Subject: [External] reading materials
Date: Tuesday, July 26, 2022 at 9:19:54 AM Uzbekistan Standard Time
From: szhitomi@math.uci.edu
To: Ibragimov, Zair
Attachments: Ch9.pdf, Ch10.pdf, RAGE.pdf, ICM-175.pdf, cdm2.pdf, SvetlanaJitomirskaya20200302.pdf

External Email Use Caution and Confirm Sender

Hi Zair, please forward to the students.

Thanks!

Svetlana

Dear all,

Here are some reading materials for those of you who are actively listening:

1. The 2022 ICM proceedings paper of my plenary talk, for motivation (I am trying to make the video available, but it may, unfortunately, take a few days)

2. my 2018 lecture notes that we will partially follow (not today though); they also briefly mention most of the necessary background (some of which we will go over in detail today/tomorrow)

3. my 2019 CDM lecture notes that also contain some further background/motivation

3. Barry Simon's 1996 Wonderland paper, over most of which we will go over today https://nam10.safelinks.protection.outlook.com/?url=https%3A%2F%2Furldefense.com%2Fv3%2F__http%3A%2F%2Fwww.math.caltech.edu%2FSimonPapers%2F234.pdf__%3B!!GF3VTAzAMGBM8A!y4Y04rUQsKELeYx0skNzfeKSf9LL8pGmjRynLWGWPH5Mf4sOf9HUfIReMTXF8JUINIrv3JChU0rKVW8lFoK8SPnEwp1i%24&data=05%7C01%7Czibragimov%40fullerton.edu%7Caa33e7b3ca484279987c08da6ebe7bd3%7C82c0b871335f4b5c9ed0a4a23565a79b%7C0%7C0%7C637944062160855171%7CUnknown%7CTWFpbGZsb3d8eyJWIjoiMC4wLjAwMDAiLCJQIjoiV2luMzIiLCJBTiI6Ik1haWwiLCJXVCI6Mn0%3D%7C3000%7C%7C%7C&sdata=lk6xJOvT1RtDIiSNN3Cjj2wtLVfAqvCLhN1twB9lnqc%3D&reserved=0

4. Two parts of Cycon, Froese, Kirsch, Simon, from which some material will be taken today and tomorrow, and an extra one that will be mentioned

5. My 1999 paper and Avila's 2010 paper that we plan to discuss to some extent

https://nam10.safelinks.protection.outlook.com/?url=https%3A%2F%2Furldefense.com%2Fv3%2F__https%3A%2F%2Fwww.jstor.org%2Fstable%2F121066__%3B!!GF3VTAzAMGBM8A!y4Y04rUQsKELeYx0skNzfeKSf9LL8pGmjRynLWGWPH5Mf4sOf9HUfIReMTXF8JUINIrv3JChU0rKVW8lFoK8SONBRRtz%24&data=05%7C01%7Czibragimov%40fullerton.edu%7Caa33e7b3ca484279987c08da6ebe7bd3%7C82c0b871335f4b5c9ed0a4a23565a79b%7C0%7C0%7C637944062160855171%7CUnknown%7CTWFpbGZsb3d8eyJWIjoiMC4wLjAwMDAiLCJQIjoiV2luMzIiLCJBTiI6Ik1haWwiLCJXVCI6Mn0%3D%7C3000%7C%7C%7C&sdata=xVPltFZn%2Fgw0ej3TZ%2F4MUX%2BSVcQ%2B%2BBoNxBDOXDcNX4E%3D&reserved=0

