_{(1)}$ solves the Hodge-transport system (243a)–(243b), where the inhomogeneous term $\mathfrak{I}_{(1)} - \overline{\mathfrak{I}_{(1)}}$ is defined by (238a). From (297a), (326a), and the initial condition (304), we deduce the pointwise identity

$$\not\mu_{(1)}(t, u, \omega) = \upsilon^{-\frac{1}{2}}(t, u, \omega) \int_u^t \left[\upsilon^{\frac{1}{2}} \mathscr{D}^{-1}(\mathfrak{I}_{(1)} - \overline{\mathfrak{I}_{(1)}}, 0)\right](\tau, u, \omega)\, d\tau. \tag{392}$$

The term $\mathfrak{I}_{(1)} - \overline{\mathfrak{I}_{(1)}}$ on RHS (392) is the same term appearing in [54]. At the start of the last paragraph in the proof of [54, Proposition 6.4] (in which the author derived bounds for the first piece of quantity denoted by $\mathbf{A}^\dagger$, which was split into two pieces), the author derived estimates for RHS (392) showing that $\|\not\mu_{(1)}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}$, which is in fact slightly better than the bound stated in (305a).

Finally, we prove the estimate (305b) for $\not\mu_{(2)}$ using the Hodge-transport system (244a)–(244b). We first define the following two scalar functions: $\mathfrak{F} :=$ RHS (244a), $\mathfrak{G} :=$ RHS (244b). From (244a)–(244b), (297a), (326a), and the initial condition (304), we deduce the pointwise identity

$$\not\mu_{(2)}(t, u, \omega) = \upsilon^{-\frac{1}{2}}(t, u, \omega) \int_u^t \left[\upsilon^{\frac{1}{2}} \mathscr{D}^{-1}(\mathfrak{F}, \mathfrak{G})\right](\tau, u, \omega)\, d\tau. \tag{393}$$

From (318) with $Q := p$ and (361), we find that $\|\mathscr{D}^{-1}(\mathfrak{F}, \mathfrak{G})\|_{L_\omega^\infty(S_{t,u})} \lesssim \|\tilde{r}(\mathfrak{F}, \mathfrak{G})\|_{L_\omega^p(S_{t,u})}$. From this estimate, (393), (297a), and the simple bound

$\tilde{r}(\tau, u)/\tilde{r}(t, u) \lesssim 1$ for $\tau \leq t$, we deduce that

$$\|\not{\mu}_{(2)}\|_{L_u^2 L_t^\infty L_\omega^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \|\tilde{r}(\mathfrak{F}, \mathfrak{G})\|_{L_u^2 L_t^1 L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})}.$$

In [54, Equation (6.37)] and the discussion below that equation, based on the bootstrap assumptions, (151), (177), and the already proven estimates (287a), (287c), (288d), (295), (301a), and (302), the author gave arguments that imply that $\|\tilde{r}(\mathfrak{F}, \mathfrak{G})\|_{L_u^2 L_t^1 L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}$ as desired, except that the terms in $\mathfrak{I}_{(2)} - \overline{\mathfrak{I}_{(2)}}$ (i.e., the first term RHS (244a)) generated by the two terms on RHS (238b) with the coefficient λ^{-1} were not present in [54]. To handle these new terms, we use (340d)–(340e). We have therefore proved (305b), which completes the proof of Proposition 10.1.

11 Summary of the reductions of the proof of the Strichartz estimate of Theorem 7.2

In this section, we outline how the Strichartz estimate of Theorem 7.2 follows as a consequence of the estimates for the eikonal function that we derived in Sect. 10. We only sketch the arguments since, given the estimates that we derived in Sect. 10, the proof of Theorem 7.2 follows from the same arguments given in [54]. For the reader's convenience, we note that the flow of the logic can be summarized as follows, although in Sects. 11.1–11.5, we will discuss the steps in the reverse order:

1. Estimates for the eikonal function, connection coefficients, and conformal factor σ obtained in Sect. 10
2. $\Longrightarrow$ Estimates for a conformal energy for solutions φ to the linear wave equation $\square_{\mathbf{g}(\vec{\Psi})}\varphi = 0$
3. $\Longrightarrow$ Dispersive-type decay estimate for the linear wave equation solution φ
4. $\Longrightarrow$ Rescaled version of the desired Strichartz estimates
5. $\Longrightarrow$ Theorem 7.2.

We remind the reader that the completion of the proof of Theorem 7.2 closes the bootstrap argument initiated in Sect. 3.5, thereby justifying the estimate (17) and completing the proof of Theorem 1.2.

11.1 Rescaled version of Theorem 7.2

From standard scaling considerations, one can easily show that Theorem 7.2 (where in (108), $\vec{\Psi}$ denotes the non-rescaled wave variables) would follow[62] from a rescaled version of it, which we state as Theorem 11.1. Here we do not provide the simple proof that Theorem 7.2 follows from Theorem 11.1; we refer readers to [54, Section 3.1] and [54, Theorem 3.3] for further discussion.

[62] More precisely, the analog of Theorem 7.2 in [54], namely [54, Theorem 3.2], was stated only in the special case $\tau = t_k$, where τ is as in the statement of Theorem 7.2. However, the case of a general $\tau \in [t_k, t_{k+1}]$ follows from the same arguments.

Theorem 11.1 (Rescaled version of Theorem 7.2). *Let P denote the Littlewood–Paley projection onto frequencies ξ with $\frac{1}{2} \leq |\xi| \leq 2$. Under the assumptions of Sect. 10.2, there is a $\Lambda_0 > 0$ such that for every $\lambda \geq \Lambda_0$, every $q > 2$ that is sufficiently close to 2, and every solution φ to the homogeneous linear wave equation*

$$\square_{\mathbf{g}(\vec{\Psi})}\varphi = 0 \tag{394}$$

on the slab $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$, the following mixed space-time estimate holds:

$$\|P\boldsymbol{\partial}\varphi\|_{L^q([0,T_{*;(\lambda)}])L^\infty_x} \lesssim \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_0)}. \tag{395}$$

We clarify that in (394), the argument "$\vec{\Psi}$" in $\mathbf{g}(\vec{\Psi})$ denotes the rescaled solution, as in Sects. 9.1 and 9.3.

11.2 Dispersive-type decay estimate

As we discussed in Sect. 11.1, to prove Theorem 7.2, it suffices to prove Theorem 11.1. Theorem 11.1 can be shown, via a technical-but-by-now-standard TT^* argument, to follow as a consequence of the dispersive-type decay estimate provided by Theorem 11.2. See [54, Appendix B] for a proof that Theorem 11.1 follows from Theorem 11.2. We remark that the proof given in [54, Appendix B] goes through almost verbatim, with only minor changes needed to handle the fact that the future-directed unit normal to Σ_t is $\mathbf{B}$ in the present article (and thus the $\mathbf{B}$-differentiation occurs on LHS (397)), while in [54], the future-directed unit normal to Σ_t is ∂_t.

We now state Theorem 11.2. In Sect. 11.3, we will discuss its proof.[63]

Theorem 11.2 (Dispersive-type decay estimate) *Let P denote the Littlewood-Paley projection onto frequencies ξ with $\frac{1}{2} \leq |\xi| \leq 2$. Under the conventions of Sect. 9.3 and the assumptions of Sect. 10.2, there exists a large $\Lambda_0 > 0$ and a function $d(t) \geq 0$ such that if $\lambda \geq \Lambda_0$ and if $q > 2$ is sufficiently close to 2, then*

$$\|d\|_{L^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1, \tag{396}$$

and for every solution φ to the homogeneous linear wave equation (394) on the slab $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$, the following decay estimate holds for $t \in [0, T_{*;(\lambda)}]$:

$$\|P\mathbf{B}\varphi\|_{L^\infty(\Sigma_t)} \lesssim \left\{\frac{1}{(1+t)^{\frac{2}{q}}} + d(t)\right\}\left\{\sum_{m=0}^{3}\|\partial^m\varphi\|_{L^1(\Sigma_0)} + \sum_{m=0}^{2}\|\partial^m\partial_t\varphi\|_{L^1(\Sigma_0)}\right\}. \tag{397}$$

63 The presence of up to three derivatives of φ on RHS (397) is not problematic because in practice, the estimate (397) is only used on functions supported near unit frequencies in Fourier space (and thus the functions' derivatives can be controlled in terms of the function itself, by Bernstein's inequality). See [54, Appendix B], especially the first estimate on page 105.

11.3 Reduction of the proof of Theorem 11.2 to the case of compactly supported data

It is convenient to reduce the proof of Theorem 11.2 to a spatially localized version in which the L^1 norms on the RHSs of the estimates are replaced with terms involving L^2 norms, which are more natural (in view of their connection to energy estimates). More precisely, the same arguments given in [54, Section 4] yield that Theorem 11.2 follows as a consequence of Proposition 11.1, which is an analog of [54, Proposition 4.1], and Lemma 11.2, which is an analog of [54, Lemma 4.2]. We will discuss the proof of Proposition 11.1 in Sect. 11.5, while we provide the simple proof of Lemma 11.2 in this subsection.

Proposition 11.1 (Spatially localized version of Theorem 11.2) *Let $R > 0$ be as in Sect. 9.1, fix any*[64] $\mathbf{z} \in \Sigma_0$*, and let* $\gamma_{\mathbf{z}}(1)$ *be the unique point on the cone-tip axis in* Σ_1 *(see Sect. 9.4.1 for the definition of the cone-tip axis). Let P denote the Littlewood–Paley projection onto frequencies ξ with $\frac{1}{2} \leq |\xi| \leq 2$. Under the assumptions of Sect. 10.2, there exists a large $\Lambda_0 > 0$ and a function $d(t) \geq 0$ such that if $\lambda \geq \Lambda_0$ and if $q > 2$ is sufficiently close to 2, then*

$$\|d\|_{L^{\frac{q}{2}}([0,T_{*;(\lambda)}])} \lesssim 1, \tag{398}$$

and for every solution φ to the homogeneous linear wave equation (394) on the slab $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ whose data on Σ_1 are supported in the Euclidean ball $B_R(\gamma_{\mathbf{z}}(1))$ of radius R centered at $\gamma_{\mathbf{z}}(1)$, the following decay estimate holds for $t \in [1, T_{*;(\lambda)}]$:

$$\|P\mathbf{B}\varphi\|_{L^\infty(\Sigma_t)} \lesssim \left\{ \frac{1}{(1+|t-1|)^{\frac{2}{q}}} + d(t) \right\} \left\{ \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_1)} + \|\varphi\|_{L^2(\Sigma_1)} \right\}. \tag{399}$$

Remark 11.1 (φ vanishes in $\left([1, T_{*;(\lambda)}] \times \mathbb{R}^3\right) \backslash \widetilde{\mathcal{M}}_1^{(Int)}$) From the definition (171) of $\widetilde{\mathcal{M}}_1^{(Int)}$, (174b), (174c), (175), and standard domain of dependence considerations, it follows that the solution φ from Proposition 11.1 satisfies $\varphi \equiv 0$ in $\left([1, T_{*;(\lambda)}] \times \mathbb{R}^3\right) \backslash \widetilde{\mathcal{M}}_1^{(Int)}$.

Lemma 11.2 (Standard energy estimate for the wave equation). *Under the bootstrap assumption (306a) for the first term on the LHS, there exists a large $\Lambda_0 > 0$ such that if $\lambda \geq \Lambda_0$, then solutions φ to the homogeneous linear wave equation (394) (where in (394), $\mathbf{g} = \mathbf{g}(\vec{\Psi})$, with $\vec{\Psi}$ the rescaled solution) verify the following estimate for $t \in [0, T_{*;(\lambda)}]$:*

$$\|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_0)}. \tag{400}$$

[64] As we highlighted in Remark 9.1, the hypersurface that we denote by "Σ_0" in this proposition corresponds to the hypersurface that we denoted by "Σ_{t_k}" in Sects. 3–8. Similar remarks apply for the other constant-time hypersurfaces appearing in this proposition.

Moreover, for $0 \leq t \leq 1$, *we have*

$$\|\varphi\|_{L^2(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\varphi\|_{L^2(\Sigma_0)} + \|\varphi\|_{L^2(\Sigma_0)}. \tag{401}$$

Proof Reasoning as in our proof of (52), but omitting the φ^2 term in the analog of the energy (49) and the energy identity (53), we find that

$$\|\boldsymbol{\partial}\varphi\|^2_{L^2(\Sigma_t)} \lesssim \|\boldsymbol{\partial}\varphi\|^2_{L^2(\Sigma_0)} + \int_0^t \|\boldsymbol{\partial}\vec{\Psi}\|_{L^\infty(\Sigma_\tau)}\|\boldsymbol{\partial}\varphi\|^2_{L^2(\Sigma_\tau)}\, d\tau.$$

(400) now follows from this estimate, Grönwall's inequality, and the estimate $\|\boldsymbol{\partial}\vec{\Psi}\|_{L^1([0,T_{*;(\lambda)}])L^\infty_x} \lesssim \lambda^{-8\epsilon_0} \leq 1$, which is a simple consequence of (151) and (306a).

(401) then follows from (400) and the fundamental theorem of calculus. □

11.4 Mild growth rate for a conformal energy

The proof of Proposition 11.1 fundamentally relies on deriving estimates for a conformal energy, which we define in this subsection. We stress that our definition coincides with the definition of the conformal energy given in [54, Definition 4.4].

11.4.1 Definition of the conformal energy

We start by fixing two smooth, non-negative cut-off functions of (t, u), denoted by W and $\underline{W}$ and satisfying $0 \leq W(t, u) \leq 1$, $0 \leq \underline{W}(t, u) \leq 1$, such that the following properties hold for $t > 0$:

$$W(t,u) = \begin{cases} 1 & \text{if } \frac{u}{t} \in [0, 1/2], \\ 0 & \text{if } \frac{u}{t} \in (-\infty, -1/4] \cup [3/4, 1], \end{cases} \qquad \underline{W}(t,u) = \begin{cases} 1 & \text{if } \frac{u}{t} \in [0, 1], \\ 0 & \text{if } \frac{u}{t} \in (-\infty, -1/4], \end{cases} \tag{402a}$$

$$W(t,u) = \underline{W}(t,u) \qquad \text{if } t \in [1, T_{*;(\lambda)}] \text{ and } \frac{u}{t} \in [-1/4, 0]. \tag{402b}$$

See Fig. 4 for a schematic depiction of the regions in the case $\mathbf{z} := 0$, where for convenience, we have suppressed the "quasilinear nature" of the geometry by depicting it as flat.

Definition 11.1 *(Conformal energy).* For scalar functions φ that vanish outside of $\widetilde{\mathcal{M}}_1^{(Int)}$ (see definition (171) and Remark 11.1), we define the conformal energy $\mathscr{C}[\varphi]$ as follows:

$$\begin{aligned} \mathscr{C}[\varphi](t) := & \int_{\widetilde{\Sigma}_t^{(Int)}} (\underline{W} - W) t^2 \left\{ |\mathbf{D}\varphi|^2 + |\tilde{r}^{-1}\varphi|^2 \right\} d\varpi_g \\ & + \int_{\widetilde{\Sigma}_t^{(Int)}} W \left\{ |\tilde{r}\mathbf{D}_L\varphi|^2 + |\tilde{r}\nabla\!\!\!/\varphi|^2_{g\!\!\!/} + |\varphi|^2 \right\} d\varpi_g. \end{aligned} \tag{403}$$

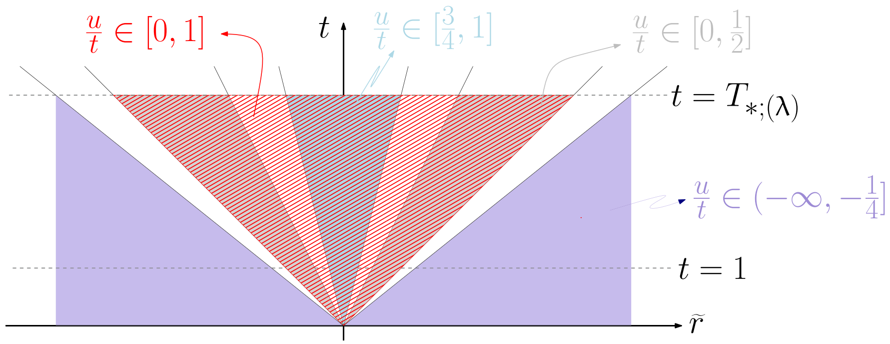

Fig. 4 Schematic illustration of the regions appearing in definition (402a) in the case $\mathbf{z} := 0$

11.4.2 The precise eikonal function and connection coefficient estimates needed for the proof of the conformal energy estimate

The following corollary is a routine consequence of Proposition 10.1. It provides all of the estimates for the eikonal function and connection coefficients that are needed to prove Theorem 11.3, which in turn provides the main estimates needed to prove Proposition 11.1. Some statements in the corollary are redundant in the sense that they already appeared in Proposition 10.1. For the reader's convenience, we have allowed for redundancies; having all needed estimates in the same corollary will facilitate our discussion of the proof of Theorem 11.3.

Corollary 11.3 (The precise estimates needed for the proof of the conformal energy estimate). *Let*

$$\mathbf{A} := \mathrm{f}_{(\vec{L})} \cdot \left(tr_{\widetilde{g}} \widetilde{\chi}^{(Small)}, \hat{\chi}, tr_{g} \chi - \frac{2}{\tilde{r}}, \zeta, \boldsymbol{\partial} \vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D}, \frac{b^{-1} - 1}{\tilde{r}}, k, L \ln b, \hat{\theta}, \theta - \frac{2}{\tilde{r}} \right), \tag{404}$$

where $\mathrm{f}_{(\vec{L})}$ *is any smooth function of the type described in Sect. 9.9.1.*

Under the assumptions of Sect. 10.2, the following estimates hold:

$$T_{*;(\lambda)} \leq \lambda^{1-8\epsilon_0} T_*, \qquad 0 \leq \tilde{r} < 2T_{*;(\lambda)}, \tag{405a}$$

$$\|b - 1\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{-\epsilon_0} \leq \frac{1}{4}, \qquad \upsilon \approx \tilde{r}^2, \qquad \tilde{r} tr_{\widetilde{g}} \widetilde{\chi} \approx 1. \tag{405b}$$

Moreover, we have the following estimates,[65] *where the norms are defined in Sects. 9.10 and 9.12, the corresponding spacetime regions such as* $\widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}$ *are defined*

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[65] Our estimates (406a) and (409b) feature the power $-1/2 - 3\epsilon_0$ on the RHS, as opposed to the power $-1/2 - 4\epsilon_0$ that appeared in the analogous estimates of [54]. This minor change has no substantial effect on the main results.

in Sect. 9.5 *(see especially* (172)*), and p is as in* (276)*:*

$$\|\mathbf{A}\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-1/2-3\epsilon_0}, \tag{406a}$$

$$\|\tilde{r}(\nabla\!\!\!/, \mathbf{D}\!\!\!/_L)\mathbf{A}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)},\ \|\mathbf{A}\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)},\ \|\tilde{r}^{1/2}\mathbf{A}\|_{L_t^\infty L_\omega^{2p}(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \tag{406b}$$

$$\|\tilde{r}^{\frac{1}{2}} L\sigma\|_{L_t^\infty L_\omega^{2p}(\widetilde{\mathcal{C}}_u)},\ \|\tilde{r}^{\frac{1}{2}-\frac{2}{p}}\nabla\!\!\!/\sigma\|_{L_{g\!\!\!/}^p L_t^\infty(\widetilde{\mathcal{C}}_u)},\ \|\tilde{r}^{\frac{1}{2}}\nabla\!\!\!/\sigma\|_{L_\omega^p L_t^\infty(\widetilde{\mathcal{C}}_u)},$$
$$\|\nabla\!\!\!/\sigma\|_{L_t^2 L_\omega^p(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2},\ \text{if}\ \widetilde{\mathcal{C}}_u \subset \widetilde{\mathcal{M}}^{(Int)}, \tag{407a}$$

$$\|\sigma\|_{L^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-8\epsilon_0}, \tag{407b}$$

$$\|\tilde{r}^{-1/2}\sigma\|_{L^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}-4\epsilon_0}, \tag{407c}$$

$$\left\|\tilde{r}^{1/2}\left(\frac{b^{-1}-1}{\tilde{r}}, tr_{g\!\!\!/}\chi - \frac{2}{\tilde{r}}, tr_{g\!\!\!/}\underline{\chi} + \frac{2}{\tilde{r}}, k_{NN}\right)\right\|_{L_t^\infty L_\omega^2(\widetilde{\mathcal{C}}_u)} \lesssim \lambda^{-1/2}, \tag{408}$$

$$\|\mu\!\!\!/_{(2)}\|_{L_u^2 L_t^\infty L_\omega^\infty(\widetilde{\mathcal{M}}^{(Int)})} \cdot \|\tilde{r}^{1/2}\nabla\!\!\!/\sigma\|_{L_u^\infty L_t^\infty L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-1-4\epsilon_0}, \tag{409a}$$

$$\|(\zeta, \widetilde{\zeta} - \mu\!\!\!/, \mu\!\!\!/_{(1)})\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})} \cdot \|\tilde{r}^{1/2}\nabla\!\!\!/\sigma\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-1-3\epsilon_0}, \tag{409b}$$

$$\|\tilde{r}^{3/2}(\check{\mu}, tr_{g\!\!\!/}\chi \cdot \mathbf{\Gamma}_{\underline{L}})\|_{L_u^2 L_t^\infty L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}, \tag{409c}$$

$$\|\tilde{r}\nabla\!\!\!/\sigma\|_{L_u^\infty L_t^\infty L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-4\epsilon_0}, \tag{409d}$$

$$\left\|\tilde{r}^{-1/2}\left\{L\left(\frac{1}{2}tr_{\tilde{g\!\!\!/}}\widetilde{\chi}\upsilon\right) - \frac{1}{4}\left(tr_{g\!\!\!/}\chi\right)^2\upsilon + \frac{1}{2}\{L\ln b\}\,tr_{\tilde{g\!\!\!/}}\widetilde{\chi}\upsilon - |\nabla\!\!\!/\sigma|_{g\!\!\!/}^2\upsilon\right\}\right\|_{L_u^\infty L_t^\infty L_\omega^{\frac{p}{2}}(\widetilde{\mathcal{M}}^{(Int)})} \lesssim \lambda^{-\frac{1}{2}}. \tag{409e}$$

Proof The bootstrap assumptions imply that $\|\mathrm{f}_{(\vec{L})}\|_{L^\infty(\widetilde{\mathcal{M}}^{(Int)})} \lesssim 1$; thus, we can ignore $\mathrm{f}_{(\vec{L})}$ throughout the rest of this proof. The estimates (405a), (405b), (406a), (406b), (407a), (407b), and (407c) are restatements of (151), (177), (307b), and of estimates derived in Propositions 10.1 and 10.4, combined with the schematic relations $L\ln b =$

$\mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, $L\sigma = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, $k = \mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\vec{\Psi}$, $\hat{\theta} = \hat{\chi} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, and $\theta - \frac{2}{\tilde{r}} = \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$ (see (197a), (201a), (233), (200), and (207)). Here we clarify that although the proof of the estimate (295) relied on Schauder-type estimates for $\hat{\chi}$ that forced us to obtain control of $\|(\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}, \mathrm{tr}_{g}\chi - \frac{2}{\tilde{r}}, \hat{\chi})\|_{L_t^2 L_u^\infty C_\omega^{0,\delta_0}(\widetilde{\mathcal{M}}^{(Int)})}$, we have stated the estimate (406a) in terms of the weaker norm $\|\cdot\|_{L_t^2 L_x^\infty(\widetilde{\mathcal{M}}^{(Int)})}$; control of this weaker norm is sufficient for the proof of Theorem 11.3.

(408) follows from (406b) and the schematic relations $\mathrm{tr}_{g}\chi - \frac{2}{\tilde{r}} = \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, $\mathrm{tr}_{g}\underline{\chi} + \frac{2}{\tilde{r}} = -\mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)} + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$, and $k_{NN} = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$ (see (197a), (200), and (207)).