https://nam10.safelinks.protection.outlook.com/?url=https%3A%2F%2Furldefense.com%2Fv3%2F__https%3A%2F%2Fprojecteuclid.org%2Fjournals%2Facta-mathematica%2Fvolume-215%2Fissue-1%2FGlobal-theory-of-one-frequency-Schr**Adinger-operators%2F10.1007%2Fs11511-015-0128-7.full__%3Bw7Y!!GF3VTAzAMGBM8A!y4Y04rUQsKELeYx0skNzfeKSf9LL8pGmjRynLWGWPH5Mf4sOf9HUfIReMTXF8JUINIrv3JChU0rKVW8lFoK8SCzg84VJ%24&data=05%7C01%7Czibragimov%40fullerton.edu%7Caa33e7b3ca484279987c08da6ebe7bd3%7C82c0b871335f4b5c9ed0a4a23565a79b%7C0%7C0%7C637944062160855171%7CUnknown%7CTWFpbGZsb3d8eyJWIjoiMC4wLjAwMDAiLCJQIjoiV2luMzIiLCJBTiI6Ik1haWwiLCJXVCI6Mn0%3D%7C3000%7C%7C%7C&sdata=QlRnP6UelbkwU56n7i9yic%2BpbAJdPWmJECKQNdIfrOY%3D&reserved=0

Technically, all that is necessary will be said in the lectures, but I do understand it is too fast for those who are seeing it for the first time. Don't forget that Simon (and also Alberto and Omar) are providing the tutorial today. It is best to be on top of the previous lecture material for better understanding going forward, especially the yesterday's lecture which contained the key preliminaries.

Finally, if anyone needs an access to Reed-Simon, please let me know

See you this afternoon!

Lana

ONE-DIMENSIONAL QUASIPERIODIC OPERATORS: GLOBAL THEORY, DUALITY, AND SHARP ANALYSIS OF SMALL DENOMINATORS

SVETLANA JITOMIRSKAYA

ABSTRACT

Spectral theory of one-dimensional discrete one-frequency Schrödinger operators is a field with the origins in and strong ongoing ties to physics. It features a fascinating competition between randomness (ergodicity) and order (periodicity), which is often resolved on a deep arithmetic level. This leads to an especially rich spectrum of phenomena, many of which we are only beginning to understand. The corresponding analysis involves, in particular, dealing with small denominator problems. It has led to the development of non-KAM methods in this traditionally KAM domain, and to results completely unattainable by the old techniques, also in a number of other settings. This article accompanies the author's lecture at the International Congress of Mathematicians 2022. It covers several related recent developments.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 47B36; Secondary 37C55, 82B26, 37D25

KEYWORDS

Quasiperiodic operators, small denominators, Lyapunov exponents, spectrual theory, localization

INTERNATIONAL CONGRESS OF MATHEMATICIANS
2022 JULY 6–14

Published by EMS Press. DOI 10.4171/ICM2022/175
Proc. Int. Cong. Math. 2022, Vol. ?, pp. 2–32

One-dimensional discrete one-frequency Schrödinger operators

$$(H_{V,\alpha,x}u)_n := u_{n-1} + u_{n+1} + V(x+n\alpha)u_n,$$
$$u \in \ell^2(\mathbb{Z}), \quad \alpha \in \mathbb{T} := \mathbb{R}\backslash\mathbb{Q},\ x \in \mathbb{T},\ V : \mathbb{T} \to \mathbb{R}, \tag{0.1}$$

and related questions of the dynamics of quasiperiodic cocycles have not been underrepresented at the ICMs. As I remember, roughly within the last 25 years, there were sectional lectures by H. Eliasson in 1998, myself in 2002, B. Fayad, R. Krikorian, and J. You in 2018, as well as plenary lectures by A. Avila in 2010 and 2014, devoted either in part or in full to this topic.

The field itself is not at all new. It may be seen as having been originated in physics when Peierls [103] and later his student Harper [61] studied the tight-binding two-dimensional electron in a uniform perpendicular magnetic field (also known as the Harper model) and derived the by now iconic family $H_{2\lambda \cos,\alpha,x}$ that we now, following Barry Simon [105], call the almost Mathieu operator. It remains hugely popular in physics, being directly linked to several remarkable experimental discoveries and Nobel prizes, providing, in particular, the theoretical underpinning of the Quantum Hall Effect, as proposed by D. J. Thouless in 1983 (see, e.g. [18,19]). A Google search for "Harper's model physics" leads to many thousands of hits.