(409a) follows from (300a) and (305b).

(409b) follows from (300a), (305a), and (406a) for ζ.

(409c) follows from (288a), (301b), (334), the estimate $\|\tilde{r}^{1/2}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{1/2-4\epsilon_0}$ guaranteed by (405a), (332a) for the second term on the LHS, and the schematic relations $\boldsymbol{\Gamma}_{\underline{L}} = \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$ and $\mathrm{tr}_{\tilde{g}}\widetilde{\chi} = \mathrm{tr}_{g}\chi + \mathrm{f}_{(\vec{L})} \cdot \boldsymbol{\partial}\vec{\Psi}$.

(409d) follows from the bound (407a) for $\|\tilde{r}^{\frac{1}{2}} \nabla\!\!\!/ \sigma\|_{L_\omega^p L_t^\infty(\widetilde{\mathcal{C}}_u)}$ and the estimate $\|\tilde{r}^{1/2}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{1/2-4\epsilon_0}$ guaranteed by (405a).

To obtain (409e), we first use (229), the estimate (405b) for υ, and the aforementioned estimate

$$\|\tilde{r}^{1/2}\|_{L^\infty(\widetilde{\mathcal{M}})} \lesssim \lambda^{1/2-4\epsilon_0}$$

to deduce that

$$\begin{aligned} \text{LHS } (409e) &\lesssim \|\tilde{r}^{1/2}(\boldsymbol{\partial}\vec{\Psi}, \vec{\mathcal{C}}, \mathcal{D})\|_{L_t^\infty L_u^\infty L_\omega^p(\widetilde{\mathcal{M}})} \\ &\quad + \lambda^{1/2-4\epsilon_0} \left\| \tilde{r}^{1/2}(\boldsymbol{\partial}\vec{\Psi}, \mathrm{tr}_{\tilde{g}}\widetilde{\chi}^{(Small)}, \hat{\chi}, \nabla\!\!\!/ \sigma) \right\|^2_{L_u^\infty L_t^\infty L_\omega^p(\widetilde{\mathcal{M}}^{(Int)})}. \end{aligned} \tag{410}$$

From the estimate (331a), the estimate (406b) for $\|\tilde{r}^{1/2}\mathbf{A}\|_{L_t^\infty L_\omega^p(\widetilde{\mathcal{C}}_u)}$, and the estimate (407a) for $\|\tilde{r}^{\frac{1}{2}} \nabla\!\!\!/ \sigma\|_{L_\omega^p L_t^\infty(\widetilde{\mathcal{C}}_u)}$, we conclude that RHS (410) $\lesssim \lambda^{-1/2}$ as desired. □

11.4.3 Mild growth estimate for the conformal energy

The main estimate needed to prove Proposition 11.1 is provided by the following theorem. The proof of the theorem is fundamentally based on the estimates for the acoustic geometry provided by Corollary 11.3.

Theorem 11.3 (Mild growth estimate for the conformal energy). *Let $R > 0$ be as in Sect. 9.1 and let $\gamma_{\mathbf{z}}(1)$ be the unique point $\gamma_{\mathbf{z}}(1)$ on the cone-tip axis in Σ_1 (see Sect. 9.4.1). Let φ be any solution to the covariant linear wave equation (394) on the slab $[0, T_{*;(\lambda)}] \times \mathbb{R}^3$ such that $(\varphi|_{\Sigma_1}, \partial_t\varphi|_{\Sigma_1})$ is supported in the Euclidean ball of radius R centered at the point $\gamma_{\mathbf{z}}(1)$ in Σ_1 (and thus Remark 11.1 applies).*

Then under the assumptions of Sect. 10.2, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ (which can blow up as $\varepsilon \downarrow 0$) such that the conformal energy of φ (which is *defined in* (403)) *satisfies the following estimate for* $t \in [1, T_{*;(\lambda)}]$*:*

$$\mathscr{C}[\varphi](t) \leq C_\varepsilon (1+t)^{2\varepsilon} \left\{ \|\boldsymbol{\partial}\varphi\|^2_{L^2(\Sigma_1)} + \|\varphi\|^2_{L^2(\Sigma_1)} \right\}. \tag{411}$$

Proof (Discussion of proof) Given the estimates that we have already derived, the proof of Theorem 11.3 is the same as the proof of [54, Theorem 4.5] given in [54, Section 7.6]. Thus, here we only clarify which estimates are needed to apply the preliminary arguments given in [54, Section 7], which are used in [54, Section 7.6] to prove Theorem 11.3.

The proof of (411) given in [54, Section 7] is carried out via a bootstrap argument, wherein one needs to establish [54, Equations (7.61)–(7.63)] to close the bootstrap; see [54, Section 7.4.1]. For the reader's convenience, we first list the main steps given in [54, Section 7], which lead to the proof of Theorem 11.3. They are a collection estimates for the linear solution φ in the statement of the theorem:

1. The most basic ingredient in the proof is that one needs a uniform bound, in terms of the data, for a standard non-weighted energy of φ along a portion of the constant-time hypersurfaces Σ_t and null cones $\widetilde{\mathcal{C}}_u$; see [54, Lemma 7.1].
2. A Morawetz-type energy estimate, which, when combined with Step 1, yields preliminary control of a coercive spacetime integral of $|\boldsymbol{\partial}\varphi|^2$ and φ^2 near the cone-tip axis. The integral involves weights with negative powers of $\tilde{r}$, and it is bounded by the data plus some error terms that are controlled later in the argument.
3. In this step, one makes preliminary progress in controlling the yet-to-be-controlled error terms mentioned above in Step 2. Specifically, one derives estimates showing that $\tilde{r}$-weighted versions of φ can be controlled in L^2 along a portion of Σ_t in terms of a weighted spacetime integral involving the square of its outgoing null derivative and an integral of φ^2 along a portion of a sound cone.
4. Comparison results for various norms and energies, some of which involve the conformal metric $\widetilde{\mathbf{g}}$ from Sect. 9.7.1 and a corresponding conformally rescaled solution variable $\widetilde{\varphi} := e^{-\sigma}\varphi$.
5. Weighted energy estimates for the wave equation $\square_{\widetilde{\mathbf{g}}}\widetilde{\varphi} = \cdots$, where the energies control the L and $\nabla\!\!\!/$ derivatives of $\widetilde{\varphi}$ along portions of Σ_t with weights involving υ (see (193)) and positive powers of $\tilde{r}$. These are obtained by multiplying the wave equation $\square_{\widetilde{\mathbf{g}}}\widetilde{\varphi} = \cdots$ by $(L\widetilde{\varphi} + \frac{1}{2}\mathrm{tr}_{\widetilde{g\!\!\!/}}\widetilde{\chi})\tilde{r}^m$ for appropriate choices of $m \geq 0$, and integrating by parts. Ultimately, when combined with the results from the previous steps, this allows one to bound the conformal energy (i.e., the terms on on RHS (403)) in the region $\{u \leq \frac{3t}{4}\} \cap \widetilde{\mathcal{M}}^{(Int)}$; see [54, Section 7.6], in particular [54, Equation (7.94)] and [54, Equation (7.95)].
6. A decay estimate for the standard non-weighted energy along Σ_t, showing in particular that it decays like $(1+t)^{-2}$; see [54, Equation (7.93)]. Ultimately, when combined with the preliminary estimates for φ provided by Step 3, this yields the desired control of the conformal energy (i.e., the terms on on RHS (403)) in the region $\{u \geq \frac{t}{2}\} \cap \widetilde{\mathcal{M}}^{(Int)}$; see [54, Section 7.6].

We now discuss precisely which of the estimates we have already derived are needed to repeat the arguments of [54, Section 7] and to carry out the above steps. We will not fully describe all of the analysis in [54, Section 7]; rather, we will describe only the part of the analysis that relies on the estimates we have derived. Specifically, we focus primarily on arguments that rely on estimates for the acoustic geometry. We again emphasize that although we have derived the same estimates for the acoustic geometry as in [54], our proof of the estimates (derived in Sect. 10) required substantial additional arguments because we had to control new source terms coming from the entropy and vorticity. The remaining arguments, not discussed here, needed to close the bootstrap—and hence establish Theorem 11.3—are the same as in [54, Section 7], to which we refer the reader for more details. We start by noting that the basic estimates (405a), (405b), and (407b) are used throughout [54, Section 7]. We also refer readers to Footnote 65 regarding a minor discrepancy between the estimates we derived here and corresponding estimates in [54]; we will not comment further on these issues.

Step 1 (see [54, Lemma 7.1]) is essentially equivalent to the basic energy estimates for the wave equations derived in the proofs of Propositions 4.1 and 6.1, differing only in that the needed estimates are spatially localized. For the proof, one needs only the bound

$$\|^{(\mathbf{B})}\boldsymbol{\pi}_{\alpha\beta}\|_{L_t^1 L_x^\infty} \lesssim \lambda^{-8\epsilon_0}, \tag{412}$$

where $^{(\mathbf{B})}\boldsymbol{\pi}_{\alpha\beta}$ are the Cartesian components of the deformation tensor of $\mathbf{B}$. Recalling that each Cartesian component $\mathbf{B}^\alpha$ satisfies $\mathbf{B}^\alpha = \mathrm{f}(\vec{\Psi})$ for some smooth function f (where $\vec{\Psi}$ is the rescaled solution), we see that the Cartesian components $^{(\mathbf{B})}\boldsymbol{\pi}_{\alpha\beta}$ satisfy $^{(\mathbf{B})}\boldsymbol{\pi}_{\alpha\beta} = \mathrm{f}(\vec{\Psi}) \cdot \boldsymbol{\partial}\Psi$ (for some other smooth function f). Hence, the desired bound (412) follows from (151), (307c), and Hölder's inequality.

The Morawetz estimate from Step 2 is provided in [54, Lemma 7.4] and [54, Lemma 7.5]. The proof relies on applying the divergence theorem (the geometric version, with respect to the rescaled metric $\mathbf{g}$) on an appropriate spacetime region to the vectorfield $^{(\mathbf{X})}\mathbf{J}^\alpha[\varphi] := \mathbf{Q}^{\alpha\beta}[\varphi]\mathbf{X}_\beta - \frac{1}{2}\left\{(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta\Theta\right\}\varphi^2 + \frac{1}{2}\Theta(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta(\varphi^2)$, where $\mathbf{Q}^{\alpha\beta}[\varphi]$ is defined in (45), $\mathbf{X} := fN$, N is the outward g-unit normal to $S_{t,u}$ in Σ_t (see (181)), $f := \epsilon_0^{-1} - \frac{\epsilon_0^{-1}}{(1+\tilde{r})^{2\epsilon_0}}$, and $\Theta := \tilde{r}^{-1}f$. The error terms involve various geometric derivatives of N that can be expressed in terms of connection coefficients of the null frame and their first derivatives. For the proof of [54, Lemma 7.4] and [54, Lemma 7.5] to go through verbatim, one needs only the estimates (406a) and (406b); see just below [54, Equation (7.18)].

In obtaining estimates for $\tilde{r}$-weighted versions φ^2 in Step 3, in the sub-step provided by [54, Lemma 7.6], one needs the estimate (406a); see below [54, Equation (7.34)].

For the comparison results from Step 4, in the sub-step provided by [54, Proposition 7.10], one needs the estimates (406b) and (407a); see below [54, Equation (7.43)] and [54, Equation (7.45)]. In the sub-step provided by [54, Lemma 7.11], one needs the estimates (406b) and (407a).

In deriving the weighted energy estimate from Step 5, in the sub-step provided by [54, Lemma 7.15], one needs the estimate (406a); see the first line of the proof. Then, in the same proof, to bound the error terms denoted on [54, page 87] by "$\mathcal{A}_i$", ($i = 1, 2, 3$), one needs, respectively, the estimates (409a), (409b), and (409c); see the analysis just below [54, Equation (7.72)].

For the energy-decay estimate provided by Step 6, in the sub-step provided by [54, Proposition 7.22], the estimates (409d)–(409e) are ingredients needed to help bound the term denoted by "$\mathcal{I}$" on [54, page 94]; see [54, page 95] for the role that (409d)–(409e) play. One also needs (407a) (see the bottom of [54, page 95]) and (406a) (see the top of [54, page 96]). □

11.5 Discussion of the proof of Proposition 11.1

Thanks to the assumptions of Sect. 10.2 and the estimates for the acoustic geometry that we obtained in (292), Proposition 11.1 follows as a consequence of Theorem 11.3 and the same arguments given in [54, Section 4.1] (see in particular [54, Proposition 4.1]) and Lemma 11.2.

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The relativistic Euler equations with a physical vacuum boundary: Hadamard local well-posedness, rough solutions, and continuation criterion

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Abstract

In this paper we provide a complete local well-posedness theory for the free boundary relativistic Euler equations with a physical vacuum boundary on a Minkowski background. Specifically, we establish the following results: (i) local well-posedness in the Hadamard sense, i.e., local existence, uniqueness, and continuous dependence on the data; (ii) low regularity solutions: our uniqueness result holds at the level of Lipschitz velocity and density, while our rough solutions, obtained as unique limits of smooth solutions, have regularity only a half derivative above scaling; (iii) stability: our uniqueness in fact follows from a more general result, namely, we show that a certain nonlinear functional that tracks the distance between two solutions (in part by measuring the distance between their respective boundaries) is propagated by the flow; (iv) we establish sharp, essentially scale invariant energy estimates for solutions; (v) a sharp continuation criterion, at the level of scaling, showing that solutions can be continued as long as the velocity is in $L^1_t Lip$ and a suitable weighted version of the density is at the same regularity level. Our entire approach is in Eulerian coordinates and relies on the functional framework developed in the companion work of the second and third authors on corresponding non relativistic problem. All our results are valid for a general equation of state $p(\varrho) = \varrho^\gamma, \gamma > 1$.

Contents

1. Introduction

In this article, we consider the relativistic Euler equations, which describe the motion of a relativistic fluid in a Minkowski background $\mathbb{M}^{d+1}$, $d \geq 1$. The fluid state is represented by the (energy) *density* $\varrho \geq 0$, and the *relativistic velocity* u. The velocity is assumed to be a forward time-like vector field, normalized by

$$u^\alpha u_\alpha = -1. \tag{1.1}$$

The equations of motion consist of

$$\partial_\alpha \mathcal{T}^\alpha_\beta = 0, \tag{1.2}$$

where $\mathcal{T}$ is the energy-momentum tensor for a perfect fluid, defined by

$$\mathcal{T}_{\alpha\beta} := (p+\varrho) u_\alpha u_\beta + p\, m_{\alpha\beta}. \tag{1.3}$$

Here m is the Minkowski metric, and p is the *pressure*, which is subject to the equation of state

$$p = p(\varrho).$$

Projecting (1.2) onto the directions parallel and perpendicular to u, using definition (1.3), and the identity (1.1), yields the system

$$\begin{cases} u^\mu \partial_\mu \varrho + (p+\varrho)\partial_\mu u^\mu = 0 \\ (p+\varrho) u^\mu \partial_\mu u_\alpha + \Pi^\mu_\alpha \partial_\mu p = 0, \end{cases} \tag{1.4}$$

with u satisfying the constraint (1.1), which is in turn preserved by the time evolution. Here Π is is the projection on the space orthogonal to u and is given by

$$\Pi_{\alpha\beta} = m_{\alpha\beta} + u_\alpha u_\beta.$$

Throughout this paper, we adopt standard rectangular coordinates in Minkowski space, denoted by $\{x^0, x^1, \ldots, x^d\}$, and we identify x^0 with a time coordinate, $t{:=}x^0$. Greek indices vary from 0 to d and Latin indices from 1 to d.

The system (1.4) can be seen as a nonlinear hyperbolic system, which in the reference frame of the moving fluid has the propagation speed

$$c_s^2(\varrho){:=}\frac{dp}{d\varrho},$$

which is subject to

$$0 \le c_s < 1,$$

implying that the speed of propagation of sound waves is always non-negative and below the speed of light (which equals to one in the units we adopted).

In this article we consider the physical situation where vacuum states are allowed, i.e. the density is allowed to vanish. The gas is located in the moving domain

$$\Omega_t{:=}\{x \in \mathbb{R}^d \mid \varrho(t,x) > 0\},$$

whose boundary Γ_t is the *vacuum boundary*, which is advected by the fluid velocity u.

The distinguishing characteristic of a gas, versus the case of a liquid, is that the density, and implicitly the pressure and the sound speed, vanish on the free boundary Γ_t,

$$\varrho = 0, \quad p = 0, \quad c_s = 0 \qquad \text{on } \Gamma_t.$$

Thus, the equations studied here provide a basic model of relativistic gaseous stars (see Section 1.6). An appropriate equation of state to describe this situation is [1] (see,e.g., [[37], Section 2.4] or [35]):

$$p(\varrho) = \varrho^{\kappa+1}, \ \text{ where } \kappa > 0 \text{ is a constant.} \tag{1.5}$$

The decay rate of the sound speed at the free boundary plays a critical role. Precisely, there is a unique, natural decay rate which is consistent with the time

[1] Observe that the requirement $0 \le c_s^2 < 1$ imposes a bound on ϱ. This occurs because power-law equations of state such as (1.5) are no longer valid if the density is very large [17].

evolution of the free boundary problem for the relativistic Euler gas, which is commonly referred to as *physical vacuum*, and has the form

$$c_s^2(t,x) \approx \operatorname{dist}(x,\Gamma_t) \qquad \text{in } \Omega_t, \tag{1.6}$$

where $\operatorname{dist}(\cdot,\cdot)$ is the distance function. Exactly the same requirement is present in the non-relativistic compressible Euler equations. As in the non-relativistic setting, (1.6) should be considered as a condition on the initial data that is propagated by the time-evolution.

There are two classical approaches in fluid dynamics, using either Eulerian coordinates, where the reference frame is fixed and the fluid particles are moving, or using Lagrangian coordinates, where the particles are stationary but the frame is moving. Both of these approaches have been extensively developed in the context of the Euler equations, where the local well-posedness problem is very well understood.

By contrast, the free boundary problem corresponding to the physical vacuum has been far less studied and understood. Because of the difficulties related to the need to track the evolution of the free boundary, all the prior work is in the Lagrangian setting and in high regularity spaces which are only indirectly defined.

Our goal in this paper is to provide the *first local well-posedness result for this problem.* Unlike previous approaches, which were limited to proving energy-type estimates at high regularity and in a Lagrangian setting [12,16], here we consider this problem fully within the Eulerian framework, where we provide a complete local well-posedness theory, in the Hadamard sense, in a low regularity setting. We summarize here the main features of our result, which mirror the results in the last two authors' prior paper devoted to the non-relativistic problem [14], referring to Section 1.5 for precise statements:

a) We prove the *uniqueness* of solutions with very limited regularity $v \in Lip$, $\varrho \in Lip$[2]. More generally, at the same regularity level we prove *stability*, by showing that bounds for a certain nonlinear distance between different solutions can be propagated in time.
b) Inspired by [14], we set up the Eulerian Sobolev *function space structure* where this problem should be considered, providing the correct, natural scale of spaces for this evolution.
c) We prove sharp, *scale invariant*[3] *energy estimates* within the above mentioned scale of spaces, which guarantee that the appropriate Sobolev regularity of solutions can be continued for as long as we have uniform bounds at the same scale $v \in Lip$.
d) We give a constructive proof of *existence* for regular solutions, fully within the Eulerian setting, based on the above energy estimates.

[2] In an appropriately weighted sense in the case of ϱ, see Theorem 1.1.
[3] While this problem does not have an exact scaling symmetry, one can still identify a leading order scaling.

e) We employ a nonlinear Littlewood-Paley type method, developed prior work [14], in order to obtain *rough solutions* as unique limits of smooth solutions. This also yields the *continuous dependence* of the solutions on the initial data.

1.1. Space-time foliations and the material derivative

The relativistic character of our problem implies that there is no preferred choice of coordinates. On the other hand, in order to derive estimates and make quantitative assertions about the evolution, we have to choose a foliation of spacetime by space-like hypersurfaces. Here, we take advantage of the natural set-up provided by Minkowski space and foliate the spacetime by $\{t = constant\}$ slices. We then define the material derivative, which is adapted to this specific foliation, as

$$D_t := \partial_t + \frac{u^i}{u^0}\partial_i. \tag{1.7}$$

The vectorfield D_t is better adapted to the study of the free-boundary evolution than working directly with $u^\mu \partial_\mu$. Indeed, in order to track the motion of fluid particles on the boundary, we need to understand their velocity relative to the aforementioned spacetime foliation. The velocity that is measured by an observer in a reference frame characterized by the coordinates $(t, x^1, \dots, x^d)$ is u^i/u^0. This is a consequence of the fact that in relativity observers are defined by their world-lines, which can be reparametrized. This ambiguity is fixed by imposing the constraint $u^\mu u_\mu = -1$. As a consequence, the d-dimensional vectorfield $(u^1, \dots, u^d)$ can have norm arbitrarily large, while the physical velocity has to have norm at most one (the speed of light).

It follows, in particular, that fluid particles on the boundary move with velocity u^i/u^0. These considerations also imply that the standard differentiation formula for moving domains holds with D_t, i.e.,

$$\frac{d}{dt}\int_{\Omega_t} f\,dx = \int_{\Omega_t} D_t f\,dx + \int_{\Omega_t} f\partial_i\left(\frac{u^i}{u^0}\right)dx. \tag{1.8}$$

This formula remains valid with the good variable v we introduce below since $v^i/v^0 = u^i/u^0$.

1.2. The good variables

The starting point of our analysis is a good choice of dynamical variables. We seek variables that are tailored to the characteristics of the Euler flow *all the way to moving boundary,* where the sound characteristics degenerate due to the vanishing of the sound speed. Our choice of good variables will

(i) better diagonalize the system with respect to the material derivative,
(ii) be associated with truly relativistic properties of the vorticity, and
(iii) lead to good weights that allow us to control the behavior of the fluid variables when one approaches the boundary.

Property (i) will be intrinsically tied with both the wave and transport character of the flow in that (a) the diagonalized equations lead to good second order equations that capture the propagation of sound in the fluid, see Section 3.2, and (b) it provides a good transport structure that will allow us to implement a time discretization for the construction of regular solution, see Section 6. Property (ii) will ensure a good coupling between the wave-part and the transport-part of the system. Finally, property (iii) will lead to the correct functional framework needed to close the estimates.[4] Our good variables, denoted by (r, v), are defined in (1.9) and (1.15). The corresponding equations of motion are (1.16), which we now derive.

Our first choice of good variables is a rescaled version of the velocity given by

$$v^\alpha = f(\varrho)u^\alpha, \tag{1.9}$$

where f is given by

$$f(\varrho) := \exp \int \frac{c_s^2(\varrho)}{p(\varrho)+\varrho}\, d\varrho. \tag{1.10}$$

Although we are interested in the case $p(\varrho) = \varrho^{\kappa+1}$, it is instructive to consider first a general barotropic equation of state; see the discussion related to the vorticity further below.

In order to understand our choice for f, compute

$$\partial_\mu v^\alpha = f'(\varrho)\partial_\mu \varrho u^\alpha + f(\varrho)\partial_\mu u^\alpha.$$

Solving for $\partial_\mu u^\alpha$ and plugging the resulting expression into the second equation of (1.4) we find

$$\frac{p+\varrho}{f} u^\mu \partial_\mu v^\alpha + c_s^2 m^{\alpha\mu}\partial_\mu \varrho + \left(-\frac{f'}{f}(p+\varrho) + c_s^2\right) u^\alpha u^\mu \partial_\mu \varrho = 0.$$

We see that the term in parenthesis vanishes if f is given by (1.10), resulting in an equation which is diagonal with respect to the material derivative, and which we write as

$$D_t v^\alpha + \frac{c_s^2 f^2}{(p+\varrho)v^0} m^{\alpha\mu}\partial_\mu \varrho = 0. \tag{1.11}$$

We notice that in terms of v, the material derivative (1.7) reads as

$$D_t = \partial_t + \frac{v^i}{v^0}\partial_i.$$

In view of the constraint (1.1), we have that v^0 satisfies

$$v^0 = \sqrt{f^2 + |v|^2}, \quad |v|^2 := v^i v_i, \tag{1.12}$$

[4] It is well known that we can think of the relativistic Euler flow as a wave-transport system. What is relevant here is that the wave evolution that comes out of the diagonalized equations allows estimates all the way to the free surface.

and in solving for v^0 we chose the positive square root because u, and thus v, is a future-pointing vectorfield.

We now show that our choice (1.9) also diagonalizes the first equation in (1.4). First, we use (1.11) with $\alpha = 0$ and solve to $\partial_t v^0$, obtaining

$$\partial_t v^0 = \frac{c_s^2 f^2}{(p+\varrho)v^0}\partial_t\varrho - \frac{v^i}{v^0}\partial_i v^0$$
$$= \frac{c_s^2 f^2}{(p+\varrho)v^0}\partial_t\varrho - \frac{ff'}{(v^0)^2}v^i\partial_i\varrho - \frac{v^i v^j}{(v^0)^2}\partial_i v_j,$$

where in the second equality we used (1.12) to compute $\partial_i v^0$. Using the above identity for $\partial_t v^0$, we find the following expression for $\partial_\mu v^\mu$:

$$\partial_\mu v^\mu = \frac{c_s^2 f^2}{(p+\varrho)v^0}\partial_t\varrho - \frac{ff'}{(v^0)^2}v^i\partial_i\varrho + \left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right)\partial_i v_j,$$

where δ is the Euclidean metric. Expressing $\partial_\mu u^\mu$ in terms of $\partial_\mu v^\mu$ (and derivatives of ϱ) and using the above expression for $\partial_\mu v^\mu$, we see that the first equation in (1.4) can be written as

$$D_t\varrho + \frac{p+\varrho}{a_0 v^0}\left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right)\partial_i v_j - c_s^2\frac{2f^2}{a_0(v^0)^3}v^i\partial_i\varrho = 0. \tag{1.13}$$

Here we are using the notation

$$a_0 := 1 - c_s^2\frac{|v|^2}{(v^0)^2}. \tag{1.14}$$

Observe that Eqs. (1.11) and (1.13) are valid for a general barotropic equation of state. We now assume the equation of state (1.5). Then the sound speed is given by $c_s^2 = (\kappa+1)\varrho^\kappa$ and f becomes $f(\varrho) = (1+\varrho^\kappa)^{1+\frac{1}{\kappa}}$ (we choose the constant of integration by setting $f(0) = 1$, so that $v = u$ when $\varrho = 0$). It turns out that it is better to adopt the sound speed squared as a primary variable instead of ϱ because it plays the role of the correct weight in our energy functionals. We thus define[5] the second component of our good variables by

$$r := \frac{1+\kappa}{\kappa}\varrho^\kappa. \tag{1.15}$$

Therefore, using (r, v) as our good variables, and $p(\varrho)$ given by (1.5) we find that the Eqs. (1.11) and (1.13) become

$$\begin{cases} D_t r + rG^{ij}\partial_i v_j + ra_1 v^i\partial_i r = 0 & (1.16a)\\ D_t v_i + a_2\partial_i r = 0, & (1.16b)\end{cases}$$

[5] The factor $\frac{1}{\kappa}$ in the definition of r is a matter of convenience. Although r and c_s^2 differ by this factor, we slightly abuse the terminology and also call r the sound speed squared.

where we have defined

$$G^{ij}:=\frac{\kappa\langle r\rangle}{a_0 v^0}\left(\delta^{ij}-\frac{v^i v^j}{(v^0)^2}\right),\quad \langle r\rangle:=1+\frac{\kappa r}{\kappa+1},$$

and the coefficients a_0, a_1 and a_2 are given by

$$a_0:=1-\kappa r\frac{|v|^2}{(v^0)^2},\qquad a_1:=-\frac{2\kappa\langle r\rangle^{2+\frac{2}{\kappa}}}{(v^0)^3 a_0},\qquad a_2:=\frac{\langle r\rangle^{1+\frac{2}{\kappa}}}{v^0}.$$

Equations (1.16) are the desired diagonal with respect to D_t equations, and the rest of the article will be based on them. In writing these equations we consider only the spatial components v^i as variables, with v^0 always given by

$$v^0=\sqrt{\langle r\rangle^{2+\frac{2}{\kappa}}+|v|^2}. \tag{1.17}$$

The specific form of the coefficients a_0, a_1, and a_2 is not very important for our argument. We essentially only use that they are smooth functions of r and v, and that $a_0,\ a_2>0$.

The operator $G^{ij}\partial_i(\cdot)_j$ can be viewed as a divergence type operator. This divergence structure is related to the fact that Equations (1.16) express the wave-like behavior of r and of the divergence part of v. The symmetric and positive-definite matrix $c_s^2 G^{ij}$ is closely related to the inverse of the acoustical metric; precisely, they agree at the leading order near the boundary.

As we will see, Equations (1.16) also have the correct balance of powers of r to allow estimates all the way to the free boundary. The r factor in the divergence of v is related to the propagation of sound in the fluid (see Section 3.2) whereas the r factor in the last term of (1.16a) will allow us to treat $ra_1 v^i\partial_i r$ essentially as a perturbation at least in elliptic estimates (see Section 5).

One can always diagonalize Eq. (1.4) by simply algebraically solving for $\partial_t(\varrho,u)$. But it is not difficult to see that this procedure will not lead to equations with good structures for the study of the vacuum boundary problem. In this regard, observe that the choice (1.9) is a nonlinear change of variables, whereas algebraically solving for $\partial_t(\varrho,u)$ is a linear procedure.

We now comment on the relation between v and the vorticity of the fluid ω. It is well-known (see, e.g., [5] Section IX.10.1) that in relativity the correct notion of vorticity is given by the following two-form in spacetim

$$\omega_{\alpha\beta}:=(d_{st}v)_{\alpha\beta}=\partial_\alpha v_\beta-\partial_\beta v_\alpha, \tag{1.18}$$

where d_{st} is the exterior derivative in spacetime. This is true not only for the power law equation of state (1.5), but also for an arbitrary barotropic equation of state.

A computation using (1.18) (see, e.g., [5]) and the equations of motion implies that

$$v^\alpha\omega_{\alpha\beta}=0, \tag{1.19}$$

and that ω satisfies the following evolution equation

$$v^\mu\partial_\mu\omega_{\alpha\beta}+\partial_\alpha v^\mu\omega_{\mu\beta}+\partial_\beta v^\mu\omega_{\alpha\mu}=0. \tag{1.20}$$

Observe that (1.20) implies that $\omega = 0$ if it vanishes initially.

Since we will consider only the spatial components of v as independent, we use (1.19) to eliminate the $0j$ components of ω from (1.20) as follows: from (1.19) we can write

$$\omega_{0j} = -\frac{v^i}{v^0}\omega_{ij}. \tag{1.21}$$

Using (1.21) into (1.20) with $\alpha, \beta = i, j$ we finally obtain

$$D_t\omega_{ij} + \frac{1}{v^0}\partial_i v^k \omega_{kj} + \frac{1}{v^0}\partial_j v^k \omega_{ik} - \frac{1}{(v^0)^2}\partial_i v^0 v^k \omega_{kj} + \frac{1}{(v^0)^2}\partial_j v^0 v^k \omega_{ki} = 0. \tag{1.22}$$

Equation (1.22) will be used to derive estimates for ω_{ij} that will complement the estimates for r and the divergence of v obtained from (1.16).

We remark that in the literature, the use of v, given by (1.9), seems to be restricted mostly to definition and evolution of the vorticity. To the best of our knowledge, this is the first time when it was observed that the same change of variables needed to define the relativistic vorticity also diagonalizes the equations of motion with respect to D_t.

1.3. Scaling and bookkeeping scheme

Although Eq. (1.16) do not obey a scaling law, it is still possible to identify a scaling law for the leading order dynamics near the boundary. This will motivate the control norms we introduce in the next section, as well as provide a bookkeeping scheme that will allow us to streamline the analysis of many complex multilinear expressions we will encounter.

As we will see, the contribution of last term in (1.16a) to our energies is negligible, due to the multiplicative r factor. Thus, we ignore this term for our scaling analysis.[6] Replacing all coefficients that are functions of (r, v) by 1, while keeping the transport and divergence structure present in the equations, we obtain the following simplified version of (1.16):

$$\begin{cases} (\partial_t + v^j\partial_j)r + r\delta^{ij}\partial_i v_j \sim 0 & (1.23a) \\ (\partial_t + v^j\partial_j)v_i + \partial_i r \sim 0. & (1.23b) \end{cases}$$

This system is expected to capture the leading order dynamics near the boundary, and also mirrors the nonrelativistic version of the compressible Euler equations, considered in the predecessor to this paper, see [14]. Equations (1.23) admit the scaling law

$$(r(t,x), v(t,x)) \mapsto (\lambda^{-2}r(\lambda t, \lambda^2 x), \lambda^{-1}v(\lambda t, \lambda^2 x)).$$

Based on this leading order scaling analysis, we assign the following order to the variables and operators in Eq. (1.16):

[6] And then indeed turns out to be lower order from a scaling perspective.

(i) r and v have order -1 and $-1/2$, respectively. More precisely, we only count v as having order $-1/2$ when it is differentiated. Undifferentiated v's have order zero.
(ii) D_t and ∂_i have order $1/2$ and 1, respectively.
(iii) G, a_0, a_1, and a_2, and more generally, any smooth function of (r, v) not vanishing at $r = 0$, have order 0.

Expanding on (iii) above, the order of a function of r is defined by the order of its leading term in the Taylor expansion about $r = 0$, being of order zero if this leading term is a constant. The order of a multilinear expression is defined as the sum of the orders of each factor. Here we remark that all expressions arising in this paper are multilinear expressions, with the possible exception of nonlinear factors as in (iii) above.

According to this convention, all terms in equation (1.16b) have order zero, and all terms in (1.16a) have order $-1/2$, except for the last term in (1.16a) which has order -1. Upon successive differentiation of any multilinear expression with respect to D_t or ∂, all terms produced are the same (highest) order, unless some of these derivatives apply to nonlinear factors as in (iii); then lower order terms are produced.

1.4. Energies, function spaces, and control norms

Here we introduce the function spaces and control norms that we need in order to state our main results. A more detailed discussion is given in Section 2. With some obvious adjustments, here we follow the lat two authors' prior work in [14]. We assume throughout that r is a positive function on Ω_t, vanishing simply on the boundary, and so that r is comparable to the distance to the boundary Γ_t.

In order to identify the correct functional framework for our problem, we start with the linearization of the Eq. (1.16). In Section 3 we show that the linearized equations admit the following energy

$$\|(s, w)\|_{\mathcal{H}}^2 = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} (s^2 + a_2^{-1} r G^{ij} w_i w_j)\, dx,$$

which defines the (time dependent) weighted L^2 space $\mathcal{H}$.

The motivation for the definition of higher order norms and spaces comes from the good second order equations mentioned in Section 1.2. From Eq. (1.16), we find that the second order evolution is governed at leading order by a wave-like operator which is essentially a variable coefficient version of $D_t^2 - r\Delta$. This points toward higher order spaces built on powers of $r\Delta$. Taking into account also the form of the linearized energy above, we are led to the following. We define $\mathcal{H}^{2k}$ as the space of pairs of functions (s, w) in Ω_t for which the norm below is finite

$$\|(s, w)\|_{\mathcal{H}^{2k}}^2 := \sum_{|\alpha|=0}^{2k} \sum_{\substack{a=0 \\ |\alpha|-a\le k}}^{k} \|r^{\frac{1-\kappa}{2\kappa}+a} \partial^\alpha s\|_{L^2}^2 + \sum_{|\alpha|=0}^{2k} \sum_{\substack{a=0 \\ |\alpha|-a\le k}}^{k} \|r^{\frac{1-\kappa}{2\kappa}+\frac{1}{2}+a} \partial^\alpha w\|_{L^2}^2.$$

The definition of $\mathcal{H}^{2k}$ for non-integer k is given in Section 2, via interpolation.

In view of the scaling analysis of Section 1.3, we introduce the critical space $\mathcal{H}^{2k_0}$ where

$$2k_0 = d + 1 + \frac{1}{\kappa}, \tag{1.24}$$

which has the property that its leading order homogeneous component is invariant with respect to the scaling discussed in Section 1.3. Associated with the exponent $2k_0$ we define the following scale invariant time dependent control norm

$$A{:=}\|\nabla r - N\|_{L^\infty} + \|v\|_{\dot{C}^{\frac{1}{2}}}.$$

here N is a given non-zero vectorfield with the following property. In each sufficiently small neighborhood of the boundary, there exists a $x_0 \in \Gamma_t$ such that $N(x_0) = \nabla r(x_0)$. The fact that we can choose such a N follows from the properties of r. The motivation for introducing N is that we can make A small by working in small neighborhood of each reference point x_0, whereas $\|\nabla r\|_{L^\infty}$ is a scale invariant quantity that cannot be made small by localization arguments.

We further introduce a second time dependent control norm that is associated with H^{2k_0+1}, given by[7]

$$B{:=}A + \|\nabla r\|_{\tilde{C}^{\frac{1}{2}}} + \|\nabla v\|_{L^\infty},$$

where

$$\|f\|_{\tilde{C}^{\frac{1}{2}}} := \sup_{x,y\in\Omega_t} \frac{|f(x) - f(y)|}{r(x)^{\frac{1}{2}} + r(y)^{\frac{1}{2}} + |x-y|^{\frac{1}{2}}}.$$

It follows that $\|\nabla r\|_{\tilde{C}^{\frac{1}{2}}}$ scales like the $\dot{C}^{\frac{3}{2}}$ norm of r, but it is weaker in that it only uses one derivative of r away from the boundary. The norm B will control the growth of our energies, allowing for a secondary dependence on A.

When the density is bounded away from zero, the relativistic Euler equations can be written as a first-order symmetric hyperbolic system (see, e.g., [1]) and standard techniques can be applied to derive local estimates. The difficulties in our case come from the vanishing of r on the boundary. Using the finite speed of propagation of the Euler flow, we can use a partition of unity to separate the near-boundary behavior, where r approaches zero, from the bulk dynamics, where r is bounded away from zero. Furthermore, we can also localize to a small set where A is small. Such a localization will be implicitly assumed in all our analysis, in order to avoid cumbersome localization weights through the proofs.

1.5. The main results

Here we state our main results. Combined, these results establish the sharp local well-posedness and continuation criterion discussed earlier. We will make all our statements for the system written in terms of the good variables (r, v), i.e.,

[7] In [14] the A component is omitted, and B is a homogeneous norm. But here, we need to also add the lower order component A in order to be able to handle lower order terms.

Eq. (1.16). Readers interested in the evolution of (1.4) should have no difficulty translating our statements to the original variables ϱ and u.

We recall that Eq. (1.16) are always considered in the moving domain given by

$$\Omega := \bigcup_{0 \le t < T} \{t\} \times \Omega_t,$$

for some $T > 0$, where the moving domain at time t, Ω_t, is given by

$$\Omega_t = \{x \in \mathbb{R}^d \mid r(t,x) > 0\}.$$

We also recall that we are interested in solutions satisfying the physical vacuum boundary condition

$$r(t,x) \approx \text{dist}(x, \Gamma_t), \tag{1.25}$$

where $\text{dist}(\cdot,\cdot)$ is the distance function. Hence, by a *solution* we will always mean a pair of functions (r, v) that satisfies Eq. (1.16) within Ω, and for which (1.25) holds.

We begin with our uniqueness result:

Theorem 1.1. (Uniqueness) *Eq. (1.16) admit at most one solution* (r, v) *in the class*

$$v \in C_x^1, \ \nabla r \in \tilde{C}_x^{\frac{1}{2}}.$$

For the next Theorem, we introduce the phase space

$$\mathbf{H}^{2k} := \{(r, v) \mid (r, v) \in \mathcal{H}^{2k}\}. \tag{1.26}$$

We refer to Section 2 for a more precise definition of $\mathbf{H}^{2k}$, including its topology. Since the $\mathcal{H}^{2k}$ norms depend on r, it is appropriate to think of $\mathbf{H}^{2k}$ in a nonlinear fashion, as an infinite dimensional manifold. We also stress that, while k was an integer in our preliminary discussion in Section 1.4, in Section 2 we extend their definition for any $k \ge 0$. Consequently, $\mathbf{H}^{2k}$ is also defined for any $k \ge 0$, and our Theorems 1.2 and 1.4 below include non-integer values of k.

Theorem 1.2. *Equations* (1.16) *are locally well-posed in* $\mathbf{H}^{2k}$ *for any data* $(\mathring{r}, \mathring{v}) \in \mathbf{H}^{2k}$ *with* $\mathring{r}$ *satisfying* (1.25), *provided that*

$$2k > 2k_0 + 1, \tag{1.27}$$

where k_0 *is given by* (1.24).

Local well-posedness in Theorem 1.2 is understood in the usual quasilinear fashion, namely:

- Existence of solutions $(r, v) \in C([0, T], \mathbf{H}^{2k})$.
- Uniqueness of solutions in a larger class, see Theorem 1.1.
- Continous dependence of solutions on the initial data in the $\mathbf{H}^{2k}$ topology.

Furthermore, in our proof of uniqueness in Section 4 we establish something stronger, namely, that a suitable nonlinear distance between two solutions is propagated under the flow. This distance functional, in particular, tracks the distance between the boundaries of the moving domains associated with different solutions. Thus, our local well-posedness also includes:

- Weak Lipschitz dependence on the initial data relative to a suitable nonlinear functional introduced in Section 4.

An important threshold for our results corresponds to the uniform control parameters A and B. Of these A is at scaling, while B is one half of a derivative above scaling. Thus, by Lemma 2.5 of Section 2, we will have the bounds

$$A \lesssim \|(r, v)\|_{\mathbf{H}^{2k}}, \qquad k > k_0 = \frac{d+1}{2} + \frac{1}{2\kappa},$$

and

$$B \lesssim \|(r, v)\|_{\mathbf{H}^{2k}}, \qquad k > k_0 + \frac{1}{2} = \frac{d+2}{2} + \frac{1}{2\kappa}.$$

Next, we turn our attention to the continuation of solutions.

Theorem 1.3. *For each integer $k \geq 0$ there exists an energy functional $E^{2k} = E^{2k}(r, v)$ with the following properties:*
a) Coercivity: as long as A remains bounded, we have

$$E^{2k}(r, v) \approx \|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

b) Energy estimates hold for solutions to (1.16), i.e.

$$\frac{d}{dt} E^{2k}(r, v) \lesssim_A B\|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

By Gronwall's inequality, Theorem 1.3 readily implies

$$\|(r, v)(t)\|^2_{\mathcal{H}^{2k}} \lesssim e^{\int_0^t C(A)B(\tau)\, d\tau} \|(\mathring{r}, \mathring{v})\|^2_{\mathcal{H}^{2k}}, \tag{1.28}$$

where $C(A)$ is a constant depending on A. The energies E^{2k} will be constructed explicitly only for integer k. Nevertheless, our analysis will show that (1.28) will also hold for any $k > 0$. This will be done using a mechanism akin to a paradifferential expansion, without explicitly constructing energy functionals for non-integer k. As a consequence, we will obtain

Theorem 1.4. *Let k be as in (1.27). Then, the $\mathbf{H}^{2k}$ solution given by Theorem 1.2 can be continued as long as A remains bounded and $B \in L^1_t(\Omega)$.*

1.6. Historical comments

The study of the relativistic Euler equations goes back to the early days of relativity theory, with the works of Einstein [7] and Schwarzschild [38]. The relativistic free-boundary Euler equations were introduced in the '30s in the classical works of Tolman, Oppenheimer, and Volkoff [34, 39, 40], where they derived the now-called TOV equations.[8] With the goal of modeling a star in the framework of relativity, Tolman, Oppenheimer, and Volkoff studied spherically symmetric static solutions to the Einstein-Euler system for a fluid body in vacuum and identified the vanishing of the pressure as the correct physical condition on the boundary. Observe that such a condition covers both the cases of a liquid, where $\varrho > 0$ on the boundary, as well as a gas, which we study here, where $\varrho = 0$ on the boundary. This distinction is related to the choice of equation of state.

Although the TOV equations have a long history and the study of relativistic stars is an active and important field of research (see, e.g., [37], Part III and [30], Part V), the mathematical theory of the relativistic free-boundary Euler equations lagged behind.

If we restrict ourselves to spherically-symmetric solutions, possibly also considering coupling to Einstein's equations, a few precise and satisfactory mathematical statements can be obtained. Lindblom [20] proved that a static, asymptotically flat spacetime, that contains only a uniform-density perfect fluid confined to a spatially compact region ought to be spherically symmetric, thus generalizing to relativity a classical result of Carleman [4] and Lichtenstein [19] for Newtonian fluids. The proof of existence of spherically symmetric static solutions to the Einstein-Euler system consisting of a fluid region and possibly a vacuum region was obtained by Rendall and Schmidt [36]. Their solutions allow for the vanishing of the density along the interface of the fluid-vacuum region, although it is also possible that the fluid occupies the entire space and the density merely approaches zero at infinity. Makino [21] refined this result by providing a general criterion for the equation of state which ensures that the model has finite radius. Makino has also obtained solutions to the Einstein-Euler equations in spherical symmetry with a vacuum boundary and near equilibrium in [22, 23], where equilibrium here corresponds to the states given by the TOV equations. In [24], Makino extended these results to axisymmetric solutions that are slowly rotating, i.e., when the speed of light is sufficiently large or when the gravitational field is sufficiently weak (see also the follow-up works [25, 26] and the preceding work in [13]). Another result within symmetry class related to the existence of vacuum regions and relevant for the mathematical study of star evolution is Hadžić and Lin's recent proof of the "turning point principle" for relativistic stars [11].

The discussion of the last paragraph was not intended to be an exhaustive account of the study of the relativistic free-boundary Euler equations under symmetry

[8] In order to provide some context, we briefly discuss the general relativistic free-boundary Euler equations, i.e., including both the cases of a gas and a liquid. We do not, however, make an overview of related works that treat the non-relativistic free-boundary Euler equations. See [14] and references therein for such a discussion.

assumptions, and we refer the reader to the above references for further discussion. Rather, the goal was to highlight that a fair amount of results can be obtained in symmetry classes. This is essentially because some of the most challenging aspects of the problem are absent or significantly simplified when symmetry is assumed. This should be contrasted with what is currently known in the general case, which we now discuss.

Local existence and uniqueness of solutions the relativistic Euler equations in Minkowski background with a compactly supported density have been obtained by Makino and Ukai [27,28] and LeFloch and Ukai [18]. These solutions, however, require some strong regularity of the fluid variables near the free boundary and, in particular, do not allow for the existence of physical vacuum states. Similarly, Rendall [35] established a local existence and uniqueness[9] result for the Einstein-Euler system where the density is allowed to vanish. Nevertheless, as the author himself pointed out, the solutions obtained are not allowed to accelerate on the free boundary and, in particular, do not include the physical vacuum case. Rendall's result has been improved by Brauer and Karp [2,3], but still without allowing for a physical vacuum boundary. Oliynyk [31] was able to construct solutions that can accelerate on the boundary, but his result is valid only in one spatial dimension. A new approach to investigate the free-boundary Euler equations, based on a frame formalism, has been proposed by Friedrich in [8] (see also [9]) and further investigated by the first author in [6], but it has not led to a local well-posedness theory.

In the case of a *liquid*, i.e., where the fluid has a free-boundary where the pressure vanishes but the density remains strictly positive, a-priori estimates have been obtained by Ginsberg [10] and Oliynyk [32]. Local existence of solutions was recently established by Oliynyk [33] whereas Miao, Shahshahani, and Wu [29] proved local existence and uniqueness for the case when the fluid is in the so-called hard phase, i.e., when the speed of sound equals to one. See also [41], where the author, after providing a proof of local existence for the non-isentropic compressible free-boundary Euler equations in the case of a liquid, discusses ideas to adapt his proof for the relativistic case.