The field may also be seen as having been originated in a numerical experiment, as the interest was picked after Douglas Hofstadter came up with what we now call the Hofstadter's butterfly [64]—a beautiful numerically produced fractal (Figure 1), discovered even before the word "fractal" was coined by Benoit Mandelbrot. Finally, the field may be seen as having been originated from the first application of KAM in the spectral theory—a pioneering work of Dinaburg and Sinai [37], that preceded Hofstadter. The field has consistently attracted top mathematical physicists (e.g., Bellissard, Deift, Simon, Sinai, Spencer), dynamicists (e.g., Avila, Eliasson, Herman, Krikorian, You), and analysts (e.g., Bourgain, Eliott, Sarnak, Schlag). Indeed, it turned out to be a fantastic ever-expanding playground for the analysts and dynamicists alike, leading to strong cross-fertilization of ideas that have a tendency to later expand to other subjects. Jean Bourgain wrote a book [28] devoted to analytic, mostly one-dimensional, quasiperiodic operators that summarized significant new understanding achieved around the turn of the century, where the work of Jean and collaborators was central.

It is therefore all the more surprising that as of the time of this writing it seems that the field is on the verge of further significant breakthroughs, with our current understanding covering just the tip of an exciting iceberg. Given the remarkable current momentum, we will refrain from making an attempt at an overview of the vast past literature, neither even very recent nor a number of important milestones, and will concentrate instead only on two selected topics that enjoyed significant recent advances and hold a particular promise to shape some of the future discourse.

For the review up to about five years ago, see [82], and for various fine issues related to continuity of the Lyapunov exponents, featuring, in particular, very important work by M. Goldstein and W. Schlag, see the recent book by P. Duarte and S. Klein [38]. The 2018

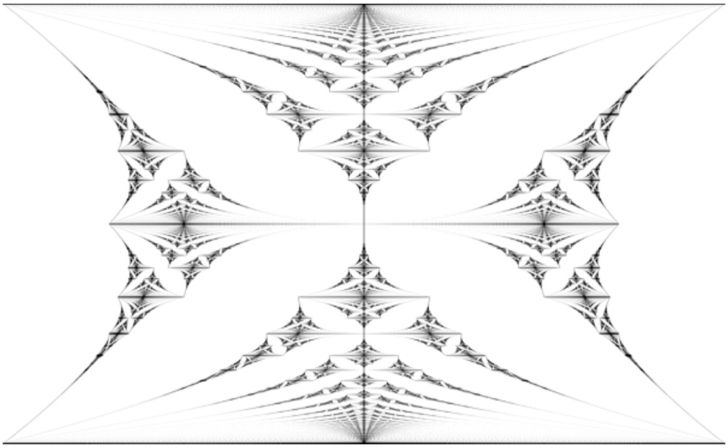

FIGURE 1
Hofstadter's butterfly.

ICM proceedings by J. You [117] summarize, among other things, the quantitative reducibility breakthrough developed in his group, that has led to a number of powerful consequences. There are also recent expositions [68, 80] that include some further remarkable results of roughly the last decade that could not make it into this article.

1. SPECTRAL THEORY MEETS (DUAL) DYNAMICS

Quasiperiodic operators (0.1) are, of course, a particular case of one-dimensional discrete ergodic Schrödinger operators

$$(H_x u)_n := u_{n-1} + u_{n+1} + V(T^n x)u_n, \quad u \in \ell^2(\mathbb{Z}), \tag{1.1}$$

where $x \in X$, and (X, μ, T) is an ergodic dynamical system. Operators with ergodic potentials (also in the continuum or in a more general multidimensional/covariant setting) always have spectra and closures of the other spectral components constant for μ-a.e. x [95, 102]. In case of the minimal underlying dynamics, such as, e.g., the irrational rotation of the circle in (0.1), the spectra [21] and absolutely continuous spectra in the one-dimensional case [97] are constant for *all* x. In contrast, the point and singular continuous parts (that are constant a.e.) can depend sensitively on x. It is an interesting problem, usually attributed to B. Simon, and open even in the setting of (0.1) whether this still holds when they are combined together (see Problem 6 in [67]).