Finally, for the case treated in this paper, i.e., the relativistic Euler equations with a *physical vacuum* boundary, the only results we are aware of are the a-priori estimates by Hadžić, Shkoller, and Speck [12] and Jang, LeFloch, and Masmoudi [15]. In particular, no local existence and uniqueness (let along a complete local well-posedness theory as we present here) had been previously established.

1.7. Outline of the paper

Our approach carefully considers the dual role of r, on the one hand, as a dynamical variable in the evolution and, on the other hand, as a defining function of the domain that, in particular, plays the role of a weight in our energies. An

[9] More precisely, only a type of partial uniqueness has been obtained, see the discussion in [35].

important aspect of our approach is to decouple these two roles. Such decoupling is what allows us to work entirely in Eulerian coordinates. When comparing different solutions (which in general will be defined in different domains), we can think of the role of r as a defining function as leading to a measure of the distance between the two domains (i.e., a distance between the two boundaries), whereas the role of r as a dynamical variable leads to a comparison in the common region defined by the intersection of the two domains. For instance, in our regularization procedure for the construction of regular solutions, the defining functions of the domains are regularized at a different scale than the main dynamical variables.

Although the relativistic and non-relativistic Euler equations, and their corresponding physical vacuum dynamics, are very different, some of our arguments here will closely follow those in the last two authors' prior work [14], where results similar to those of Section 1.5 were established for the non-relativistic Euler equations in physical vacuum. Thus, when it is appropriate, we will provide a brief proof, or quote directly from [14]. This is particularly the case for Sects. 6 and 7.

The paper is organized as follows:

1.7.1. Function spaces, Sect. 2 This section presents the functional framework needed to study Eq. (1.16). These are spaces naturally associated with the degenerate wave operator

$$D_t^2 - rG^{ij}\partial_i\partial_j$$

that is key to our analysis. Similar scales of spaces have been introduced in [14] treating the non-relativistic case and also in [16] where the non-relativistic problem had been considered in Lagrangian coordinates and in high regularity spaces.

Our function spaces $\mathcal{H}^{2k}$ are Sobolev-type spaces with weights r. Since the fluid domain is determined by $\Omega_t := \{r > 0\}$, the state space $\mathbf{H}^{2k}$ is nonlinear, having a structure akin to an infinite dimensional manifold.

Interpolation plays two key roles in our work. Firstly, it allows us to define $\mathcal{H}^{2k}$ for non-integer k without requiring us to establish direct energy estimates with fractional derivatives. This is in particular important for our low regularity setting since the critical exponent (1.24) will in general not be an integer. Secondly, we interpolate between $\mathcal{H}^{2k}$ and the control norms A and B. For this we use some sharp interpolation inequalities presented in Section 2.3. These inequalities are proven in the last two authors' prior work [14] and, to the best of our knowledge, have not appeared in the literature before. In fact, it is the use of these inequalities that allows us to work at low regularity, to obtain sharp energy estimates, and a continuation criterion at the level of scaling.

1.7.2. The linearized equation and the corresponding transition operators, Sect. 3 The linearized equation and its analysis form the foundation of our work, rather than direct nonlinear energy estimates. Besides allowing us to prove nonlinear energy estimates for single solutions, basing our analysis on the linearized equation will also allow us to get good quantitative estimates for the difference of two solutions. The latter is important for our uniqueness result and for the construction of rough solutions as limits of smooth solutions. We observe that there are no boundary conditions that need to be imposed on the linearized variables. This is related to the aforementioned decoupling of the roles of r and signals a good choice of functional framework.

Using the linearized equation we obtain transition operators L_1 and L_2 that act at the level of the linearized variables s and w. These transition operators are roughly the leading elliptic part of the wave equations for s and the divergence part of w. Note that since the wave evolution for the fluid degenerates on the boundary due to the vanishing of the sound speed, so do the transition operators L_1 and L_2. We refer to L_1 and L_2 as transition operators because they relate the spaces $\mathcal{H}^{2k+2}$ and $\mathcal{H}^{2k}$ in a coercive, invertible manner. Because of that, these operators play an important role in our regularization scheme used to construct high-regularity solutions.

1.7.3. Difference estimates and uniqueness, Sect. 4 In this section we construct a nonlinear functional that allows us to measure the distance between two solutions. We show that bounds for this functional are propagated by the flow, which in particular implies uniqueness. A fundamental difficulty is that, since we are working in Eulerian coordinates, different solutions are defined in different domains. This difficulty is reflected in the nonlinear character of our functional, which could be thought of as measuring the distance between the boundaries of two different solutions. The low regularity at which we aim to establish uniqueness leads to some technical complications that are dealt with by a careful analysis of the problem.

1.7.4. Energy estimates and coercivity, Sect. 5 The energies that we use contain two components, a wave component and a transport component, in accordance with the wave-transport character of the system. The energy is constructed after identifying Alinhac-type "good variables" that can be traced back to the structure of the linearized problem. This connection with the linearized problem is also key to establish the coercivity of the energy in that it relies on the transition operators L_1 and L_2 mentioned above.

1.7.5. Existence of regular solutions, Sect. 6 This section establishes the existence of regular solutions. It heavily relies on the last two authors' prior work [14], to which the reader is referred for several technical points.

Our construction is based essentially on an Euler scheme to produce good approximating solutions. Nevertheless, a direct implementation of Euler's method loses derivatives. We overcome this by preceding each iteration with a regularization at an appropriate scale and a separate transport step. The main difficulty is to control the growth of the energies at each step.

1.7.6. Rough solutions as limits of regular solutions, Sect. 7 In this section we construct rough solutions as limits of smooth solutions, in particular establishing the existence part of Theorem 1.2. We construct a family of dyadic regularizations of the data, and control the corresponding solutions in higher $\mathcal{H}^{2k}$ norms with our energy estimates, and the difference of solutions in $\mathcal{H}$ with our nonlinear stability bounds. The latter allow us to establish the convergence of the smooth solutions to the desired rough solution in weaker topologies. Convergence in $\mathbf{H}^{2k}$ is obtained with more accurate control using frequency envelopes. A similar argument then also gives continuous dependence on the data.

1.8. Notation for v, ω and the use of Latin indices

In view of Eq. (1.16) and the corresponding vorticity evolution (1.22), we have now written the dynamics solely in terms of r and the spatial components of v, i.e., v^i. We henceforth consider v as a d-dimensional vector field, so that whenever referring to v we always mean $(v^1, \dots, v^d)$. v^0 is always understood as a shorthand for the RHS of (1.17). Similarly, by ω will stand for ω_{ij}.

Recalling that indices are raised and lowered with the Minkowski metric and that $m_{0i} = 0 = m^{0i}$, $m_{ij} = \delta_{ij}$, we see that tensors containing only Latin indices have indices equivalently raised and lowered with the Euclidean metric.

2. Function spaces

Here we define the function spaces that will play a role in our analysis. They are weighted spaces with weights given by the sound speed squared r which, in view of (1.25), is comparable to the distance to the boundary. More precisely, since a solution to (1.16) is not a-priori given, in the definitions below we take r to be a fixed non-degenerate defining function for the domain Ω_t, i.e., proportional to the distance to the boundary Γ_t. In turn, the boundary Γ_t is assumed to be Lipschitz.

We denote the L^2-weighted spaces with weights h by $L^2(h)$ and we equip them with the norm

$$\|f\|_{L^2(h)} := \int_{\Omega_t} h|f|^2 \, dx.$$

With these notations the base L^2 space of pairs of functions in Ω for our system, denoted by $\mathcal{H}$, is defined as

$$\mathcal{H} = L^2(r^{\frac{1-\kappa}{\kappa}}) \times L^2(r^{\frac{1}{\kappa}}).$$

This space depends only on the choice of r. However, we will often use an equivalent norm that also depends on v, which corresponds to the energy space for the linearized problem and will also be important in the construction of our energies:

$$\|(s, w)\|^2_{\mathcal{H}} = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} (s^2 + a_2^{-1} r G^{ij} w_i w_j) \, dx. \tag{2.1}$$

This uses G to measure the pointwise norm of the one form w. The $\mathcal{H}$ norm is equivalent to the $\mathcal{H}^0$ norm (see the definition of $\mathcal{H}^{2k}$ below) since G is equivalent to the the Euclidean inner product with constants depending on the L^∞ norm of (r, v).

We continue with higher Sobolev norms. We define $H^{j,\sigma}$, where $j \geq 0$ is an integer and $\sigma > -\frac{1}{2}$, to be the space of all distributions in Ω_t whose norm

$$\|f\|^2_{H^{j,\sigma}} := \sum_{|\alpha| \leq j} \|r^\sigma \partial^\alpha f\|^2_{L^2}$$

is finite. Using interpolation, we extend this definition, thus defining $H^{s,\sigma}$ for all real $s \geq 0$.

To measure higher regularity we will also need higher Sobolev spaces where the weights depend on the number of derivatives. More precisely, we define $\mathcal{H}^{2k}$ as the space of pairs of functions (s, w) defined inside Ω_t, and for which the norm below is finite :

$$\|(s, w)\|^2_{\mathcal{H}^{2k}} := \sum_{|\alpha|=0}^{2k} \sum_{\substack{a=0 \\ |\alpha|-a\leq k}}^{k} \|r^{\frac{1-\kappa}{2\kappa}+a} \partial^\alpha s\|^2_{L^2} + \sum_{|\alpha|=0}^{2k} \sum_{\substack{a=0 \\ |\alpha|-a\leq k}}^{k} \|r^{\frac{1-\kappa}{2\kappa}+\frac{1}{2}+a} \partial^\alpha w\|^2_{L^2}.$$

We extend the definition of $\mathcal{H}^{2k}$ to non-integer k using interpolation. An explicit characterization of $\mathcal{H}^{2k}$ for non-integer k, based on interpolation, was given in the last two authors' prior work [14]. Using the embedding theorems given below, we can show that the $\mathcal{H}^{2k}$ norm is equivalent to the $H^{2k,\frac{1-\kappa}{2\kappa}+k} \times H^{2k,\frac{1}{2\kappa}+k}$ norm.

2.1. The state space $\mathbf{H}^{2k}$.

As already mentioned in the introduction, the state space $\mathbf{H}^{2k}$ is defined for $k > k_0$ (i.e. above scaling) as the set of pairs of functions (r, v) defined in a domain Ω_t in $\mathbb{R}^d$ with boundary Γ_t with the following properties:

a) Boundary regularity: Γ_t is a Lipschitz surface.
b) Nondegeneracy: r is a Lipschitz function in $\bar{\Omega}_t$, positive inside Ω_t and vanishing simply on the boundary Γ_t.
c) Regularity: The functions (r, v) belong to $\mathcal{H}^{2k}$.

Since the domain Ω_t itself depends on the function r, one cannot think of $\mathbf{H}^{2k}$ as a linear space, but rather as an infinite dimensional manifold. However, describing a manifold structure for $\mathbf{H}^{2k}$ is beyond the purposes of our present paper, particularly since the trajectories associated with our flow are merely expected to be continuous with values in $\mathbf{H}^{2k}$. For this reason, here we will limit ourselves to defining a topology on $\mathbf{H}^{2k}$.

Definition 2.1. A sequence (r_n, v_n) converges to (r, v) in $\mathbf{H}^{2k}$ if the following conditions are satisfied:

i) Uniform nondegeneracy, $|\nabla r_n| \geq c > 0$.
ii) Domain convergence, $\|r_n - r\|_{Lip} \to 0$. Here, we consider the functions r_n and and r as extended to zero outside their domains, giving rise to Liptschitz functions in $\mathbb{R}^d$.
iii) Norm convergence: for each $\epsilon > 0$ there exist a smooth function $(\tilde{r}, \tilde{v})$ in a neighbourhood of Ω so that

$$\|(r, v) - (\tilde{r}, \tilde{v})\|_{\mathcal{H}^{2k}(\Omega)} \leq \epsilon, \qquad \limsup_{n\to\infty} \|(r_n, v_n) - (\tilde{r}, \tilde{v})\|_{\mathcal{H}^{2k}(\Omega_n)} \leq \epsilon.$$

The last condition in particular provides both a uniform bound for the sequence (r_n, v_n) in $\mathcal{H}^{2k}(\Omega_n)$ as well as an equicontinuity type property, insuring that a nontrivial portion of their $\mathcal{H}^{2k}$ norms cannot concentrate on thinner layers near the boundary. This is akin to the the conditions in the Kolmogorov-Riesz theorem for compact sets in L^p spaces.

This definition will enable us to achieve two key properties of our flow:

- Continuity of solutions (r, v) as functions of t with values in $\mathbf{H}^{2k}$.
- Continuous dependence of solutions $(r, v) \in C(\mathbf{H}^{2k})$ as functions of the initial data $(\mathring{r}, \mathring{v}) \in \mathbf{H}^{2k}$.

2.2. Regularization and good kernels

In what follows we outline the main steps developed in Section 2 of [14], and in which, for a given state (r, v) in $\mathbf{H}^{2k}$, we construct regularized states, denoted by (r^h, v^h), to our free boundary evolution, associated to a dyadic frequency scale 2^h, $h \geq 0$. This relies on having good regularization operators associated to each dyadic frequency scale 2^h, $h \geq 0$. We denote these regularization operators by Ψ^h, with kernels K^h. These are the same as in [14], and their exact definition can be found in there as well. A brief description on how one should envision these regularization operators is in order.

It is convenient to think of the domain Ω_t as partitioned in dyadic boundary layers, denoting by $\Omega^{[j]}$ the layer at distance 2^{-2j} away from the boundary. Within each boundary layer we need to understand which is the correct spatial regularization scale. The principal part of the second order elliptic differential operator associated to our system is the starting point. Given a dyadic frequency scale h, our regularizations will need to select frequencies ξ with the property that $r\xi^2 \lesssim 2^{2h}$, which would require kernels on the dual scale

$$\delta x \approx r^{\frac{1}{2}} 2^{-h}.$$

However, if we are too close to the boundary, i.e. $r \ll 2^{-2h}$, then we run into trouble with the uncertainty principle, as we would have $\delta x \gg r$. To remedy this issue we select the spatial scale $r \lesssim 2^{-2h}$ and the associated frequency scale 2^{2h} as cutoffs in this analysis. Then the way the regularization works is as follows: (i) for $j < h$, the regularizations (r^h, v^h) in $\Omega^{[j]}$ are determined by (r, v) also in $\Omega^{[j]}$, and (ii) for $j = h$, the values of (r, v) in $\Omega^{[h]}$ determine (r^h, v^h) in a full neighborhood $\tilde{\Omega}^{[>h]}$ of Γ, of size 2^{-2h}. The regularized state is obtained by restricting the full regularization to the domain $\Omega_h := \left\{ r^h > 0 \right\}$.

For completeness we state the result in [14], and refer the reader there for the proof:

Proposition 2.2. *Assume that $k > k_0$. Then given a state $(r, v) \in \mathbf{H}^{2k}$, there exists a family of regularizations $(r^h, v^h) \in \mathbf{H}^{2k}$, so that the following properties hold for a slowly varying frequency envelope $c_h \in \ell^2$ which satisfies*

$$\|c_h\|_{\ell^2} \lesssim_A \|(r, v)\|_{\mathbf{H}^{2k}}. \tag{2.2}$$

i) *Good approximation,*

$$(r^h, v^h) \to (r, v) \quad in\ C^1 \times C^{\frac{1}{2}} \quad as\ h \to \infty, \tag{2.3}$$

and

$$\|r^h - r\|_{L^\infty(\Omega)} \lesssim 2^{-2(k-k_0+1)h}. \tag{2.4}$$

ii) *Uniform bound,*

$$\|(r^h, v^h)\|_{\mathbf{H}^{2k}} \lesssim_A \|(r, v)\|_{\mathbf{H}^{2k}}. \tag{2.5}$$

iii) *Higher regularity*

$$\|(r^h, v^h)\|_{\mathbf{H}_h^{2k+2j}} \lesssim 2^{2hj} c_h, \qquad j > 0. \tag{2.6}$$

iv) *Low frequency difference bound:*

$$\|(r^{h+1}, v^{h+1}) - (r^h, v^h)\|_{\mathcal{H}_{\tilde{r}}} \lesssim 2^{-2hk} c_h, \tag{2.7}$$

for any defining function $\tilde{r}$ with the property $|\tilde{r} - r| \ll 2^{-2h}$.

2.3. Embedding and interpolation theorems

In this section we state some embedding and interpolation results that will be used throughout.They have been proved in the last two authors' prior paper [14], to which the reader is referred to for the proofs.

Lemma 2.3. *Assume that $s_1 > s_2 \geq 0$ and $\sigma_1 > \sigma_2 > -\frac{1}{2}$ with $s_1 - s_2 = \sigma_1 - \sigma_2$. Then we have*

$$H^{s_1,\sigma_1} \subset H^{s_2,\sigma_2}.$$

As a corollary of the above lemma we have embeddings into standard Sobolev spaces.

Lemma 2.4. *Assume that $\sigma > 0$ and $\sigma \leq j$. Then we have*

$$H^{j,\sigma} \subset H^{j-\sigma}.$$

In particular, by standard Sobolev embeddings, we also have Morrey type embeddings into C^s spaces:

Lemma 2.5. *We have*

$$H^{j,\sigma} \subset C^s, \qquad 0 \leq s \leq j - \sigma - \frac{d}{2},$$

where the equality can hold only if s is not an integer.

Next, we state the interpolation bounds.

Proposition 2.6. *Let* $\sigma_0, \sigma_m \in \mathbb{R}$ *and* $1 \le p_0, p_m \le \infty$. *Define*

$$\theta_j = \frac{j}{m}, \qquad \frac{1}{p_j} = \frac{1-\theta_j}{p_0} + \frac{\theta_j}{p_m}, \qquad \sigma_j = \sigma_0(1-\theta_j) + \sigma_m\theta_j,$$

and assume that

$$m - \sigma_m - d\left(\frac{1}{p_m} - \frac{1}{p_0}\right) > -\sigma_0, \qquad \sigma_j > -\frac{1}{p_j}.$$

Then for $0 < j < m$ *we have*

$$\|r^{\sigma_j}\partial^j f\|_{L^{p_j}} \lesssim \|r^{\sigma_0} f\|_{L^{p_0}}^{1-\theta_j} \|r^{\sigma_m}\partial^m f\|_{L^{p_m}}^{\theta_j}.$$

Remark 2.7. One particular case of the above proposition which will be used later is when $p_0 = p_1 = p_2 = 2$, with the corresponding relation in between the exponents of the r^{σ_j} weights.

As the objective here is to interpolate between the L^2 type $H^{m,\sigma}$ norm and L^∞ bounds, we will need the following straightforward consequence of Proposition 2.6:

Proposition 2.8. *Let* $\sigma_m > -\frac{1}{2}$ *and*

$$m - \sigma_m - \frac{d}{2} > 0.$$

Define

$$\theta_j = \frac{j}{m}, \qquad \frac{1}{p_j} = \frac{\theta_j}{2}, \qquad \sigma_j = \sigma_m\theta_j.$$

Then for $0 < j < m$ *we have*

$$\|r^{\sigma_j}\partial^j f\|_{L^{p_j}} \lesssim \|f\|_{L^\infty}^{1-\theta_j} \|r^{\sigma_m}\partial^m f\|_{L^2}^{\theta_j}.$$

We will also need the following two variations of Proposition 2.8:

Proposition 2.9. *Let* $\sigma_m > -\frac{1}{2}$ *and*

$$m - \frac{1}{2} - \sigma_m - \frac{d}{2} > 0.$$

Define

$$\sigma_j = \sigma_m\theta_j, \quad \theta_j = \frac{2j-1}{2m-1}, \quad \frac{1}{p_j} = \frac{\theta_j}{2}.$$

Then for $0 < j < m$ *we have*

$$\|r^{\sigma_j}\partial^j f\|_{L^{p_j}} \lesssim \|f\|_{\dot C^{\frac{1}{2}}}^{1-\theta_j} \|r^{\sigma_m}\partial^m f\|_{L^2}^{\theta_j}.$$

Proposition 2.10. *Let* $\sigma_m > \frac{m-2}{2}$ *and*

$$m - \frac{1}{2} - \sigma_m - \frac{d}{2} > 0.$$

Define

$$\sigma_j = \sigma_m \theta_j - \frac{1}{2}(1-\theta_j), \quad \theta_j = \frac{j}{m}, \quad \frac{1}{p_j} = \frac{\theta_j}{2}.$$

Then for $0 < j < m$ *we have*

$$\|r^{\sigma_j} \partial^j f\|_{L^{p_j}} \lesssim \|f\|_{\tilde{C}^{\frac{1}{2}}}^{1-\theta_j} \|r^{\sigma_m} \partial^m f\|_{L^2}^{\theta_j}.$$

3. The linearized equation

Consider a one-parameter family of solutions (r_τ, v_τ) for the main system (1.16) such that $(r_\tau, v_\tau)|_{\tau=0} = (r, v)$, Then formally the functions $\frac{d}{d\tau}(r_\tau, v_\tau)\big|_{\tau=0} = (s, w)$, defined in the moving domain Ω_t, will solve the corresponding linearized equation. Precisely, a direct computation shows that, for (s, w) in Ω_t, the linearized equation can be written in the form

$$\begin{cases} D_t s + \frac{1}{\kappa} G^{ij} \partial_i r w_j + r G^{ij} \partial_i w_j + r a_1 v^i \partial_i s = f & (3.1a) \\ D_t w_i + a_2 \partial_i s = g_i, & (3.1b) \end{cases}$$

where f and g_i represent perturbative terms of the form

$$f = V_1 s + r W_1 w, \qquad g = V_2 s + W_2 w$$

with potentials $V_{1,2}$ and $W_{1,2}$ which are linear in $\partial(r, v)$, with coefficients which are smooth functions of r and v.

Importantly, we remark that for the above system we do not obtain or require any boundary conditions on the free boundary Γ_t. This is related to the fact that our one parameter family of solutions are not required to have the same domain, as it would be the case if one were working in Lagrangian coordinates.

For completeness, we also provide the explicit expressions for the potentials $V_{1,2}$ and $W_{1,2}$, though this will not play any role in the sequel. We have

$$\begin{aligned} V_1 &= \frac{v^j}{(v^0)^3} \langle r \rangle^{1+\frac{2}{\kappa}} \partial_j r - G^{ij} \partial_i v_j - r \frac{\partial G^{ij}}{\partial r} \partial_i v_j, \\ W_2^l &= -\frac{\partial G^{ij}}{\partial v^l} \partial_i v_j - r a_3 G^{il} \partial_i r, \\ V_{2,i} &= -\frac{v^j}{(v^0)^3} \langle r \rangle^{1+\frac{2}{\kappa}} \partial_j v_i + \frac{\partial a_2}{\partial r} \partial_i r, \\ (W_2)_i^l &= -\frac{a_0}{\kappa \langle r \rangle} G^{jl} \partial_j v_i + \frac{\partial a_2}{\partial v^l} \partial_i r, \end{aligned}$$

where a_3 is a smooth function of (r, v), given by

$$\frac{a_0}{\kappa\langle r\rangle} - \frac{1}{\kappa} = ra_3, \qquad a_3 = -\frac{1}{\langle r\rangle}\left(\frac{1}{2} + \frac{|v|^2}{(v^0)^2}\right). \tag{3.2}$$

As for the other coefficients, the particular form of a_3 is not relevant, but we wrote it here for completeness.

3.1. Energy estimates and well-posedness

We now consider the well-posedness of the linearized problem (3.1) in the time dependent space $\mathcal{H}$. For the purpose of this analysis, we will view $\mathcal{H}$ as a Hilbert space whose squared norm plays the role of the energy functional for the linearized equation,

$$E_{lin}(s, w) := \|(s, w)\|_{\mathcal{H}}^2 = \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}}(s^2 + a_2^{-1} rG^{ij} w_i w_j)\, dx. \tag{3.3}$$

We will use this space for both the linearized equation and its adjoint. Our main result here is as follows:

Proposition 3.1. *Let (r, v) be a solution to (1.16). Assume that both r and v are Lipschitz continuous and that r vanishes simply on the free boundary. Then, the linearized Eq. (3.1) are well-posed in $\mathcal{H}$, and the following estimate holds for solutions (s, w) to (3.1):*

$$\left|\frac{d}{dt}\|(s, w)\|_{\mathcal{H}}^2\right| \lesssim B\|(s, w)\|_{\mathcal{H}}^2. \tag{3.4}$$

Proof. We first remark that (f, g) are indeed perturbative terms, as they satisfy the estimate

$$\|(f, g)\|_{\mathcal{H}} \lesssim B\|(s, w)\|_{\mathcal{H}}. \tag{3.5}$$

This in turn follows from a trivial pointwise bound on the corresponding potentials,

$$\|V_{1,2}\|_{L^\infty} + \|W_{1,2}\|_{L^\infty} \lesssim B.$$

We multiply (3.1a) by $r^{\frac{1-\kappa}{\kappa}} s$ and contract (3.1b) with $a_2^{-1} r^{\frac{1}{\kappa}} G^{ij} w_j$ to find

$$\frac{1}{2} r^{\frac{1-\kappa}{\kappa}} D_t s^2 + \frac{1}{\kappa} r^{\frac{1-\kappa}{\kappa}} G^{ij} \partial_i r w_j s + r^{\frac{1}{\kappa}} G^{ij} \partial_i w_j s + \frac{1}{2} r^{\frac{1}{\kappa}} a_1 v^i \partial_i s^2 = f r^{\frac{1-\kappa}{\kappa}} s,$$
$$\frac{1}{2} a_2^{-1} r^{\frac{1}{\kappa}} G^{ij} D_t(w_i w_j) + r^{\frac{1}{\kappa}} G^{ij} w_j \partial_i s = a_2^{-1} r^{\frac{1}{\kappa}} G^{ij} g_i w_j.$$

Next, we add the two equations above, noting that the second and third terms on the LHS of the first equation combine with the second term on the LHS of the second equation to produce

$$\begin{aligned}
&\frac{1}{\kappa} r^{\frac{1-\kappa}{\kappa}} G^{ij} \partial_i r w_j s + r^{\frac{1}{\kappa}} G^{ij} \partial_i w_j s + r^{\frac{1}{\kappa}} G^{ij} w_j \partial_i s \\
&\quad = \partial_i (r^{\frac{1}{\kappa}}) G^{ij} w_j s + r^{\frac{1}{\kappa}} G^{ij} \partial_i w_j s + r^{\frac{1}{\kappa}} G^{ij} w_j \partial_i s \\
&\quad = G^{ij} \partial_i (r^{\frac{1}{\kappa}} w_j s)
\end{aligned}$$

This yields

$$\begin{aligned}
&\frac{1}{2} r^{\frac{1-\kappa}{\kappa}} D_t s^2 + \frac{1}{2} a_2^{-1} r^{\frac{1}{\kappa}} G^{ij} D_t (w_i w_j) + \frac{1}{2} r^{\frac{1}{\kappa}} a_1 v^i \partial_i s^2 + G^{ij} \partial_i (r^{\frac{1}{\kappa}} w_j s) \\
&\quad = f r^{\frac{1-\kappa}{\kappa}} s + a_2^{-1} r^{\frac{1}{\kappa}} G^{ij} g_i w_j .
\end{aligned}$$

We now integrate the above identity over Ω_t, using the formula (1.8) to produce a time derivative of the energy. For this, we need to write the terms on the left as perfect derivatives or material derivatives. When we do so the zero order coefficients do not cause any harm. We only need to be careful with the terms where a derivative falls on $r^{\frac{1-\kappa}{\kappa}}$ because this could potentially produce a term with the wrong weight (i.e., one less power of r). However, this does not occur because we can solve for $D_t r$ in (1.16a):

$$D_t r^{\frac{1-\kappa}{\kappa}} = \frac{1-\kappa}{\kappa} r^{\frac{1}{\kappa}-2} D_t r = r^{\frac{1-\kappa}{\kappa}} O(\partial(r, v)).$$

Using the above observations, we obtain

$$\left| \frac{d}{dt} \|(s, w)\|_{\mathcal{H}}^2 \right| \lesssim B \|(s, w)\|_{\mathcal{H}}^2 + \|(s, w)\|_{\mathcal{H}} \|(f, g)\|_{\mathcal{H}} \lesssim B \|(s, w)\|_{\mathcal{H}}^2 .$$

We now compute the adjoint equation to (3.1) with respect to the duality relation defined by the $\mathcal{H}$ inner product determined by the norm (3.3). The terms f and g on the RHS of (3.1) are linear expressions in s and rw and and in s and w, respectively, with $\partial(r, v)$ coefficients. Thus, the source terms in the adjoint equation have the same structure as the original equation. Let us write the LHS of (3.1) as

$$D_t \begin{pmatrix} s \\ w \end{pmatrix} + \mathsf{A}^i \partial_i \begin{pmatrix} s \\ w \end{pmatrix} + \mathsf{B} \begin{pmatrix} s \\ w \end{pmatrix},$$

where

$$\mathsf{A}^i = \begin{bmatrix} a_1 r v^i & r G^{ij} \\ a_2 \delta^{il} & 0_{3\times 3} \end{bmatrix}$$

and

$$\mathsf{B} = \begin{bmatrix} 0_{1\times 1} & \frac{1}{\kappa} G^{ij} \partial_i r \\ 0_{3\times 1} & 0_{3\times 3} \end{bmatrix}.$$

With respect to the $\mathcal{H}$ inner product, the adjoint term corresponding to $\mathsf{A}^i \partial_i$ is

$$\tilde{\mathsf{A}}^i \partial_i = - \begin{bmatrix} a_1 r v^i & r G^{ij} \\ a_2 \delta^{il} & 0_{3\times 3} \end{bmatrix} \partial_i - \begin{bmatrix} 0 & \frac{1}{\kappa} G^{ij} \partial_i r \\ \frac{1}{\kappa} r^{-1} a_2 \partial_l r & 0_{3\times 3} \end{bmatrix}$$

modulo terms that are linear expressions in $\tilde{s}$ and $r\tilde{w}$ and in $\tilde{s}$ and $\tilde{w}$ (with $\partial(r, v)$ coefficients) in the first and second components, respectively, where $\tilde{s}$ and $\tilde{w}$ are elements of the dual. Similarly, the adjoint term corresponding to B is

$$\tilde{\mathsf{B}} = \begin{bmatrix} 0_{1\times 1} & 0_{1\times 3} \\ \frac{1}{\kappa} r^{-1} a_2 \partial_l r & 0_{3\times 3} \end{bmatrix}.$$

Combining these expressions, we see that the bad term on the lower left corner of the second matrix in $\tilde{\mathsf{A}}^i \partial_i$ cancels with the corresponding terms in $\tilde{B}$. Therefore, the adjoint problem is the same as the original one, modulo perturbative terms, and it therefore admits an energy estimate similar to the energy estimate (3.4) we have for the linearized Eq. (3.1).

In a standard fashion, the forward energy estimate for the linearized equation and the backward in time energy estimate for the adjoint linearized equation yield uniqueness, respectively existence of solutions for the linearized equation, as needed. This guarantees the well-posedness of the linearized problem.

3.2. Second order transition operators

An alternative approach the linearized equations is to rewrite the linearized Eq. (3.1) as a second order system which captures the wave-like part of the fluid associated with the propagation of sound. Applying D_t to (3.1a) and using (3.1b) and vice-versa, and ignoring perturbative terms, we find

$$D_t^2 s \approx \hat{L}_1 s, \tag{3.6a}$$

$$D_t^2 w_i \approx (\hat{L}_2 w)_i, \tag{3.6b}$$

where

$$\hat{L}_1 s := r\partial_i \left(a_2 G^{ij} \partial_j s \right) + \frac{a_2}{\kappa} G^{ij} \partial_i r \partial_j s, \tag{3.7a}$$

$$(\hat{L}_2 w)_i := a_2 \left(\partial_i (r G^{ml} \partial_m w_l) + \frac{1}{\kappa} G^{ml} \partial_m r \partial_i w_l \right). \tag{3.7b}$$

Equations (3.6a) and (3.6b) are akin to wave equations in that the operators $\hat{L}_1$ and $\hat{L}_2$ satisfy elliptic estimates as proved in Section 5.2. More precisely, the operator $\hat{L}_2$ is associated with the divergence part of w, and it satisfies elliptic estimates once it is combined with a matching curl operator $\hat{L}_3$.

Even though in this paper we do not use the operators $\hat{L}_1$ and $\hat{L}_2$ directly in connection to the corresponding wave equation, they nevertheless play an important role at two points in our proof. Because slightly different properties of $\hat{L}_1$ and $\hat{L}_2$ are needed at these two points, we will take advantage of the fact that only their principal part is uniquely determined in order to make slightly different choices for $\hat{L}_1$ and $\hat{L}_2$. Precisely, these operators will be needed as follows:

I. In the proof of our energy estimates in Section 5.2, in order to establish the coercivity of our energy functionals. There we will need the coercivity of $\hat{L}_1$ and $\hat{L}_2 + \hat{L}_3$, but we also want their coefficients to involve only r, ∇r and undifferentiated v.

II. In the constructive proof of existence of regular solutions, in our paradifferential style regularization procedure. There we use functional calculus for both $\hat{L}_1$ and $\hat{L}_2 + \hat{L}_3$, so we need them to be both coercive and self-adjoint, but we no longer need to impose the previous restrictions on the coefficients.

The two sets of requirements are not exactly compatible, which is why two choices are needed.[10]

We begin with the case (I), where we modify $\hat{L}_1$ and $\hat{L}_2$ as follows:

$$\tilde{L}_1 s := G^{ij} a_2 \left(r \partial_i \partial_j s + \frac{1}{\kappa} \partial_i r \partial_j s \right), \tag{3.8a}$$

$$(\tilde{L}_2 w)_i := a_2 G^{ml} \left(\partial_i (r \partial_m w_l) + \frac{1}{\kappa} \partial_m r \partial_i w_l \right). \tag{3.8b}$$

To $\tilde{L}_2$ we associate an operator $\tilde{L}_3$ of the form

$$(\tilde{L}_3 w)^i := r^{-\frac{1}{\kappa}} a_2 G^{ij} \partial^l [r^{1+\frac{1}{\kappa}} (\partial_l w_j - \partial_j w_l)]. \tag{3.9}$$

For case (II), we keep the first of the operators, setting $L_1 = \hat{L}_1$ but make some lower order changes to $\hat{L}_2$ and $\hat{L}_3$ as follows:

$$(L_2 w)_i := \partial_i \left(a_2^2 (r \partial_m + \frac{1}{\kappa} \partial_m r)(a_2^{-1} G^{ml} w_l) \right). \tag{3.10}$$

$$(L_3 w)_i := r^{-\frac{1}{\kappa}} a_2 F_{ij} \partial_l [G^{lm} G^{jp} r^{1+\frac{1}{\kappa}} (\partial_m w_p - \partial_p w_m)], \tag{3.11}$$

where F is the inverse of the matrix G, i.e., $F = G^{-1}$.

It is not difficult to show that L_1 is a self-adjoint operator in $L^2(r^{\frac{1-\kappa}{\kappa}})$ with respect to the inner product defined by the first component of the $\mathcal{H}$ norm in (2.1), and

$$\mathcal{D}(L_1) = \left\{ f \in L^2(r^{\frac{1-\kappa}{\kappa}}) \,|\, L_1 f \in L^2(r^{\frac{1-\kappa}{\kappa}}) \text{ in the sense of distributions} \right\}.$$

Similarly, both L_2 and L_3 are self-adjoint operators in $L^2(r^{\frac{1}{\kappa}})$ with respect to the inner product defined by the second component of the $\mathcal{H}$ norm in (2.1) and

$$\mathcal{D}(L_2) = \left\{ f \in L^2(r^{\frac{1}{\kappa}}) \,|\, L_2 f \in L^2(r^{\frac{1}{\kappa}}) \text{ in the sense of distributions} \right\}.$$

and similarly for L_3. We further note that $L_2 L_3 = L_3 L_2 = 0$ and that the output of L_2 is a gradient, whereas $L_3 w$ depends only on the curl of w.

[10] Heuristically, both would be fulfilled by an appropriate Weyl type paradifferential quantization, but that would be very cumbersome to use in the presence of the free boundary.

As seen above, it is the operator $\hat{L}_2$ that naturally come out of the equations of motion rather than L_2 (recall that $L_1 = \hat{L}_1$). Thus, we need to compare these operators; we also compare L_1 and $\tilde{L}_1$ for later reference. We have

$$\begin{cases} (L_2 w)_i = (\hat{L}_2 w)_i + \partial_i a_2 r \partial_m (G^{ml} w_l) + \partial_i (r a_2^3 \partial_m a_2^{-1} G^{ml} w_l) + \dfrac{a_2}{\kappa} \partial_i (G^{ml} \partial_m r) w_l \\ L_1 s = \tilde{L}_1 s + r \partial_i (a_2 G^{ij}) \partial_j s. \end{cases} \tag{3.12}$$

We will establish coercive estimates for L_1, L_2, and L_3 (see Sects. 5 and 6), from which follows that the above domains can be characterized as

$$\mathcal{D}(L_1) = H^{2, \frac{1+\kappa}{2\kappa}}, \qquad \mathcal{D}(L_2 + L_3) = H^{2, \frac{1+3\kappa}{2\kappa}}.$$

4. The uniqueness theorem

In this Section we establish Theorem 1.1. It will be a direct consequence of the more general Theorem 4.2 below, which establishes that a suitable functional that measures the difference between two solutions is propagated by the flow.

We consider two solutions (r_1, v_1) and (r_2, v_2) defined in the respective domains Ω_t^1 and Ω_t^2. Put $\Omega_t := \Omega_t^1 \cap \Omega_t^2$, $\Gamma_t := \partial \Omega_t$. If the boundaries of the domains Ω_t^1 and Ω_t^2 are sufficiently close, which will be the case of interest here, then Ω_t will have a Lipschitz boundary. Let D_t^1 and D_t^2 be the material derivatives associated with the domains Ω_t^1 and Ω_t^2, respectively. In Ω_t define the averaged material derivative

$$D_t := \frac{D_t^1 + D_t^2}{2}$$

and the averaged G,

$$G^{ij}_{mid} := G^{ij} \left(\frac{r_1 + r_2}{2}, \frac{v_1 + v_2}{2} \right).$$

We note that the above averaged material derivative is not exactly advecting the domain Ω_t. Fortunately exact advection is not at all needed for what follows. See also Remark 4.3 below.

To measure the difference between two solutions on the common domain Ω_t, we introduce the following distance functional:

$$\mathcal{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) := \int_{\Omega_t} (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} ((r_1 - r_2)^2 + (r_1 + r_2)|v_1 - v_2|^2)\, dx \tag{4.1}$$

which is the same as in [14]. We could have used G to measure $|v_1 - v_2|$, but the Euclidean metric suffices. This is not only because both metrics are comparable but also because we will not use (4.1) directly in conjunction with the equations.

We will, however, use G further below when we introduce another functional for which the structure of the equations will be relevant.

We observe the following Lemma, which has been proved in [14]:

Lemma 4.1. *Assume that r_1 and r_2 are uniformly Lipschitz and nondegenerate, and close in the Lipschitz topology. Then we have*

$$\int_{\Gamma_t} |r_1 + r_2|^{\frac{1}{\kappa}+2} d\sigma \lesssim \mathcal{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)). \tag{4.2}$$

One can view the integral in (4.2) as a measure of the distance between the two boundaries, with the same scaling as $\mathcal{D}_{\mathcal{H}}$.

We now state our main estimate for differences of solutions:

Theorem 4.2. *Let (r_1, v_1) and (r_2, v_2) be two solutions for the system (1.16) in $[0, T]$, with regularity $\nabla r_j \in \tilde{C}^{\frac{1}{2}}$, $v_j \in C^1$, so that r_j are uniformly nondegenerate near the boundary and close in the Lipschitz topology, $j = 1, 2$. Then we have the uniform difference bound*

$$\sup_{t\in[0,T]} \mathcal{D}_{\mathcal{H}}((r_1, v_1)(t), (r_2, v_2)(t)) \lesssim \mathcal{D}_{\mathcal{H}}((r_1, v_1)(0), (r_2, v_2)(0)). \tag{4.3}$$

We remark that

$$\mathcal{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) = 0 \quad \text{iff} \quad (r_1, v_1) = (r_2, v_2),$$

which implies our uniqueness result.

The remaining of this Section is dedicated to proving Theorem 4.2.

4.1. A degenerate energy functional

We will not work directly with the functional $D_{\mathcal{H}}$ because it is non-degenerate, so we cannot take full advantage of integration by parts when we compute its time derivative. We thus consider the modified difference functional

$$\begin{aligned}\tilde{\mathcal{D}}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) := \int_{\Omega_t} & (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} \big(a(r_1 - r_2)^2 \\ & + b(a_{21} + a_{22})^{-1} G^{ij}_{mid} (v_1 - v_2)_i (v_1 - v_2)_j\big)\, dx,\end{aligned} \tag{4.4}$$

where a_{21} and a_{22} are the coefficient a_2 corresponding to the solutions (r_1, v_1) and (r_2, v_2), respectively, and a and b are functions of $\mu := r_1 + r_2$ and $\nu = r_1 - r_2$ with the following properties

(1) They are smooth, nonnegative functions in the region $\{0 \leq |\nu| < \mu\}$, even in ν, and homogeneous of degree 0, respectively 1.
(2) They are connected by the relation $\mu a = b$.
(3) They are supported in $\{|\nu| < \frac{1}{2}\mu\}$, with $a = 1$ in $\{|\nu| < \frac{1}{4}\mu\}$.

Remark 4.3. The choice of the weights a and b above guarantees that the integrand in (4.4) above vanishes polynomially on the boundary of the common domain Ω_t. This is why we refer to this difference functional as degenerate, and is also the reason why we are able to use the averaged material derivative D_t to propagate bounds for $\tilde{\mathcal{D}}_{\mathcal{H}}$ in time even though it does not exactly advect Ω_t.

From [14] we also borrow the equivalence property of the two distance functionals defined above:

Lemma 4.4. *Assume that $A = A_1 + A_2$ is small. Then*

$$\mathcal{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) \approx_A \tilde{\mathcal{D}}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)). \tag{4.5}$$

4.2. The energy estimate

To prove the Theorem 4.2 it remains to track the time evolution of the degenerate distance functional $\tilde{\mathcal{D}}_{\mathcal{H}}$. This is the content of the next result, which immediately implies Theorem 4.2 after an application of Gronwall's inequality.

Proposition 4.5. *We have*

$$\frac{d}{dt}\tilde{\mathcal{D}}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)) \lesssim (B_1 + B_2)\mathcal{D}_{\mathcal{H}}((r_1, v_1), (r_2, v_2)). \tag{4.6}$$

Proof. The difference of the two solutions to (1.16) in the common domain Ω_t satisfies

$$\begin{aligned} 2D_t(r_1 - r_2) = &- (D_t^1 - D_t^2)(r_1 + r_2) - (r_1(G_1)^{ij} + r_2(G_2)^{ij})\partial_i((v_1)_j - (v_2)_j) \\ &- (r_1(G_1)^{ij} - r_2(G_2)^{ij})\partial_i((v_1)_j + (v_2)_j) \\ &- (r_1 a_{11} v_1^i + r_2 a_{12} v_2^i)\partial_i(r_1 - r_2) \\ &- (r_1 a_{11} v_1^i - r_2 a_{12} v_2^i)\partial_i(r_1 + r_2), \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} 2D_t((v_1)_i - (v_2)_i) = &- (D_t^1 - D_t^2)((v_1)_i + (v_2)_i) - (a_{2,1} + a_{2,2})\partial_i(r_1 - r_2) \\ &- (a_{21} - a_{22})\partial_i(r_1 + r_2). \end{aligned} \tag{4.8}$$

Above, G_i and a_{1i} correspond to G and a_1 for the solutions (r_i, v_i), $i = 1, 2$. We observe that the difference of material derivatives can be written as

$$(D_t^1 - D_t^2) = (\tilde{v}_1 - \tilde{v}_2) \cdot \nabla, \qquad \tilde{v}^i = \frac{v^i}{v^0}.$$

To compute the time derivative of the degenerate distance we use the standard formula for differentiation in a moving domain Ω_t,

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)\,dx = \int_{\Omega_t} D_t f + f\nabla\cdot v(t)\,dx, \tag{4.9}$$

where v is in our case the average velocity. Classically this holds under the assumption that the domain Ω_t is advected by D_t. But in our case we replace this assumption with the alternative condition that f vanishes on the boundary of Ω_t. Using this formula we obtain

$$\frac{d}{dt}\tilde{\mathcal{D}}_{\mathcal{H}}(t) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + O(B_1+B_2)D_{\mathcal{H}}(t),$$

where the integrals I_i, $i=\overline{1,6}$, represent contributions as follows:

(a) I_1 represents the contribution where the averaged material derivative falls on a or b,

$$\begin{aligned} I_1 = &\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}}[a_\mu(r_1-r_2)^2 + b_\mu(a_{21}+a_{22})^{-1}G^{ij}_{mid}(v_1-v_2)^2\,]D_t(r_1+r_2)\,dx \\ &+ \int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}}[a_\nu(r_1-r_2)^2 + b_\nu(a_{21}+a_{22})^{-1}G^{ij}_{mid}(v_1-v_2)^2\,]D_t(r_1-r_2)\,dx. \end{aligned}$$

Here the derivatives of a and b are homogeneous of order -1, respectively 0. We get Gronwall terms when they get coupled with factors of r_1+r_2 or r_1-r_2 from the material derivatives. We discuss I_1 later.

(b) I_2 gathers the contributions where the averaged material derivative is applied to $(a_{21}+a_{22})^{-1}$ and to G^{ij}_{mid}. These expressions are smooth functions of (r_1, v_1), (r_2, v_2), and thus their derivatives are bounded by B_1+B_2,

$$I_2 = O(B_1+B_2)D_{\mathcal{H}}(t).$$

(c) I_3 represents the main contribution of the averaged material derivative that falls onto (r_1-r_2) respectively on v_1-v_2 which consists of the first and second terms on the RHS in (4.7), and the second term in (4.8):

$$\begin{aligned} I_3 = &-\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} a(r_1-r_2)\big[(\tilde{v}^i_1-\tilde{v}^i_2)\partial_i(r_1+r_2) \\ &+ (r_1(G_1)^{ij} + r_2(G_2)^{ij})\partial_i((v_1)_j-(v_2)_j)\big]\,dx \\ &- \int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} bG^{ij}_{mid}((v_1)_j-(v_2)_j)\partial_i(r_1-r_2)\,dx. \end{aligned}$$

This term will need further discussion.

(d) In I_4 we place the contribution of the forth term on the RHS of (4.7):

$$I_4 = -\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} a(r_1-r_2)(r_1a_{11}v^i_1 + r_2a_{12}v^i_2)\partial_i(r_1-r_2)\,dx.$$

This term will be discussed later.

(e) I_5 is given by the third and fifth terms on the RHS in (4.7) and the third term on the RHS from (4.8)

$$\begin{aligned} I_5 = &\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} a(r_1-r_2)(r_1(G_1)^{ij} - r_2(G_2)^{ij})\partial_i((v_1)_j + (v_2)_j)\,dx \\ &- \int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} a(r_1-r_2)(r_1 a_{11} v_1^i - r_2 a_{12} v_2^i)\partial_i(r_1+r_2)\,dx \\ &- \int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} b(a_{21}+a_{22})^{-1} G^{ij}_{mid}((v_1)_j - (v_2)_j)(a_{21}-a_{22})\partial_i(r_1+r_2)\,dx. \end{aligned}$$

All of the terms in I_5 are straightforward Gronwall terms.