The spectral theory of one-dimensional ergodic Schrödinger operators (1.1) is deeply connected to the study of linear cocycles over corresponding underlying dynamics. By an $\mathrm{SL}(2, \mathbb{R})$ cocycle, we mean a pair (T, A), where $T : X \to X$ is ergodic, A is a measurable 2×2 matrix-valued function on X and $\det A = 1$.

We can regard it as a dynamical system on $X \times \mathbb{R}^2$ with

$$(T, A) : (x, f) \mapsto \big(Tx, A(x)f\big), \quad (x, f) \in X \times \mathbb{R}^2.$$

A one-parameter family of Schrödinger cocycles over (X, μ, T), indexed by the energy $E \in \mathbb{C}$, is given by $(T, A) : (X, \mathbb{R}^2) \mapsto (X, \mathbb{R}^2)$ where $(T, A) : (x, y) \mapsto (Tx, A(x, E)y)$, and $A \in \mathrm{SL}(2, \mathbb{C})$ is the transfer-matrix

$$A(x, E) := \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix},$$

with $x \in X$, $y \in \mathbb{R}^2$, and $E \in \mathbb{C}$. The eigenvalue equation $Hu = Eu$ can be rewritten dynamically as

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A\big(T^n x, E\big) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}.$$

The (top) Lyapunov exponent is then defined as $L(E) := \lim_{n\to\infty} \int \frac{1}{n} \ln \|A_n(x, E)\| \, d\mu$, where

$$A_n(x, E) := \prod_{i=n-1}^{0} A(T^i x, E). \tag{1.2}$$

Two classical results link dynamics/Lyapunov exponents to the spectral theory of ergodic operators:

- (Johnson's theorem [91]) For minimal (X, μ, T), the spectrum $\sigma(H)$ (which is constant in $x \in X$) is given by the set of $E \in \mathbb{R}$ such that the Schrödinger cocycle $(T, A(\cdot, E))$ is not uniformly hyperbolic.
- (Kotani theory [94]) The absolutely continuous spectrum $\sigma_{ac}(H)$ (μ- a.e. constant for any ergodic (X, μ, T) and constant for minimal systems [97]) is given by the essential closure of the set $\{E : L(E) = 0\}$.

Therefore, for minimal, and in particular quasiperiodic, underlying dynamics, spectrum and absolutely continuous spectrum of H_x are encoded by the dynamics of the one-parameter family $A(x, E)$ of transfer-matrix cocycles, indexed by the energy E, but, for the spectrum, not by any explicit quantity. One recent surprising development is that for analytic one-frequency quasiperiodic Schrodinger operators, the spectrum (and therefore absence of uniform hyperbolicity of the corresponding cocycles) can be characterized more directly. In [47] we introduce a new object, dual Lyapunov exponent $\hat{L}(E)$, and prove

Theorem 1.1 ([47]). *For quasiperiodic operators* (0.1) *with analytic* V*,*

$$\sigma(H) = \big\{E : L(E)\hat{L}(E) = 0\big\}. \tag{1.3}$$

Exponent $\hat{L}(E)$ is defined as the limit of lowest Lyapunov exponents of dual high-dimensional cocycles (see Sections 2 and 4) which is proved to exist. There are interesting questions of varying levels of difficulty on whether this can be appropriately extended to higher-dimensional analytic one-frequency quasiperiodic Schrodinger cocycles, corresponding to operators on the strips, to multifrequency analytic cocycles, to nonanalytic potentials, or even other underlying dynamics. Perhaps the most natural question is whether one can find an analytic characterization of the absence of uniform hyperbolicity for all analytic one-frequency quasiperiodic cocycles. For the latter, there is a topological obstruction, but one can reduce the question, say, to cocycles homotopic to the identity.