(f) I_6 contains the terms where D_t falls on $(r_1+r_2)^{\frac{1-\kappa}{\kappa}}$:

$$\begin{aligned} I_6 = \frac{1-\kappa}{\kappa}\int_{\Omega_t} &(r_1+r_2)^{\frac{1}{\kappa}-2}\Big(a(r_1-r_2)^2 + b(a_{21}+a_{22})^{-1} \\ &\times G^{ij}_{mid}(v_1-v_2)_i(v_1-v_2)_j\Big) D_t(r_1+r_2)\,dx. \end{aligned}$$

We will analyze I_6 later.

It remains to take a closer look at the integrals I_1, I_3, I_4, and I_6. We consider them in succession.

The bound for I_1. Here we write

$$2D_t(r_1+r_2) = 2D_t^1 r_1 + D_t^2 r_2 - (\tilde{v}_1 - \tilde{v}_2)\cdot\nabla(r_1-r_2),$$

and

$$2D_t(r_1-r_2) = 2D_t^1 r_1 - D_t^2 r_2 - (\tilde{v}_1 - \tilde{v}_2)\cdot\nabla(r_1+r_2).$$

The first two terms have size $O(B(r_1+r_2))$ so their contribution is a Gronwall term. We are left with the contribution of the last terms, which yields the expressions

$$\begin{aligned} I_1^a = &\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} a_\mu (r_1-r_2)^2(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1-r_2)\,dx \\ &+ \int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} b_\mu (a_{21}+a_{22})^{-1} G^{ij}_{mid}((v_1)_i - (v_2)_i)((v_1)_j \\ &- (v_2)_j)(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1-r_2)\,dx, \\ I_1^b = &\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} a_\nu (r_1-r_2)^2(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1+r_2)\,dx \\ &+ \int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} b_\nu (a_{21}+a_{22})^{-1} G^{ij}_{mid}((v_1)_i - (v_2)_i)((v_1)_j \\ &- (v_2)_j)(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1+r_2)\,dx. \end{aligned}$$

For the second integral in both expressions, we bound $|\tilde{v}_1-\tilde{v}_2| \lesssim |r_1-r_2|+|v_1-v_2|$ and estimate their part by

$$\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} |r_1-r_2||v_1-v_2|^2\,dx \lesssim D_{\mathcal{H}}(t)$$

and

$$J_2 = \int_{\Omega_t} (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} |v_1 - v_2|^3 \, dx,$$

which is discussed later.

We are left with the first integrals in I_1^a and I_1^b, which we record as

$$J_1^a = \int_{\Omega_t} (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} a_\mu (r_1 - r_2)^2 (\tilde{v}_1 - \tilde{v}_2) \cdot \nabla (r_1 - r_2) \, dx$$

and

$$J_1^b = \int_{\Omega_t} (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} a_\nu (r_1 - r_2)^2 (\tilde{v}_1 - \tilde{v}_2) \cdot \nabla (r_1 + r_2) \, dx.$$

These integrals are also discussed later.

The bound for I_3. For I_3, we seek to capture the same cancellation that it is seen in the analysis of the linearized equation. We look at the last term in I_3, use $b = a(r_1 + r_2)$, and integrate by parts; if the derivatives falls on G then this is a straightforward Gronwall term. We are left with three contributions, two of which we pair with the first two terms in I_3. We obtain

$$\begin{aligned} I_3 = &- \int_{\Omega_t} (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} a (r_1 - r_2) \left[(\tilde{v}_1^i - \tilde{v}_2^i) - \frac{1}{\kappa} G^{ij}_{mid} ((v_1)_j - (v_2)_j) \right] \partial_i (r_1 + r_2) \, dx \\ &- \int_{\Omega_t} (r_1 + r_2)^{\frac{1-\kappa}{\kappa}} a (r_1 - r_2) \left[(r_1 (G_1)^{ij} + r_2 (G_2)^{ij}) - (r_1 + r_2) G^{ij}_{mid} \right] \partial_i ((v_1)_j - (v_2)_j) \, dx \\ &+ \int_{\Omega_t} (r_1 + r_2)^{\frac{1}{\kappa}} \partial_i a G^{ij}_{mid} ((v_1)_j - (v_2)_j)(r_1 - r_2) \, dx + O(B_1 + B_2) D_{\mathcal{H}} \\ &= I_3^1 + I_3^2 + J_1^c + O(B_1 + B_2) D_{\mathcal{H}}. \end{aligned}$$

For the first integral, I_3^1 we expand the difference $\tilde{v}_1^i - \tilde{v}_2^i$, seen as a function of v_1 and v_2, in a Taylor series around the center $(v_1 + v_2)/2$. We have

$$\frac{\partial \tilde{v}_i}{\partial v_j} = \frac{1}{v^0} \left(\delta^{ij} - \frac{v_i v_j}{(v^0)^2} \right),$$

where we recognize the matrix on the right as being the main part in G. Thus, we can write

$$\left| (\tilde{v}_1^i - \tilde{v}_2^i) - \frac{1}{\kappa} G^{ij}_{mid} ((v_1)_j - (v_2)_j) \right| \lesssim |r_1 - r_2| + (r_1 + r_2)|v_1 - v_2| + |v_1 - v_2|^3,$$

where the quadratic $v_1 - v_2$ terms cancel because we are expanding around the middle, and we used (3.2) to get the second term on the right. The contributions of all of the terms in the last expansion are Gronwall terms.

For the second integral I_3^2 we have a simpler expansion

$$\left| (r_1 (G_1)^{ij} + r_2 (G_2)^{ij}) - (r_1 + r_2) G^{ij}_{mid} \right| \lesssim |r_1 - r_2| + (r_1 + r_2) |(v_1)_j - (v_2)_j|^2,$$

where all contributions qualify again as Gronwall terms.

Finally, the last integral, J_1^c, is estimated below.

The bound for I_4. After an integration by parts we have

$$I_4 = \frac{1}{2}\int_{\Omega_t} (r_1+r_2)^{\frac{1-\kappa}{\kappa}} \partial_i a\,(r_1 a_{11} v_1^i + r_2 a_{12} v_2^i)(r_1-r_2)^2\,dx$$
$$+ \frac{1-\kappa}{2\kappa}\int_{\Omega_t} (r_1+r_2)^{\frac{1}{\kappa}-2}\partial_i(r_1+r_2)a(r_1 a_{11} v_1^i + r_2 a_{12} v_2^i)(r_1-r_2)^2\,dx$$
$$+ O(B_1+B_2)D_{\mathcal{H}}.$$

Writing

$$|r_1 a_{11} v_1^i + r_2 a_{12} v_2^i| \lesssim r_1 + r_2,$$

both integrals are bounded by $O(B_1+B_2)D_{\mathcal{H}}$.

The bound for I_6. We use $D_t(r_1+r_2) = D_t^1 r_1 + D_t^2 r_2 - (\tilde{v}_1 - \tilde{v}_2)\cdot\nabla(r_1-r_2)$ where the first two terms are bounded by $(B_1+B_2)(r_1+r_2)$ and yield Gronwall contributions. Then we write

$$I_6 \lesssim J_1^d + J_2 + O(B_1+B_2)D_{\mathcal{H}},$$

where

$$J_1^d = \frac{1-\kappa}{\kappa}\int_{\Omega_t}(r_1+r_2)^{\frac{1}{\kappa}-2}a(r_1-r_2)^2(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1-r_2)\,dx.$$

To summarize the outcome of our analysis so far, we have proved that

$$\frac{d}{dt}\tilde{D}_{\mathcal{H}}(t) \le J_1^a + J_1^b + J_1^c + J_1^d + O(J_2) + O(B_1+B_2)D_{\mathcal{H}}.$$

It remains to estimate J_2, J_1^a, J_1^b, J_1^c, and J_1^d, which we write here again for convenience:

$$J_2 = \int_{\Omega_t}(r_1+r_2)^{\frac{1-\kappa}{\kappa}}|v_1-v_2|^3\,dx,$$
$$J_1^a = \int_{\Omega_t}(r_1+r_2)^{\frac{1-\kappa}{\kappa}}a_\mu(r_1-r_2)^2(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1-r_2)\,dx,$$
$$J_1^b = \int_{\Omega_t}(r_1+r_2)^{\frac{1-\kappa}{\kappa}}a_\nu(r_1-r_2)^2(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1+r_2)\,dx,$$
$$J_1^c = \int_{\Omega_t}(r_1+r_2)^{\frac{1}{\kappa}}\partial_i a G^{ij}_{mid}((v_1)_j-(v_2)_j)(r_1-r_2)\,dx,$$
$$J_1^d = \frac{1-\kappa}{\kappa}\int_{\Omega_t}(r_1+r_2)^{\frac{1}{\kappa}-2}a(r_1-r_2)^2(\tilde{v}_1-\tilde{v}_2)\cdot\nabla(r_1-r_2)\,dx.$$

The integral J_2 is the same as in [14] and can be estimated accordingly, using interpolation inequalities; see Lemma 4.4 in [14].

The bound for the integrals J_1^a, J_1^b, J_1^c and J_1^d matches estimates for similar integrals in [14]. Precisely, the integrals J_1^a and J_2^b are estimated as the integrals called J_1^b and J_1^c in [14], respectively, see Lemmas 4.6–4.13 in [14]. The integral J_1^c is estimated as the integral J_1^d in [14], see Lemmas 4.6–4.13 in [14]. The integral J_1^d is estimated as the integral J_1^a in [14], see Lemmas 4.6–4.13 in [14].

We caution the reader that the arguments in [14] are not straightforward, and involve peeling off a carefully chosen boundary layer, with separate estimates inside the boundary layer and outside it. The only difference in the present paper is the presence of additional weights in the integrals, which are smooth functions of r_1, r_2, v_1, v_2. For instance, the difference $\tilde{v}_1 - \tilde{v}_2$ can be expanded as

$$\tilde{v}_1 - \tilde{v}_2 = f(r, v)(r_1 - r_2) + g(r, v)(v_1 - v_2),$$

where r, v stand for r_1, r_2, v_1, v_2 and f, g are smooth. The contribution of the first term admits a straightforward Gronwall bound, and the contribution of the second term is akin to the corresponding integral in [14] but with the added smooth weight. The point is that every time we integrate by parts and the derivative falls on the smooth weight, the corresponding contribution is a straightforward Gronwall term. Hence such smooth weights make no difference if added in the arguments in [14]. □

5. Energy estimates for solutions

Our goal in this section is to establish uniform control over the $\mathbf{H}^{2k}$ norm of the solutions (r, v) to (1.16) with growth given by the norms A and B. For this, we will use appropriate energy functionals $E^{2k} = E^{2k}(r, v)$ constructed out of vector fields naturally associated with the evolution. Our functionals will be associated with the wave and transport parts of the system, which will be considered at matched regularity.

The vector fields we will consider are:

- The material derivative D_t, which has order $1/2$.
- All regular derivatives ∂, which have order 1.
- Multiplication by r, which has order -1.

The wave part of the energy will be associated mainly with D_t, whereas the transport part will be associated with all of the above vector fields.

5.1. Good variables and the energy functional

Heuristically, higher order energy functionals should be obtained by applying an appropriate number of vector fields to the equation, and then verifying that the output solves the linearized equation modulo perturbative terms. In the absence of the free boundary, there are two equally good choices, (i) to spatially differentiate the equation, using the ∂_j vector fields, or (ii) to differentiate the equation in time, using the ∂_t vector field.

However, in the free boundary setting, both of the above choices have issues, as neither ∂_j nor ∂_t are adapted to the boundary. For ∂_t we do have a seemingly better choice, namely to replace it by the material derivative D_t. However, this has the downside that it does not arise from a symmetry of the equations, and consequently the expressions $(D_t^{2k} r, D_t^{2k} v)$ are not good approximate solutions to the linearized equations. To address this matter, we add suitable corrections to these expressions, obtaining what we call the *good variables* (s_{2k}, w_{2k}). Precisely, motivated by the linearized equations (3.1), we introduce

$$
\begin{aligned}
s_0 &:= r,\\
w_0 &:= v,\\
s_1 &:= \partial_t r,\\
w_1 &:= \partial_t v,\\
s_2 &:= D_t^2 r + \frac{1}{2}\frac{a_0 a_2}{\kappa\langle r\rangle} G^{ij}\partial_i r\,\partial_j r,\\
(w_k)_i &:= D_t^k v_i, \qquad k \geq 2,\\
s_k &:= D_t^k r - \frac{a_0}{\kappa\langle r\rangle} G^{ij}\partial_i r\, D_t^{k-1} v_j, \qquad k \geq 3.
\end{aligned}
\tag{5.1}
$$

Here, we use the full Eq. (1.16) to interpret (s_j, w_j) as multilinear expressions in (r, v), with coefficients which are functions of undifferentiated (r, v). Observe that it would be compatible with the linearized equations to define s_k with $\frac{1}{\kappa} G^{ij}\partial_i r D_t^{k-1} v_j$ instead of $\frac{a_0}{\kappa\langle r\rangle} G^{ij}\partial_i r D_t^{k-1} v_j$. The difference between the two cases, however, is a perturbative term due to (3.2), and the definition we adopt here is more convenient because $\frac{a_0}{\kappa\langle r\rangle}$ is what appears in the commutator $[D_t, \partial]$.

Using equations (1.16), we find that for $k \geq 1$, our good variables (s_{2k}, w_{2k}) can be seen as approximate solutions to the linearized equation (3.1). Precisely, they satisfy the following equations in Ω (compare with (3.1)):

$$
\begin{cases}
D_t s_{2k} + \frac{1}{\kappa} G^{ij}\partial_i r (w_{2k})_i + r G^{ij}\partial_i (w_{2k})_j + r a_1 v^i \partial_i s_{2k} = f_{2k} & (5.2a)\\
D_t (w_{2k})_i + a_2 \partial_i s_{2k} = (g_{2k})_i, & (5.2b)
\end{cases}
$$

with source terms (f_{2k}, g_{2k}) which will be shown to be perturbative, see Lemma 5.7. For later use we compute the expressions for the source terms (f_{2k}, g_{2k}), which are given by

$$
\begin{aligned}
f_{2k} = {} & [r G^{ij}\partial_i, D_t^{2k}] v_j + [r a_1 v^i \partial_i, D_t^{2k}] r - r\frac{a_0 a_1}{\kappa\langle r\rangle} G^{pq}\partial_q r v^i \partial_i D_t^{2k-1} v_p \\
& - D_t\left(\frac{a_0}{\kappa\langle r\rangle} G^{ij}\partial_i r\right) D_t^{2k-1} v_j - r a_1 v^i \partial_i \left(\frac{a_0}{\kappa\langle r\rangle} G^{pq}\partial_p r\right) D_t^{2k-1} v_q \\
& - r a_3 G^{ij}\partial_i r (w_{2k})_j,
\end{aligned}
\tag{5.3a}
$$

$$
\begin{aligned}
(g_{2k})_i = {} & D_t^{2k-1}[a_2\partial_i, D_t] r + [a_2 \partial_i, D_t^{2k-1}] D_t r - \frac{a_0 a_2}{\kappa\langle r\rangle} G^{jl}\partial_j r \partial_i D_t^{2k-1} v_l \\
& - a_2 \partial_i \left(\frac{a_0}{\kappa\langle r\rangle} G^{ml}\partial_m r\right) D_t^{2k-1} v_l,
\end{aligned}
\tag{5.3b}
$$

where we used that

$$[A, BC] = [A, B]C + B[A, C].$$

to write

$$[a_2\partial_i, D_t^{2k}] = [a_2\partial_i, D_t^{2k-1}] D_t + D_t^{2k-1}[a_2\partial_i, D_t].$$

We also define

$$\omega_{2k} = r^a \partial^b \omega, \ |b| \leq 2k-1, \ |b| - a = k-1, \tag{5.4}$$

which we think of as the vorticity counterpart to (s_{2k}, w_{2k}). These we will think of as solving approximate transport equations; using (1.22) we find

$$D_t(\omega_{2k})_{ij} + \frac{1}{v^0}(\partial_i v^l (\omega_{2k})_{lj} + \partial_j v^l (\omega_{2k})_{il}) - \frac{v^l}{(v^0)^2}(\partial_i v^0 v(\omega_{2k})_{lj} - \partial_j v^0 (\omega_{2k})_{li}) \\ = (h_{2k})_{ij}, \tag{5.5}$$

where h_{2k} is given by

$$(h_{2k})_{ij} = \left[D_t, r^a \partial^b\right] \omega_{ij} + \left[\frac{1}{v^0}\partial_i v^l, r^a \partial^b\right] \omega_{lj} + \left[\frac{1}{v^0}\partial_j v^l, r^a \partial^b\right] \omega_{il} \\ - \left[\frac{1}{(v^0)^2}\partial_i v^0 v^l, r^a \partial^b\right] \omega_{lj} + \left[\frac{1}{(v^0)^2}\partial_j v^0 v^l, r^a \partial^b\right] \omega_{li}. \tag{5.6}$$

We introduce the wave energy

$$E^{2k}_{\text{wave}}(r, v) := \sum_{j=0}^{k} \|(s_{2j}, w_{2j})\|^2_{\mathcal{H}},$$

the transport energy

$$E^{2k}_{\text{transport}}(r, v) := \|\omega\|^2_{H^{2k-1, k+\frac{1}{2\kappa}}},$$

and the total energy

$$E^{2k}(r, v) := E^{2k}_{\text{wave}}(r, v) + E^{2k}_{\text{transport}}(r, v). \tag{5.7}$$

5.2. *Energy coercivity*

Our goal in this section is to show that the energy (5.7) measures the $\mathbf{H}^{2k}$ size of (r, v). To do so, we would like to consider the energy as a functional of (r, v) defined at a fixed time. This can be done by using Eq. (1.16) to algebraically solve for spatial derivatives of (r, v).

Theorem 5.1. *Let* (r, v) *be smooth functions in* $\overline{\Omega}$. *Assume that* r *is positive in* Ω *and uniformly non-degenerate on the* Γ. *Then*

$$E^{2k} \approx_A \|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

Proof. We begin with the $\lesssim$ part. We consider the wave part of the energy and the corresponding expressions for (s_{2k}, w_{2k}). Using use Eq. (1.16a) and (1.16b) to successively solve for $D_t(r, v)$, we obtain that each (s_{2k}, w_{2k}) is a linear combination of multilinear expressions in r and ∂v (with order zero coefficients).

We will use our bookkeeping scheme of Section 1.3 to understand the expressions for (s_{2k}, w_{2k}). It is useful to record here the order and structure of the linear-in-derivatives top order terms obtained by using the equations to inductively compute $D_t^{2k}(r, v)$ and $D_t^{2k-1}(r, v)$, which involve $2k$ and $2k-1$ derivatives, respectively:

$$D_t^{2k} r \approx r^k \partial^{2k} r + r^{k+1} \partial^{2k} v \approx r^k \partial^{2k} r, \tag{5.8a}$$

$$(k-1) \approx (k-1) + (k - \frac{3}{2}) \approx (k-1),$$

$$D_t^{2k} v \approx r^k \partial^{2k} v + r^k \partial^{2k} r \approx r^k \partial^{2k} v, \tag{5.8b}$$

$$(k - \frac{1}{2}) \approx (k - \frac{1}{2}) + (k-1) \approx (k - \frac{1}{2}),$$

$$D_t^{2k-1} r \approx r^k \partial^{2k-1} r + r^k \partial^{2k-1} v \approx r^k \partial^{2k-1} v, \tag{5.8c}$$

$$(k - \frac{3}{2}) \approx (k-2) + (k - \frac{3}{2}) \approx (k - \frac{3}{2}),$$

$$D_t^{2k-1} v \approx r^k \partial^{2k-1} v + r^{k-1} \partial^{2k-1} r \approx r^{k-1} \partial^{2k-1} r, \tag{5.8d}$$

$$(k-1) \approx (k - \frac{3}{2}) + (k-1) \approx (k-1).$$

Expressions (5.8) are obtained by successively solving for $D_t(r, v)$ in (1.16a)–(1.16b). Below each expression in (5.8a)-(5.8d) we have written the orders of the corresponding terms. The terms of order $k-3/2$, $k-1$, $k-2$, and $k-3/2$ in (5.8a), (5.8b), (5.8c), and (5.8d), respectively have orders less than the other terms in the same expressions, despite having the same number of derivatives, and hence are dropped in the second $\approx$ on each line. Such terms have smaller order, even though they have the same number of derivatives, because of extra powers of r, and come from the term $r a_1 v^i \partial_i r$ in (1.16a).

We begin with the expressions of highest order (see Section 1.3), thus we first focus on the multilinear expressions that come from ignoring the last term on LHS (1.16a) and also where no derivative lands on G, a_1, and a_2. We also consider first the case when every time we commute D_t with ∂, the derivative lands on v^i and not on r via v^0.

In this case, the corresponding multilinear expressions for (s_{2k}, w_{2k}) have the following properties:

- They have order $k-1$ and $k-\frac{1}{2}$, respectively.
- They have exactly $2k$ derivatives.
- They contain at most $k+1$ and k factors of r, respectively.

Thus, a multilinear expression for s_{2k} in this case has the form

$$M = r^a \prod_{j=1}^{J} \partial^{n_j} r \prod_{l=1}^{L} \partial^{m_l} v, \tag{5.9}$$

where $n_j, m_l \geq 1$, and subject to

$$\begin{aligned} \sum n_j + \sum m_l &= 2k, \\ a + J + L/2 &= k + 1, \end{aligned} \tag{5.10}$$

and when $J = 0$ or $L = 0$ the corresponding product is omitted. We claim that it is possible to choose b_j and c_l such that

$$0 \leq b_j \leq (n_j - 1)\frac{k}{2k-1}, \; 0 \leq c_l \leq (m_l - 1)\frac{k+1/2}{k-1/2}, \; a = \sum b_j + \sum c_l.$$

This follows from observing that

$$\begin{aligned} &\sum (n_j - 1)\frac{k}{2k-1} + \sum (m_l - 1)\frac{k+1/2}{k-1/2} \leq \left(\sum n_j + \sum m_l - J - L\right)\frac{k}{2k-1} \\ &= (2k - J - L)\frac{k}{2k-1} \leq (a + k - 1)\frac{k}{2k-1} \leq a, \end{aligned}$$

since $a \leq k$. Equality holds only if $a = k$, $J = 1$ and $L = 0$ (i.e., for the leading linear case). This shows that it is possible to make such a choice of b_j and c_l, which allows us to use our interpolation theorems

$$\begin{aligned} \|r^{b_j}\partial^{n_j} r\|_{L^{p_j}(r^{\frac{1-\kappa}{\kappa}})} &\lesssim (1 + A)^{1-\frac{2}{p_j}} \|(r, v)\|_{\mathcal{H}^{2k}}^{\frac{2}{p_j}}, \\ \|r^{c_l}\partial^{m_l} v\|_{L^{q_l}(r^{\frac{1-\kappa}{\kappa}})} &\lesssim A^{1-\frac{2}{q_l}} \|(r, v)\|_{\mathcal{H}^{2k}}^{\frac{2}{q_l}}, \end{aligned}$$

where

$$\frac{1}{p_j} = \frac{n_j - 1 - b_j}{2(k-1)}, \; \frac{1}{q_l} = \frac{m_l - 1/2 - c_l}{2(k-1)}.$$

Observe that the numerators in $1/p_j$ and $1/q_l$ correspond to the orders of the expressions being estimated and they add up to $k - 1$ (as needed).

In addition to the principal part discussed above, we also obtain lower order terms in our expression for s_{2k}. There are three sources of such terms:

i) The terms from the commutator $[D_t, \partial]$ where derivatives apply to r via v^0. This corresponds to the second term in the formal expansion

$$[D_t, \partial] \approx (\partial v)\partial + (\partial r)\partial,$$

whose order is easily seen to be $1/2$ lower.

ii) If any derivatives are applied to either r or v via a_0, a_1, a_2 or G, this increases the order of the resulting expression by 0, respectively $1/2$, compared to the full order of the derivative which is 1.

iii) Contributions arising from the last term in (1.15), whose order is, to start with, $1/2$ lower than the rest of the terms in the (1.15).

The contributions of all such terms to s_{2k} have lower order. More precisely, they contain expressions of the form (5.9) but with (5.10) replaced by

$$\begin{aligned} \sum n_j + \sum m_l &= 2k, \\ a + J + L/2 &= k + 1 + \frac{j}{2}, \end{aligned} \tag{5.11}$$

where $j > 0$, and which have lower order $k - 1 - \frac{j}{2}$. All such lower order terms can be estimated in a similar fashion, but using lower Sobolev norms for (r, v).

We continue with the $\gtrsim$ part. Applying D_t to (5.2a) and (5.2b) and using definitions (5.1) we find the following recurrence relations

$$s_{2j} = \tilde{L}_1 s_{2j-2} + F_{2j}, \tag{5.12a}$$

$$(w_{2j})_i = (\tilde{L}_2 w_{2j-2})_i + G_{2j}, \tag{5.12b}$$

where $\tilde{L}_1$ and $\tilde{L}_2$ have been defined in Section 3. The next Lemma characterizes the error terms on the RHS of (5.12), and the lemma that follows gives a quantitative relation between the $2j$ and $2j - 2$ quantities.

Lemma 5.2. *For $j \geq 2$, the terms F_{2j} and G_{2j} in (5.12) are linear combinations of multilinear expressions in r and ∂v with $2j$ derivatives and of order at most $j - 1$ and $j - \frac{1}{2}$, respectively. Moreover, they are either*

(i) *non-endpoint, by which we mean multilinear expressions of order $j - 1$ and $j - \frac{1}{2}$, respectively, containing at most $j + 1$ and j factors of r, respectively, and whose products contain at least two factors of $\partial^{\geq 2} r$ or $\partial^{\geq 1} v$, or*

(ii) *they have order strictly less than $j - 1$ and $j - \frac{1}{2}$, respectively, and contain at most $j + 2$ and $j + 1$ factors of r, respectively.*

Proof. We begin with $j \geq 3$. We will analyze

$$s_{2j} = D_t^{2j} r - \frac{a_0}{\kappa \langle r \rangle} G^{lm} \partial_l r D_t^{2j-1} v_m. \tag{5.13}$$

In order to keep track of terms according to the statement of the Lemma, we observe that s_{2j} has order $j - 1$. We will make successive use of the commutator

$$[D_t, \partial_l] = -\frac{a_0}{\kappa \langle r \rangle} G^{pq} \partial_l v_q \partial_p + \frac{\langle r \rangle^{1+\frac{2}{\kappa}}}{(v^0)^3} v^p \partial_l r \partial_p.$$

We begin with the first term on RHS (5.13). From (1.16), we compute.

$$\begin{aligned} D_t^2 r = {} & r G^{ml} \partial_l (a_2 \partial_m r) - [D_t, r G^{ml} \partial_l] v_m - [D_t, r a_1 v^l \partial_l] r + r a_1 v^i \partial_i \left(r G^{ml} \right) \partial_l v_m \\ & + r a_1 v^i \partial_i (r a_1 v^l \partial_l) r + r^2 a_1^2 v^l v^m \partial_l \partial_m r + r^2 a_1 G^{ml} v^i \partial_i \partial_l v_m. \end{aligned}$$

Then,

$$\begin{aligned} D_t^{2j} r = D_t^{2j-2} D_t^2 r = D_t^{2j-2} \Big(& r G^{ml} \partial_l (a_2 \partial_m r) \\ & - [D_t, r G^{ml} \partial_l] v_m - [D_t, r a_1 v^l \partial_l] r \\ & + r a_1 v^i \partial_i \left(r G^{ml} \right) \partial_l v_m + r a_1 v^i \partial_i (r a_1 v^l \partial_l) r \\ & + r^2 a_1^2 v^l v^m \partial_l \partial_m r + r^2 a_1 G^{ml} v^i \partial_i \partial_l v_m, \Big). \end{aligned} \tag{5.14}$$

and we will consider each term on RHS (5.14) separately.

The terms $\partial_i \left(r G^{ml} \right)$ and $\partial_i (r a_1 v^l \partial_l) r$ have order at most zero, thus

$$D_t^{2j-2} \left(r a_1 v^i \partial_i \left(r G^{ml} \right) \partial_l v_m + r a_1 v^i \partial_i (r a_1 v^l \partial_l) r \right)$$

has order at most $j - 3/2$ and belongs to F_{2j}. Next,

$$[D_t, r G^{ml} \partial_l] v_m = [D_t, r G^{ml}] \partial_l v_m + r G^{ml} [D_t, \partial_l] v_m.$$

For the first term on the RHS, we have

$$[D_t, r G^{ml}] \partial_l v_m = D_t r G^{lm} \partial_l v_m + r \frac{\partial G^{lm}}{\partial r} D_t r \partial_l v_m + r \frac{\partial G^{lm}}{\partial v_i} D_t v_i \partial_l v_m.$$

The second and third terms have order ≤ -1 and $-1/2$, respectively, thus they belong to F_{2j} after differentiation by D_t^{2j-2}. For the first term, we have

$$D_t r G^{lm} \partial_l v_m = -r G^{ij} \partial_i v_j G^{lm} \partial_l v_m - r a_1 v^i \partial_i r G^{lm} \partial_l v_m.$$

The first term satisfies the non-endpoint property while the second has order $-1/2$, thus both terms belong to F_{2j} after differentiation by D_t^{2j-2}. Next,

$$r G^{ml} [D_t, \partial_l] v_m = -\frac{r a_0}{\kappa (1 + \langle r \rangle} G^{pq} \partial_l v_q G^{ml} \partial_p v_m + \frac{r \langle r \rangle^{1+\frac{2}{\kappa}}}{(v^0)^3} G^{ml} v^p \partial_l r \partial_p v_m.$$

The second term has order $-1/2$ so it belongs to F_{2j} upon differentiation by D_t^{2j-2}. The first term has order zero, thus producing a top order (i.e., $j - 1$) term when differentiated by D_t^{2j-2}. Nevertheless, it has two $\partial^{\geq 1} v$ terms so it satisfies the non-endpoint property and hence it also belongs to F_{2j}.

We now turn to the other commutator in (5.14):

$$\begin{aligned} [D_t, r a_1 v^l \partial_l] r & = [D_t, r a_1 v^l] \partial_l r + r a_1 v^l [D_t, \partial_l] r \\ & = D_t r a_1 v^l \partial_l r + r \tfrac{\partial a_1}{\partial r} D_t r v^l \partial_l r + r \tfrac{\partial a_1}{\partial v_i} D_t v_i v^l \partial_l r + r a_1 D_t v^l \partial_l r \\ & \quad - \tfrac{r a_0 a_1}{\kappa \langle r \rangle} G^{pq} v^l \partial_l v_q \partial_p r + \tfrac{r a_1 \langle r \rangle^{1+\frac{2}{\kappa}}}{(v^0)^3} v^p v^l \partial_l r \partial_p r. \end{aligned}$$

The terms on the RHS have orders $\leq -1/2, -3/2, -1, -1, -1/2, -1$, respectively, so they all belong to F_{2j} upon differentiation by D_t^{2j-2}.

For the last two terms on RHS (5.14),

$$r^2 a_1^2 v^l v^m \partial_l \partial_m r + r^2 a_1 G^{ml} v^i \partial_i \partial_l v_m,$$

we see that they have orders -1 and $-1/2$, thus also belong to F_{2j} after differentiation by D_t^{2j-2}.

Therefore, writing $\approx$ for equality modulo terms that belong to F_{2j}, (5.14) becomes

$$\begin{aligned}
D_t^{2j} r &= D_t^{2j-2} D_t^2 r \\
&\approx D_t^{2j-2}\Big(r G^{ml} \partial_l (a_2 \partial_m r)\Big) \\
&= \sum_{\ell=0}^{2j-2} \binom{2j-2}{\ell} D_t^{2j-2-\ell} r D_t^{\ell}\Big(G^{ml} \partial_l (a_2 \partial_m r)\Big) \\
&= r D_t^{2j-2}\Big(G^{ml} \partial_l (a_2 \partial_m r)\Big) + \sum_{\ell=0}^{2j-2-1} \binom{2j-2}{\ell} D_t^{2j-2-\ell} r D_t^{\ell}\Big(G^{ml} \partial_l (a_2 \partial_m r)\Big).
\end{aligned}$$

In the second sum, we can further write

$$D_t^{2j-2-\ell} r D_t^{\ell}\Big(G^{ml} \partial_l (a_2 \partial_m r)\Big) = D_t^{2j-2-\ell} r D_t^{\ell}\Big(G^{ml} a_2 \partial_l \partial_m r + G^{ml} \partial_l a_2 \partial_m r\Big)$$

The term $D_t^{2j-2-\ell} r D_t^{\ell}(G^{ml} \partial_l a_2 \partial_m r)$ has order at most $j-3/2$ and can be absorbed into F_{2j}. For the first term, if D_t^{ℓ} hits $G^{ml} a_2$ we again obtain a term of order strictly less than $j-1$ that is part of F_{2j}. Finally, for the term

$$D_t^{2j-2-\ell} r G^{ml} a_2 D_t^{\ell} \partial_l \partial_m r,$$

we use that $\ell \leq 2j-2-1$ and (1.16a) to write

$$\begin{aligned}
D_t^{2j-2-\ell} r G^{ml} a_2 D_t^{\ell} \partial_l \partial_m r = &-G^{ml} a_2 D_t^{\ell} \partial_l \partial_m r D_t^{2j-3-\ell}\left(r G^{pq} \partial_p v_q\right) \\
&- G^{ml} a_2 D_t^{\ell} \partial_l \partial_m r D_t^{2j-3-\ell}\left(r a_1 v^p \partial_p r\right).
\end{aligned}$$

The first term contains a $\partial^{\geq 1} v$ and a $\partial^{\geq 2} r$ so it belongs to F_{2j}, whereas the second term has order at most $j-3/2$ so it belongs to F_{2j} as well. Hence, we have that

$$\begin{aligned}
D_t^{2j} r &\approx r D_t^{2j-2}\Big(G^{ml} \partial_l (a_2 \partial_m r)\Big) \\
&= r D_t^{2j-3}\Big(G^{ml} \partial_l (a_2 \partial_m D_t r)\Big) + r D_t^{2j-3}\Big([D_t, G^{lm} \partial_l (a_2 \partial_m \cdot)] r\Big).
\end{aligned}$$

We now compute the commutator on the second term on the RHS:

$$\begin{aligned}
[D_t, G^{lm} \partial_l (a_2 \partial_m \cdot)] r &= D_t\left(G^{lm} \partial_l (a_2 \partial_m r)\right) - G^{lm} \partial_l\left(a_2 \partial_m D_t r\right) \\
&= \frac{\partial G^{lm}}{\partial r} D_t r \partial_l (a_2 \partial_m r) + \frac{\partial G^{lm}}{\partial v_i} D_t v_i \partial_l (a_2 \partial_m r) \\
&\quad + G^{lm} D_t \partial_l (a_2 \partial_m r) - G^{lm} \partial_l (a_2 \partial_m D_t r).
\end{aligned}$$

The first and second terms on the RHS of the second equality have orders $\leq 1/2$ and 1, respectively, so they produce terms of order at most $j-3/2$ when hit by rD_t^{2j-3} and thus can be discarded. Continuing

$$\begin{aligned}[D_t, G^{lm}\partial_l(a_2\partial_m\cdot)]r &\approx G^{lm}D_t\partial_l(a_2\partial_m r) - G^{lm}\partial_l(a_2\partial_m D_t r)\\ &= G^{lm}\Big(a_2 D_t\partial_l\partial_m r - a_2\partial_l\partial_m D_t r\Big)\\ &\quad + G^{lm}\Big(D_t\partial_l a_2\partial_m r + \partial_l a_2 D_t\partial_m r + D_t a_2\partial_m\partial_l r - \partial_l a_2\partial_m D_t r\Big).\end{aligned}$$

All the terms inside the second parenthesis have orders at most 1 (thus giving order at most $j-3/2$ when hit by rD_t^{2j-3}) and can be discarded. The terms in the first parenthesis give $G^{lm}a_2[D_t,\partial_l\partial_m]r$. Continuing

$$\begin{aligned}[D_t, G^{lm}\partial_l(a_2\partial_m\cdot)]r \approx G^{lm}a_2[D_t,\partial_l\partial_m]r &= G^{lm}a_2\Big([D_t,\partial_l]\partial_m r + \partial_l\left([D_t,\partial_m]r\right)\Big)\\ &= G^{lm}a_2\Big(-\frac{a_0}{\kappa\langle r\rangle}G^{pq}\partial_l v_q\partial_p\partial_m r + \frac{\langle r\rangle^{1+\frac{2}{\kappa}}}{(v^0)^3}v^p\partial_l r\partial_p\partial_m r\Big)\\ &\quad + G^{lm}a_2\partial_l\Big(-\frac{a_0}{\kappa\langle r\rangle}G^{pq}\partial_m v_q\partial_p r + \frac{\langle r\rangle^{1+\frac{2}{\kappa}}}{(v^0)^3}v^p\partial_m r\partial_p r\Big).\end{aligned}$$

The second term in the first parenthesis has order 1. The second term in the second parenthesis produces, after differentiation by ∂_l, terms of order at most 1. Hence, the second terms in both parenthesis give order at most $j-3/2$ after we apply rD_t^{2j-3} and belong to F_{2j}. Moreover, when ∂_l in front of the second parenthesis hits the zero order coefficients in the first term it gives terms of order at most 1 which can again be discarded; when it hits $\partial_p r$ it produces a term that can be combined with the first term in the first parenthesis. Therefore, we have

$$\begin{aligned}D_t^{2j}r &\approx rD_t^{2j-3}\Big(G^{ml}\partial_l(a_2\partial_m D_t r)\Big) - rD_t^{2j-3}\Big(\frac{a_0 a_2}{\kappa\langle r\rangle}G^{lm}\partial_l\partial_m v_q G^{pq}\partial_p r\Big)\\ &\quad - 2rD_t^{2j-3}\Big(\frac{a_0 a_2}{\kappa\langle r\rangle}G^{lm}G^{pq}\partial_m v_q\partial_p\partial_l r\Big).\end{aligned} \tag{5.15}$$

The last term on RHS (5.15) has a $\partial v\partial^2 r$ factor. Hence it produces, after application of rD_t^{2j-3} either non-endpoint terms or terms of order $< j-1$, so it belongs to F_{2j}.

We now analyze the second term on RHS (5.15). We distribute D_t^{2j-3}. Whenever at least one D_t hits one of the zero order factors it results in a term of order $\leq j-3/2$ that can be absorbed into F_{2j}. Hence we are left with

$$\begin{aligned}&-r\frac{a_0 a_2}{\kappa\langle r\rangle}G^{lm}G^{pq}D_t^{2j-3}\Big(\partial_l\partial_m v_q\partial_p r\Big)\\ &\quad = -r\frac{a_0 a_2}{\kappa\langle r\rangle}G^{lm}G^{pq}\sum_{\ell=0}^{2j-3}\binom{2j-3}{\ell}D_t^{2j-3-\ell}\partial_l\partial_m v_q D_t^{\ell}\partial_p r.\end{aligned}$$

The terms in the sum with $l \neq 0$ belong to F_{2j}. For, after commuting D_t with ∂, we obtain either lower order terms or $\partial v \partial^2 r$ factors, so we are left with

$$\begin{aligned}-r\frac{a_0a_2}{\kappa\langle r\rangle}G^{lm}G^{pq}D_t^{2j-3}\partial_l\partial_m v_q\partial_p r = &-r\frac{a_0a_2}{\kappa\langle r\rangle}G^{lm}G^{pq}\partial_l\partial_m D_t^{2j-3}v_q\partial_p r\\ &- r\frac{a_0a_2}{\kappa\langle r\rangle}G^{lm}G^{pq}[D_t^{2j-3},\partial_l\partial_m]v_q\partial_p r.\end{aligned}$$

The first term on the RHS belongs to $\tilde{L}_1 s_{sj-2}$. The second term on the RHS belongs to F_{2j}. This can be seen by computing the commutator in similar fashion to what we did to compute $[D_t, G^{lm}\partial_l(a_2\partial_m\cdot)]$ (in fact, $[D_t^{2j-3}, G^{lm}\partial_l(a_2\partial_m\cdot)]r$ and $[D_t^{2j-3},\partial_l\partial_m]$ are the same modulo lower terms).

It remains to analyze the first term on RHS (5.15). We have

$$rD_t^{2j-3}\Big(G^{ml}\partial_l(a_2\partial_m D_t r)\Big) = G^{ml}\partial_l(a_2\partial_m D_t^{2j-2}r) + r[D_t^{2j-3}, G^{lm}\partial_l(a_2\partial_m\cdot)]D_t r.$$

The first term on the RHS belongs to $\tilde{L}_1 s_{2j-2}$. The term of order $j-1$ from the second term on the RHS is non-endpoint, as it comes from combining ∂v from the commutator with ∂v from $D_t r$.

We next consider the second term on (5.13). We have

$$\begin{aligned}-\frac{a_0}{\kappa\langle r\rangle}G^{lm}\partial_l r D_t^{2j-1}v_m &= -\frac{a_0}{\kappa\langle r\rangle}G^{lm}\partial_l r D_t^{2j-3}D_t(-a_2\partial_m r)\\ &= \frac{a_0}{\kappa\langle r\rangle}G^{lm}\partial_l r D_t^{2j-3}\Big(a_2\partial_m D_t r + [D_t, a_2\partial_m]r\Big).\end{aligned} \tag{5.16}$$

Consider the second term on RHS (5.16). Using arguments similar to above, we can show that all terms belong to F_{2j}, except for the term that corresponds to all D_t^{2j-3} hitting the ∂v from the commutator $[D_t,\partial_m]$, i.e., except for

$$\begin{aligned}-a_2\left(\frac{a_0}{\kappa\langle r\rangle}\right)^2 G^{lm}\partial_l r G^{pq}D_t^{2j-3}\partial_m v_q\partial_p r = &-a_2\left(\frac{a_0}{\kappa\langle r\rangle}\right)^2 G^{lm}\partial_l r G^{pq}\partial_m D_t^{2j-3}v_q\partial_p r\\ &- a_2\left(\frac{a_0}{\kappa\langle r\rangle}\right)^2 G^{lm}\partial_l r G^{pq}[D_t^{2j-3},\partial_m]v_q\partial_p.\end{aligned}$$

The commutator term can again be shown to belong to F_{2j} using the same sort of calculations as above. Modulo terms that can be absorbed into F_{2j}, the remaining term can be written as

$$\begin{aligned}a_2\frac{a_0}{\kappa\langle r\rangle}G^{lm}\partial_l r\partial_m\Big(-\frac{a_0}{\kappa\langle r\rangle}G^{pq}D_t^{2j-3}v_q\partial_p r\Big) = &\frac{a_2}{\kappa}G^{lm}\partial_l r\partial_m\Big(-\frac{a_0}{\kappa\langle r\rangle}G^{pq}D_t^{2j-3}v_q\partial_p r\Big)\\ &+ ra_3G^{lm}\partial_l r\partial_m\Big(-\frac{a_0}{\kappa\langle r\rangle}G^{pq}D_t^{2j-3}v_q\partial_p r\Big),\end{aligned}$$

where we used (3.2). The first term on the RHS belongs to $\tilde{L}_1 s_{2j-2}$ and the second one can be absorbed into F_{2j}.

The first term on RHS (5.16) is treated with similar ideas. We notice that the top order term in that expression is

$$\frac{a_0}{\kappa\langle r\rangle} G^{lm}\partial_l r a_2 \partial_m D_t^{2j-2} r = \frac{a_2}{\kappa} G^{lm}\partial_l r \partial_m D_t^{2j-2} r + r a_3 G^{lm}\partial_l r \partial_m D_t^{2j-2} r.$$

The first term belongs to $\tilde{L}_1 s_{sj-2}$ and the second one to F_{2j}.

The case $j = 2$ is done separately (since the definition of s_2 is different, recall (5.1)), but it follows essentially the same steps as above. Finally, the proof for G_{2j} is done with the same type of calculations employed above and we omit it for the sake of brevity.

To continue our analysis, we need some coercivity estimates for the $\tilde{L}_1$, respectively $\tilde{L}_2 + \tilde{L}_3$. □

Lemma 5.3. *Assume that A is small. Then*

$$\|s\|_{H^{2,\frac{1}{2\kappa}+\frac{1}{2}}} \lesssim \|\tilde{L}_1 s\|_{H^{0,\frac{1}{2\kappa}-\frac{1}{2}}} + \|s\|_{L^2(r^{\frac{1-\kappa}{\kappa}})}, \tag{5.17a}$$

$$\|w\|_{H^{2,\frac{1}{2\kappa}+1}} \lesssim \|(\tilde{L}_2 + \tilde{L}_3)w\|_{H^{0,\frac{1}{2\kappa}}} + \|w\|_{L^2(r^{\frac{1}{\kappa}})}. \tag{5.17b}$$

Here we remark that the lower order terms on the right play no role in the proof, and can be omitted if (s, w) are assumed to have small support (by Poincare's inequality), or if we use homogeneous norms on the left.

As a consequence of the second estimate above, we have

Corollary 5.4. *Assume that A is small. Then*

$$\|w\|_{H^{2,\frac{1}{2\kappa}+1}} \lesssim \|\tilde{L}_2 w\|_{H^{0,\frac{1}{2\kappa}}} + \|\mathrm{curl}\, w\|_{H^{1,\frac{1}{2\kappa}+1}} + \|w\|_{L^2(r^{\frac{1}{\kappa}})}.$$

In Section 6 will also need the following straightforward alternative form of the above result:

Corollary 5.5. *Assume that B is small. Then the same result as in Lemma 5.3 holds for the operators L_1, respectively $L_2 + L_3$.*

Here the smallness condition on B allows us to treat the differences $\tilde{L}_1 - L_1$, $\tilde{L}_2 - L_2$, $\tilde{L}_3 - L_3$ perturbatively.

Proof. We start with two simple observations. First of all, using a partition of unity one can localize the estimates to a small ball. We will assume this is done, and further we will consider the interesting case where this ball is around a boundary point x_0; the analysis is standard elliptic otherwise. We can assume that at x_0 on the boundary we have $\nabla r(x_0) = e_n$ so that in our small ball we have

$$|\nabla r - e_n| \lesssim A \ll 1. \tag{5.18}$$

Secondly, the smallness condition on A guarantees that the coefficients G and a_2 have a small variation in a small ball, and we can freeze these coefficients modulo perturbative errors. Hence, we will simply freeze them, and assume that a_2 and G

are constant. Then a_2 only plays a multiplicative role, and will be set to 1 for the rest of the argument.

A preliminary step in the proof is to observe that we have the weaker bounds

$$\|s\|_{H^{2,\frac{1}{2\kappa}+\frac{1}{2}}} \lesssim \|\tilde{L}_1 s\|_{H^{0,\frac{1}{2\kappa}-\frac{1}{2}}} + \|s\|_{H^{1,\frac{1}{2\kappa}-\frac{1}{2}}},$$
$$\|w\|_{H^{2,\frac{1}{2\kappa}+1}} \lesssim \|(\tilde{L}_2+\tilde{L}_3)w\|_{H^{0,\frac{1}{2\kappa}}} + \|w\|_{H^{1,\frac{1}{2\kappa}}}.$$

These bounds can be proved in a standard elliptic fashion by integration by parts, e.g. in the case of the first bound one simply starts with the integral representing $\|\tilde{L}_1 s\|^2_{H^{0,\frac{1}{2\kappa}}}$ and exchange derivatives between the two factors. The details are left for the reader.