2. AUBRY DUALITY AND HIGHER-DIMENSIONAL COCYCLES

The early work of Dinaburg–Sinai [37] notwithstanding, it is fair to say that the study of the spectral theory of quasiperiodic operators has been largely shaped around and driven by several explicit models, all coming from physics. The most prominent of those is the almost Mathieu family $H_{2\lambda\cos,\alpha,x}$, which can be argued to be the tight-binding analogue of a harmonic oscillator. Besides being the main model in the related physics studies and that featured in the Hofstadter's butterfly, it is also the simplest, in many ways, analytic case, yet it seems to represent most of the nontrivial properties expected to be encountered in the more general situation. In some sense, it plays the same role in the theory of quasiperiodic operators that the Ising model plays in statistical mechanics, and similarly to the latter, it does have an important additional symmetry.

Namely, we define the Aubry dual of the one-frequency Schrödinger operator (0.1) as

$$(\hat{H}_{V,\alpha,\theta}u)_n = \sum_{k=-\infty}^{\infty} V_k u_{n+k} + 2\cos 2\pi(\theta + n\alpha)u_n, \quad n \in \mathbb{Z}, \tag{2.1}$$

where V_k is the kth Fourier coefficient of V.[1] It can be useful to view this as a transformation of the entire family indexed by x for fixed V, α. In this regard, this transform can be viewed as a unitary conjugation on $\mathcal{H} = L^2(\mathbb{T} \times \mathbb{Z})$, via

$$U\psi(x,n) = \hat{\psi}(n, x + \alpha n), \tag{2.2}$$

where $\hat{\psi} : L^2(\mathbb{Z} \times \mathbb{T}) \to L^2(\mathbb{T} \times \mathbb{Z})$ is the Fourier transform. The almost Mathieu family is self-dual with respect to this transformation $\hat{H}_{2\lambda\cos,\alpha,x} = H_{\frac{2}{\lambda}\cos,\alpha,\theta}$, and, in particular, $H_{2\cos,\alpha,x}$, that is, $H_{2\lambda\cos,\alpha,x}$ with $\lambda = 1$, is the self-dual (also called critical) point.

Aubry duality can be explained by the magnetic nature and corresponding gauge invariance of two-dimensional magnetic Laplacians that lead to $H_{V,\alpha,x}$ [101]. In particular, spectra and integrated densities of states of $H_{V,\alpha,x}$ and $\hat{H}_{V,\alpha,x}$ coincide. However, it is not the case for the spectral type, and indeed it is natural to expect that a Fourier-type transform would take localized eigenfunctions (point spectrum!) into extended ones (absolutely continuous spectrum!), and vice versa. That was the basis for several predictions by physicists Aubry and Andre [1] about the almost Mathieu family with irrational α, namely that the spectrum of $H_{2\lambda\cos,\alpha,x}$ is absolutely continuous for $\lambda < 1$ (called subcritical) and pure point for $\lambda > 1$ (called supercritical). This was described in the paper where transformation (2.1)

1 There is a more general, multidimensional definition, but we stick to the one-dimensional case for this exposition.

was introduced in the context of the almost Mathieu family, leading to the name Aubry duality. This problem, along with a few others related to this family, was heavily popularized by Barry Simon in [106, 108], fueling an increased interest in the mathematics community.

Aubry duality has been formulated and explored on different levels, e.g., [10, 55, 101]. It has consistently played a central role in the analysis of quasiperiodic operators, in proving absolutely continuous spectrum and reducibility [10, 31], point spectrum [17, 24, 50, 57, 70],[2] or its absence [11, 69].