In view of the above bounds, it suffices to show that

$$\|s\|_{H^{1,\frac{1}{2\kappa}-\frac{1}{2}}} \lesssim \|\tilde{L}_1 s\|_{H^{0,\frac{1}{2\kappa}-\frac{1}{2}}} + \|s\|_{L^2(r^{\frac{1-\kappa}{2\kappa}})}, \tag{5.19a}$$
$$\|w\|_{H^{1,\frac{1}{2\kappa}}} \lesssim \|(\tilde{L}_2+\tilde{L}_3)w\|_{H^{0,\frac{1}{2\kappa}}} + \|w\|_{L^2(r^{\frac{1}{\kappa}})}. \tag{5.19b}$$

For (5.19a), compute

$$\begin{aligned}
\int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} \partial_n s \tilde{L}_1 s\, dx &= \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} \partial_n s G^{ij} a_2 \left(r\partial_i\partial_j s + \frac{1}{\kappa}\partial_i r \partial_j s\right) dx \\
&= -\frac{1}{2}\int_{\Omega_t} r^{\frac{1}{\kappa}} a_2 \partial_n \left(G^{ij}\partial_i s \partial_j s\right) dx + \frac{1}{2}\int_{\Omega_t} r^{\frac{1}{\kappa}} a_2 \partial_n G^{ij}\partial_i s\partial_j s\, dx \\
&\quad - \int_{\Omega_t} r^{\frac{1}{\kappa}} \partial_n s \partial_i (a_2 G^{ij})\partial_j s\, dx \\
&\gtrsim \int_{\Omega_t} r^{\frac{1-\kappa}{\kappa}} a_2 G^{ij}\partial_i s\partial_j s\, dx + \int_{\Omega_t} r r^{\frac{1-\kappa}{\kappa}} |\partial s|^2\, dx,
\end{aligned}$$

which suffices, by the Cauchy-Schwarz inequality.

Now we consider (5.17b). As discussed above, we set $a_2 = 1$ and assume G is a constant matrix. We recall that $\tilde{L}_2$ has the form

$$(\tilde{L}_2 w)_i = G^{ml}\left(\partial_i(r\partial_m w_l) + \frac{1}{\kappa}\partial_m r \partial_i w_l\right) \tag{5.20}$$

while $\tilde{L}_3$ is given by

$$(\tilde{L}_3 w)_i = r^{-\frac{1}{\kappa}} G^{ml}\partial_l \left(r^{1+\frac{1}{\kappa}}(\partial_m w_i - \partial_i w_m)\right) \tag{5.21}$$

Then a direct computation shows that

$$r^{\frac{1}{\kappa}}((\tilde{L}_2+\tilde{L}_3)w)_i = \partial_l(G^{ml} r^{1+\frac{1}{\kappa}}\partial_m w_i) + r^{\frac{1}{\kappa}}(\partial_l r G^{lm}\partial_m w_i - \partial_l G^{lm}\partial_i w_m)$$

We will take advantage of the covariant nature of this operator in order to simplify it. Interpreting G as a dual metric and w as a one form, we see that the above operator viewed as a map from one forms to one forms is invariant with respect to linear changes of coordinates. Here we are interested in changes of coordinates which preserve the surfaces $x_n = const$. But even with this limitation, it is possible to choose a linear change of coordinates, namely the semigeodesic coordinates relative to the surface $x_n = 0$,

$$y' = Ax' + bx_n, \qquad y_n = x_n$$

so that the metric G becomes a multiple of the identity. Then the estimate (5.19b) reduces to its euclidean counterpart, which is discussed in detail in [14] in the corresponding nonrelativistic context.

To finish the proof of Theorem 5.1, we will establish

$$\|(s_{2j-2}, w_{2j-2})\|_{\mathcal{H}^{2k-2j+2}} \lesssim \|(s_{2j}, w_{2j})\|_{\mathcal{H}^{2k-2j}} + \varepsilon\|(r, v)\|_{\mathcal{H}^{2k}}, \ 1 \leq j \leq k, \tag{5.22}$$

where $\varepsilon > 0$ is sufficiently small. We are using ε here to include two types of small error terms: (a) the terms that we estimate using $O(A)$ as well as (b) the terms that have an extra factor of r and for which we can use smallness of r near the boundary; the latter type arise from the last term of (1.16a). Concatenating these estimates we then obtain the conclusion of the theorem.

To prove (5.22), we first consider $\|(F_{2j}, G_{2j})\|_{\mathcal{H}^{2k-2j}}$. Using our interpolation inequalities, the non-endpoint property, and the structure of (F_{2j}, G_{2j}) described in in Lemma 5.2, we obtain

$$\|(F_{2j}, G_{2j})\|_{\mathcal{H}^{2k-2j}} \lesssim \varepsilon\|(r, v)\|_{\mathcal{H}^{2k}}.$$

It remains to handle the term $\|(s_{2j}, w_{2j})\|_{\mathcal{H}^{2k-2j}}$. For $j = k$ the desired estimate is a direct consequence of Lemma 5.3.

We move to treat the case $2 \leq j < k$. The idea is to apply Lemma 5.3 with s_{2j-2} and w_{2j-2} replaced by suitable weighted derivatives of themselves. More precisely, we set

$$\begin{cases} s := L s_{2j-2} \\ w := L w_{2j-2}, \end{cases}$$

where

$$L = r^a \partial^b, \quad 2a \leq b \leq 2(k-j).$$

Applying L to (5.12a), we obtain

$$L s_{2j} = \tilde{L}_1 L s_{2j-2} + [L, \tilde{L}_1] L s_{2j-2} + L F_{2j}.$$

The term $L F_{2j}$ can again be dealt with using Lemma 5.2, as above. Thus we focus on the commutator. To analyze it, we consider induction on a, starting at $a = 0$, and observe the following:

- All terms where at least one r factor gets differentiated twice are non-endpoint terms and can be estimated by interpolation.
- The terms where two r factors are differentiated are handled by the induction on a.
- Terms where only one r gets differentiated are also handled by induction on a unless $a = 0$.

Therefore, all terms in the commutator where $a > 0$ are perturbative terms. We now focus on the case $a = 0$.

Consider a frame (x', x^n) in Minkowski space that is adapted to a point near the boundary in the sense that

$$|\partial' r| \lesssim A, \quad |\partial_n r - 1| \lesssim A.$$

Then, all terms in the commutator with tangential derivatives only are error terms. For terms involving ∂_n, we find

$$\begin{aligned}[\partial_n^b, \tilde{L}_1]s &\approx b a_2 G^{ij} \partial_i \partial_j \partial_n^{b-1} s \\ &\approx b a_2 G^{ij} \partial_i r \partial_j \partial_n^b s + b a_2 G^{ni'} \partial_{i'} \partial_n^b s + b a_2 G^{i'j'} \partial_{i'} \partial_{j'} \partial_n^{b-1} s,\end{aligned}$$

where primed indices run from 1 to $n-1$. The last two terms on the RHS can be treated by yet another induction, this time over b. The first term on the RHS can be combined back with $\tilde{L}_1$, yielding $\partial_n^b \tilde{L}_1 \approx \tilde{L}_1^b \partial_n^b$, where

$$\tilde{L}_1^b = r a_2 G^{ij} \partial_i \partial_j s + a_2 \left(\frac{1}{\kappa} + b \right) G^{ij} \partial_i r \partial_j s.$$

The operator $\tilde{L}_1^b$ has a similar structure to $\tilde{L}_1$, and an inspection in the proof of Lemma 5.3 shows that the corresponding coercive estimate for s holds with $\tilde{L}_1^b$ in place of $\tilde{L}_1$.

The above argument works for $j \geq 2$ in that (5.12a) is valid only for $j \geq 2$. However, a minor change in the above using the definition s_2 yields the result also for $j = 1$. This takes care of the s terms in (5.22); the proof for the w terms is similar. □

5.3. Energy estimates

In this Section we establish

Theorem 5.6. *The energy functional E^{2k} defined in (5.7) satisfies the following estimate:*

$$\frac{d}{dt} E^{2k}(r, v) \lesssim_A B \|(r, v)\|^2_{\mathcal{H}^{2k}}.$$

Proof. In view of Eqs. (5.2a)–(5.2b) and (5.5), the the energy estimates for the linearized equation in Section 3, and estimates for transport equations, it suffices to show that the terms f_{2k}, g_{2k} and h_{2k}, given by (5.3a), (5.3b), and (5.6), respectively, are perturbative, i.e., they satisfy the estimate

$$\|(f_{2k}, g_{2k})\|_{\mathcal{H}} + \|h_{2k}\|_{L^2(r^{\frac{1}{\kappa}})} \lesssim B \|(r, v)\|_{\mathcal{H}^{2k}}.$$

To prove this bound we need to understand the structure of (f_{2k}, g_{2k}), respectively h_{2k}. □

Lemma 5.7. *Let $k \geq 1$. Then source terms f_{2k} and g_{2k} in the linearized Eq. (5.2) for (s_{2k}, w_{2k}), given by (5.3a)-(5.3b) are multilinear expressions in $(r, \nabla v)$, with coefficients which are smooth functions of (r, v), which have order $\leq k - \frac{1}{2}$, respectively $\leq k$, with exactly $2k + 1$ derivatives, and which are not endpoint, in the sense that there is no single factor in f_{2k}, respectively g_{2k} which has order larger that $k - 1$, respectively $k - \frac{1}{2}$.*

Similarly, the source term h_{2k} in the vorticity transport Eq. (5.5), given by (5.6), has the same properties as g_{2k} above.

Once the lemma is proved, arguing similarly to Section 5.2, we see that this suffices to apply our interpolation results in Propositions 2.6, 2.9 and 2.10 and obtain the desired bound. Here we remark that a scaling analysis shows that in the interpolation estimates we need to use at most one B control norm, with equality exactly in the case of terms of highest order. One should also compare with the situation in the similar computation in [14], where no lower order terms appear. Hence, the poof of the theorem is concluded once we prove the above lemma.

Proof of Lemma 5.7. Consider first f_{2k}. The fact that all terms in f_k have order at most $k - \frac{1}{2}$ is obvious. The non-endpoint property can be understood as asking that there are no derivatives of order $2k + 1$, and that, in addition, for the terms of maximum order, they have at least two factors of the form $\partial^{2+} r$ or $\partial^{1+} v$. Notably, this excludes any terms of the form

$$f(r, v) r^{k+1-j} (\nabla r)^j \partial^{2k+1-j} v, \qquad 0 \leq j \leq k + 1.$$

A similar reasoning applies for g_{2k} and h_{2k}, where the forbidden terms are those with a factor with $2k + 1$ derivatives, as well as those of maximum order of the form

$$f(r, v) r^{k-j} (\nabla r)^j \partial^{2k+1-j} r, \qquad 0 \leq j \leq k.$$

We start with a simple observation, which is that, if in (5.3a) or (5.3b), any derivative falls on a coefficient such as G, a_0, a_1, or a_2, then we obtain lower order terms which automatically satisfy the above criteria. Thus, for the purpose of this Lemma we can treat these coefficients as constants.

A second observation is that there are no factors with $2k+1$ derivatives in either s_{2k} or w_{2k}, due to the commutator structures present in both (5.3a) or (5.3b). This directly allows us to discard all lower order terms, and in particular those containing a_1 and a_3. By the same token we can set $a_0 = 1$ and $\langle r \rangle = 1$.

Given the above observations, it suffices to consider the reduced expressions

$$f_{2k}^{reduced} = G^{ij} [r \partial_i, D_t^{2k}] v_j - \frac{1}{\kappa} G^{ij} D_t \, (\partial_i r) \, D_t^{2k-1} v_j \tag{5.23}$$

$$\begin{aligned} (g_{2k}^{reduced})_i = \; & a_2 (D_t^{2k-1} [\partial_i, D_t] r - \frac{a_0}{\kappa \langle r \rangle} G^{jl} \partial_j r \partial_i D_t^{2k-1} v_l) \\ & + a_2 ([\partial_i, D_t^{2k-1}] D_t r - \frac{1}{\kappa} G^{ml} \partial_i \partial_m r D_t^{2k-1} v_l), \end{aligned} \tag{5.24}$$

Consider $f_{2k}^{reduced}$ first. When commuting ∂ and D_t^{2k}, this produces at least one ∂v, so $[r\partial_i, D_t^{2k}]v_j$ is not an endpoint term. Similarly, $D_t\,(\partial_i r)$ has order $1/2$ so the second expression is also not an endpoint term.

We now investigate $g_{2k}^{reduced}$. Neither of the first two terms is perturbative, but we have a leading order cancellation between them, based on the relations

$$[D_t, \partial_i] = -\partial_i\left(\frac{v^j}{v^0}\right)\partial_j,$$

and

$$\partial_i\left(\frac{v^j}{v^0}\right) = \frac{a_0}{\kappa\langle r\rangle}G^{jl}\partial_i v_l - \frac{\langle r\rangle^{1+\frac{2}{\kappa}}}{(v^0)^3}v^j\partial_i r. \tag{5.25}$$

The contribution of the second term is lower order and thus perturbative. The contribution of the first term is combined with the second term in (5.24) to obtain a commutator structure

$$\left[D_t^{2k-1}, \frac{a_0}{\kappa\langle r\rangle}G^{jl}\partial_j r\partial_i\right]v_l,$$

which yields only balanced terms.

The third term in (5.24) is also balanced due to the commutator structure, while the last term has a direct good factorization.

We next move to h_{2k}. From (5.6) we see that we are commuting $r^a\partial^b$ with either D_t or ∂v, so we always obtain ∂v factors that give non-endpoint terms. The only possible exception is when all derivatives in the commutator with D_t are applied to the r term in v^0. But this yields a lower order term. □

6. Construction of regular solutions

In this section we provide the first step in our proof of local well-posedness, namely, here we present a constructive proof of regular solutions. The rough solutions are obtained in the last section as unique limits of regular solutions.

Given an initial data $(\mathring{r}, \mathring{v})$ with regularity

$$(\mathring{r}, \mathring{v}) \in \mathbf{H}^{2k},$$

where k is assumed to be sufficiently large, we will construct a local in time solution, bounded in $\mathbf{H}^{2k}$, with a lifespan depending on the $\mathbf{H}^{2k}$ size of the data.

6.1. Construction of approximate solutions

We discretize the problem with a time-step $\epsilon > 0$. Then, given an initial data $(\mathring{r}, \mathring{v}) \in \mathbf{H}^{2k}$, our objective is to produce a discrete approximate solution $(r(j\epsilon), v(j\epsilon))$, with properties as follows:

- (Norm bound) We have

$$E^{2k}(r((j+1)\epsilon), v((j+1)\epsilon)) \leq (1 + C\epsilon)E^{2k}(r((j\epsilon), v(j\epsilon)).$$

- (Approximate solution)

$$\begin{cases} r((j+1)\epsilon) - r(j\epsilon) + \epsilon \left[v^m \partial_m r + rG^{ml} \partial_m v_l + r a_1 v^l \partial_l r \right](j\epsilon) = O(\epsilon^2) \\ v_i((j+1)\epsilon) - v_i(j\epsilon) + \epsilon \left[v^m \partial_m v_i + a_2 \partial_i r \right](j\epsilon) = O(\epsilon^2). \end{cases}$$

The first property will ensure a uniform energy bound for our sequence. The second property will guarantee that in the limit we obtain an exact solution. There we use a weaker topology, where the exact choice of norms is not so important (e.g. C^2).

Having such a sequence of approximate solutions, it is straightforward to produce, as the limit on a subsequence, an exact solution (r, v) on a short time interval which stays bounded in the above topology. The key point is the construction of the above sequence. It suffices to carry out a single step:

Theorem 6.1. *Let k be a large enough integer. Let $(\mathring{r}, \mathring{v}) \in \mathbf{H}^{2k}$ with size*

$$E^{2k}(\mathring{r}, \mathring{v}) \leq M,$$

and $\epsilon \ll_M 1$. Then there exists a one step iterate $(\check{r}, \check{v})$ with the following properties:

(1) (Norm bound) We have

$$E^{2k}(\check{r}, \check{v}) \leq (1 + C(M)\epsilon) E^{2k}(\mathring{r}, \mathring{v}),$$

(2) (Approximate solution)

$$\begin{cases} \check{r} - \mathring{r} + \epsilon[\mathring{v}^i \partial_i r + \mathring{r}\mathring{G}^{ij} \partial_i \mathring{v}_j + \mathring{r}\mathring{a}_1 \mathring{v}^i \partial_i \mathring{r}] = O(\epsilon^2) \\ \check{v}_i - \mathring{v}_i + \epsilon[\mathring{v}^j \partial_j \mathring{v}_i + \mathring{a}_2 \partial_i \mathring{r}] = O(\epsilon^2), \end{cases}$$

where $\mathring{G}$, $\mathring{a}_1$, and $\mathring{a}_2$ are G, a_1, and a_2 evaluated at $(\mathring{r}, \mathring{v})$.

The strategy for the proof of the theorem is the same as in the last two authors' previous paper [14], by splitting the time step into three:

- Regularization,
- Transport,
- Euler's method,

where the role of the first two steps is to improve the error estimate in the third step. The regularization step is summarized in the next Proposition:

Proposition 6.2. *Given $(\mathring{r}, \mathring{v}) \in \mathbf{H}^{2k}$, there exist regularized versions (r, v) with the following properties:*

$$r - \mathring{r} = O(\epsilon^2), \qquad v - \mathring{v} = O(\epsilon^2),$$

respectively

$$E^{2k}(r, v) \leq (1 + C\epsilon) E^{2k}(\mathring{r}, \mathring{v}),$$

and

$$\|(r, v)\|_{\mathcal{H}^{2k+2}} \lesssim \epsilon^{-1} M.$$

Proof. We repeat the construction in [14]. There are only a few minor differences, namely

- The self-adjoint operators L_1, L_2 and L_3 there are replaced by their counterparts in this paper, i.e., (3.7a), (3.10), and (3.9) (recall that $L_1 = \hat{L}_1$ and $L_3 = \tilde{L}_3$).
- Using (3.12), relations similar to (5.12) continue to hold for the self-adjoint operators. Thus, the approximate relations between (s_{2k}, w_{2k}) and (s_{2k}^-, w_{2k}^-) in Section 6 of [14] also hold here.
- The elliptic estimates of Lemma 5.3 hold for L_1 and L_2, L_3, with essentially the same proof.

Aside from the above minor differences, the most important observation in invoking the proof given in [14] is that the counterpart of Lemma 6.3 in [14] still holds with a minor change. For convenience we state here its counterpart (below, Ds_{2k} and Dw_{2k} are the differentials of s_{2k} and w_{2k} as functions of r and v):

Lemma 6.3. *We have the algebraic relations*

$$\begin{cases} Ds_{2k}(\check{r}, \check{v})(\mathring{r} - \check{r}, \mathring{v} - \check{v}) = (L_1(\check{r}))^k(\mathring{r} - \check{r}) + \tilde{F}_{2k} \\ Dw_{2k}(\check{r}, \check{v})(\mathring{r} - \check{r}, \mathring{v} - \check{v}) = (L_2(\check{r}))^k(\mathring{v} - \check{v}) + \tilde{G}_{2k}, \end{cases}$$

where the error terms $(\tilde{F}_{2k}, \tilde{G}_{2k})$ are linear in $(\mathring{r} - \check{r}, \mathring{v} - \check{v})$,

$$\tilde{F}_{2k} = D^1_{2k}(\check{r}, \check{v})(\mathring{r} - \check{r}, \mathring{r} - \check{v}), \qquad \tilde{G}_{2k} = D^2_{2k}(\check{r}, \check{v})(\mathring{r} - \check{r}, \mathring{r} - \check{v}).$$

Their coefficients are multilinear differential expressions in $(\check{r}, \check{v})$, have order at most $k-1$, respectively $k - \frac{1}{2}$, and whose monomials fall into one of the following two classes:

i) *Have maximal order but contain at least one factor with order > 0, i.e. $\partial^{2+}\check{r}$ or $\partial^{1+}\check{v}$, or*
ii) *Have order strictly below maximum.*

By comparison, the similar relations in Lemma 6.3 in [14] are homogeneous, so only terms of type (i) arise in the error terms. Here our equations are no longer homogeneous, and lower order terms do appear. In particular, we note that all the contributions coming from the last term in the first equation (1.16a) belong to the class (ii) above. This is correlated with and motivates the fact that this term was neglected in our definition of the operator L_1.

With these observations in mind, the proof given in [14] applies directly here. We now use Proposition 6.2 in order to prove Theorem 6.1.

Proof of Theorem 6.1. For the transport step, we define

$$\check{x}^i = x^i + \varepsilon \frac{v^i(x)}{v^0(x)},$$

where, in agreement with the our definition of the material derivative, we iterate the coordinates by flowing with v^i/v^0, and not simply v^i.

Then we carry out the Euler step, and define (r^0, v^0) by

$$\begin{cases} \check{r}(\check{x}) = r(x) - \varepsilon\Big[rG^{ij}\partial_i v_j + ra_1 v^i \partial_i r\Big](x) \\ \check{v}_i(\check{x}) = v_i(x) - \varepsilon\,[a_2\partial_i r]\,(x). \end{cases}$$

To show that $(\check{r}, \check{v})$ have the properties in the Theorem, the argument is completely identical to the one in [14]. □

7. Rough solutions and continuous dependence

The last task of the current work is to construct rough solutions as limits of smooth solutions, and conclude the proof of Theorem 1.2. Fortunately, the arguments in the preceding paper [14] by the last two authors for the similar part of the results apply word for word. This is despite the fact there are several differences between the two problems that play a role on how the energy estimates are obtained, as well as on how uniqueness is proved. However, the functional framework developed in [14] and also implemented here does not see these differences. Furthermore, the proof of the similar result in [14] only uses (i) the regularization procedure in Section 2, (ii) the difference bounds of Theorem 1.1, and (iii) the energy estimates of Theorem 1.3, without any reference to their proof.

Thus, in our current result we rely on the same succession of steps as in the non-relativistic companion work of the last two authors [14], which we briefly outline here for the reader. These steps are

1. Regularization of the initial data. We regularize the initial data; this is achieved by considering a family of dyadic regularizations of the initial data as described in Section 2. These data generate corresponding smooth solutions by Theorem 1.2. For these smooth solutions we control on the one hand higher Sobolev norms $\mathcal{H}^{2k+2j}$ using our energy estimates in Theorem 1.3, and on the other hand the L^2-type distance between consecutive solutions, which is at the level of the $\mathcal{H}$ norms, by Theorem 1.1.

2. Uniform bounds for the regularized solutions. To prove these bounds we use a bootstrap argument on our control norm B, where B is time dependent. The need for an argument of this kind is obvious. Once we have the regularized data sets $(\mathring{r}^h, \mathring{v}^h)$, we also have the corresponding smooth solutions (r^h, v^h) generated by the smooth data $(\mathring{r}^h, \mathring{v}^h)$. A-priori these solutions exist on a time interval that depends on h. Instead, we would like to have a lifespan bound which is independent of h. This step requires closing the bootstrap argument via the energy estimates already obtained in Section 5.

3. Convergence of the regularized solutions. We obtain the convergence of the regular solutions (r^h, v^h) to the rough solution (r, v) by combining the high and the low regularity bounds directly. This yields rapid convergence in all $\mathbf{H}^{2k'}$ spaces below the desired threshold, i.e. for $k' < k$. Here we rely primarily on results in Section 4, namely Theorem 1.1.

4. Strong convergence. Here we prove the convergence of the smooth solutions to the rough limit in the strong topology $\mathbf{H}^{2k}$. To gain strong convergence in $\mathbf{H}^{2k}$ we use frequency envelopes to more accurately control both the low and the high Sobolev norms above. This allows us to bound differences in the strong $\mathbf{H}^{2k}$ topology. A similar argument yields continuous dependence of the solutions in terms of the initial data, also in the strong topology. For more details we refer the reader to [14].

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