In general, operator (2.1) is long-range. If V is a trigonometric polynomial of degree d, the transfer-matrix $A(x, E)$ of the eigenvalue equation $\hat{H}_{V,\alpha,x}\Psi = E\Psi$ gives rise to a $2d$-dimensional cocycle, which has a complex-symplectic structure [60], so we will view it as an $\mathrm{Sp}(2d, \mathbb{C})$ cocycle (α, A), $A \in \mathrm{Sp}(2d, \mathbb{C})$, a linear skew product

$$(\alpha, A) : \left\{ \begin{array}{ccc} \mathbb{T} \times \mathbb{C}^{2d} & \to & \mathbb{T} \times \mathbb{C}^{2d} \\ (x, v) & \mapsto & (x + \alpha, A(x, E) \cdot v) \end{array} \right\}.$$

The Lyapunov exponents $L_1(\alpha, A) \geq L_2(\alpha, A) \geq \cdots \geq L_{2d}(\alpha, A)$, repeated according to their multiplicity, are defined by

$$L_k(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \ln\big(\sigma_k\big(A_n(x)\big)\big) dx,$$

where for a matrix $B \in M_m(\mathbb{C})$, $\sigma_1(B) \geq \cdots \geq \sigma_m(B)$ denote its singular values (eigenvalues of $\sqrt{B^*B}$). Since for real E the transfer-matrix $A(x, E)$ of the eigenvalue equation $\hat{H}_{V,\alpha,x}\Psi = E\Psi$ is symplectic, its Lyapunov exponents come in the opposite pairs $\{\pm L_i(\alpha, A)\}_{i=1}^{d}$. We will now denote

$$\hat{L}_i = L_{d-i}(\alpha, A), \tag{2.3}$$

so that $0 \leq \hat{L}_1 \leq \hat{L}_2 \leq \cdots \leq \hat{L}_d$.

In general, Lyapunov exponents are not nicely behaved with respect to parameter changes. They can be (and most likely, typically are) discontinuous in α at $\alpha \in \mathbb{Q}$ (the almost Mathieu cocycle is one example), are generally discontinuous in A in C^0, and can be discontinuous in A even in C^∞ [35, 81, 113, 114]. It is a remarkable fact, enabling much of the related theory, that Lyapunov exponents are continuous in the analytic category.

Theorem 2.1 ([12, 29, 31, 73]). *The functions* $\mathbb{R} \times C^\omega(\mathbb{T}, M_m(\mathbb{C})) \ni (\alpha, A) \mapsto L_k(\alpha, A) \in [-\infty, \infty)$ *are continuous at any* (α', A') *with* $\alpha' \in \mathbb{R}\backslash\mathbb{Q}$.[3]

For the almost Mathieu operator, it leads to the exact formula for the Lyapunov exponent for energies E in the spectrum of $H_{2\lambda \cos,\alpha,x}$. We have $L_{\lambda,\alpha}(E) = \max\{\ln|\lambda|, 0\}$ [30].

For Diophantine α, this continuity extends to sufficiently smooth Gevrey spaces [35, 92], and it is a remarkable recent result [48] that for certain α the transition in the topology

---

2 Made possible with the development of recent powerful methods [7, 14, 65, 118] to establish nonperturbative reducibility directly and independently of localization for the dual model.

3 In dimension one, it extends to the Lyapunov exponents of multifrequency cocycles $\mathbb{R} \times C^\omega(\mathbb{T}^b, \mathrm{SL}_2(\mathbb{C})) \ni (\alpha, A) \mapsto L(\alpha, A) \in [0, \infty)$.

for continuity of L occurs sharply at the Gevrey space G^2. It should be noted that both the original spectacular counterexample [113] and its refinements [48, 114] require α to be a fixed irrational of bounded type, i.e., having a continued fraction expansion with bounded coefficients. This set includes the golden mean but forms a set of zero Lebesgue measure. The authors of all these papers also vary the cocycle, i.e., the potential. This still leaves open the question whether continuous behavior of the Lyapunov exponents at least for Schrödinger cocycles with regularity lower than G^2 is possible if α is not of bounded type. Another open question is whether it is true that for a fixed potential of lower than G^2 regularity, the Lyapunov exponent is necessarily a continuous function of energy.

3. AVILA'S GLOBAL THEORY AND CLASSIFICATION OF ANALYTIC ONE-FREQUENCY COCYCLES

While many results exist in lower regularity, the analyticity of V in (0.1) brings on board powerful ideas related to subharmonicity (leading, in particular, to the crucially important for other developments continuity results) and the technique of semialgebraic sets introduced to the field by J. Bourgain [28]. As a result, a lot more can be said about analytic quasiperiodic operators. Particularly, while Kotani theory based its characterization of the absolutely continuous spectrum on compexifying the energy, for analytic quasiperiodic operators there is one more natural parameter to complexify, namely the phase. This idea goes back to M. Herman [63], and has been fruitfully used to prove positivity (and later continuity) of the Lyapunov exponent in [29, 63, 110]. Avila [5] discovered a remarkable related structure that has served as a foundation of his global theory (later extended to the high-dimensional cocycles in [12]). Define

$$L_\epsilon(E) := \lim_{n\to\infty} \int \frac{1}{n} \ln \left\| \prod_{j=n-1}^{0} A_j(x + j\alpha + i\epsilon, E) \right\| d\mu.$$

Avila observed that, for a given cocycle, L_ϵ is a convex function of ϵ, and proved that it has quantized derivative in ϵ.

Theorem 3.1 ([5]). *For any complex-analytic one-frequency cocycle,*

$$\omega(A) = \lim_{\epsilon\to 0^+} \frac{L_\epsilon(A) - L_0(A)}{2\pi\epsilon} \in \mathbb{Z}.$$

This was enabled through approximation by the rationals due to the continuity of the Lyapunov exponent in the analytic category [32]. The fact that such continuity does not hold even for higher Gevrey cocycles [48, 113, 114] complicates potential nonanalytic extensions.

Theorem 3.1 already enables full analytic computation of the Lyapunov exponents for E in the spectrum, as well as of their complexifications L_ϵ and further analysis for several models originating and relevant in physics: the almost Mathieu operator [5], the extended Harper's model [81], recently discovered models with mobility edges [112] and unitary almost Mathieu operator [34], models arising in the study of the quantum graph graphene [23], and others.

Avila classified analytic cocycles $A(x)$ depending on the behavior of the Lyapunov exponent L_ϵ of the complexified cocycle $A(x + i\epsilon)$. Namely, he distinguishes three cases, with the terminology inspired by the almost Mathieu family:

(Subcritical) $L_\epsilon = 0, \epsilon < \delta,\ \delta > 0$, or, alternatively, $L_0 = \omega(A) = 0$.

(Critical) $L_0 = 0, L_\epsilon > 0, \epsilon > 0$, or, alternatively, $L_0 = 0, \omega(A) > 0$.

(Supercritical) $L_0 > 0$.

For the almost Mathieu family, these three regimes are uniform over the spectrum, corresponding to the supercritical ($\lambda > 1$), subcritical ($\lambda < 1$), and critical ($\lambda = 1$) values of the coupling constant. Spectrally, there is purely absolutely continuous spectrum for all x and all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ in the subcritical case [3], purely singular continuous spectrum for all x and all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ in the critical case [69], and pure point spectrum for a.e. x, α with sharp spectral transitions depending on the arithmetics of both α and x between pure point spectrum and singular continuous spectrum in the supercritical case (see Section 5). Remarkably, the critical almost Mathieu operators appear at the boundary of the two other regimes.

For general quasiperiodic operators, this classification leads to the corresponding division of energies in the spectrum, depending on (sub/super)criticality of the cocycle $A(\cdot, E)$. For convenience we will call the energy in the spectrum (super/sub)critical according to whether the corresponding transfer-matrix cocycle is such. It is expected that the key spectral properties of spectra in the three above regimes follow those of the corresponding almost Mathieu operators.

Indeed, pure point spectrum for a.e. x, α holds through the *supercritical* set of energies, for any analytic potential [30]. It is an important open problem to make this result arithmetic, and it is expected that certain universal features of the transitions and structure of the eigenfunctions discovered in [77, 78] will hold globally, throughout the supercritical regime, see Section 6.3.

The *subcritical* regime is subject to the almost reducibility conjecture (ARC) which claims that subcritical cocycles are almost reducible, that is, have constant cocycles in the closure of their analytic conjugacy class (note that since almost reducibility implies subexponential growth of the iterates of the cocycle that is uniform in the (complexified) phase, the converse is obviously true). The idea of reducing nonperturbative (global) to perturbative (local) results originated from an earlier work by Avila and Krikorian [14]. ARC was first formulated in [10], and first established for the almost Mathieu operator [3, 10]. It was solved by Avila for the Liouville case in [4], and the solution for the complementary Diophantine case has been announced [5] to appear in [2]. Also, L. Ge has recently found a different proof [46].

Almost reducible (and therefore subcritical) cocycles enjoy all the dynamical and spectral consequences of the Eliasson's perturbative regime [39]. In particular, there is purely absolutely continuous spectrum throughout the subcritical regime. Moreover, reducibility can be made quantitative [117], and even arithmetically so [50], allowing for a wealth of conclusions. However, it remains true that the absolutely continuous spectrum is fully characterized by the subcritical regime, with no delicate dependence, as far as the spectral decomposition goes, on any other parameters.

The *critical* regime is expected (see [11, 82]) to support only singular continuous spectrum (again, no dependence on the other parameters, as long as α is irrational) but fully establishing it even for the critical almost Mathieu operator took decades and was only accomplished recently [69].

On the other hand, the key result of Avila's global theory [5] is that operators with critical energies throughout the spectrum, like the critical almost Mathieu operator, are an anomaly, that does not happen typically. In fact, for prevalent (in a certain measure-theoretic sense) potentials, there are no critical energies, and the spectrum is contained in finitely many intervals, with either only subcritical or only supercritical regime within each.[4] Moreover, the set of potentials and energies (V, E) such that E is critical is contained in a countable union of codimension-one analytic submanifolds of $C^\omega(\mathbb{T};\mathbb{R}) \times \mathbb{R}$. Another remarkable related fact is that Lyapunov exponent enjoys even much stronger regularity when restricted to potentials and energies with a fixed value of acceleration: it becomes real-analytic on this (typically rather irregular) set, in both the energy E and any parameter λ ranging in a real analytic manifold Λ, if V_λ in $C^\omega(\mathbb{T};\mathbb{R})$ is a family real-analytic in parameter λ.

From the point of view of the global theory, it becomes particularly important to study the universal features of the two prevalent regimes, subcritical and supercritical. As mentioned above, the absolutely continuous spectrum is fully characterized by the subcritical regime, with no delicate dependence, as far as the spectral decomposition goes, on any other parameters. The picture for the supercritical regime is a lot more interesting, and is in a certain sense at the beginning of its development.

Going back to the complexified cocycle L_ϵ, quantizatization of acceleration means that as a function of $\epsilon > 0$, L_ϵ is convex, piecewise affine, and thus is fully characterized by $L = L_0$ and monotone increasing sequences of turning points b_i and slopes $n_i \in 2\pi\mathbb{Z}_+$, so that the slope of L_ϵ between b_i and b_{i+1} is n_i. Clearly, sequences b_i and n_i present a very important intrinsic characterization of the cocycle and the corresponding Schrödinger operator. What information do they give us?

4. DUAL LYAPUNOV EXPONENTS OR GLOBAL THEORY DEMYSTIFIED

It turns out that Aubry duality not only provides a new proof of quantization of acceleration, but holds key to the mystery of the global theory. We have

Theorem 4.1 ([47]). *Assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $V \in C^\omega(\mathbb{T}, \mathbb{R})$. Then there exist nonnegative $\{\hat{L}_i(E)\}$ such that for any $E \in \mathbb{R}$,*

$$\hat{L}_i(E) = \lim_{d\to\infty} \hat{L}_i^d(E),$$

4 A part of this picture was previously established in the semiclassical regime in the continuum in [40].

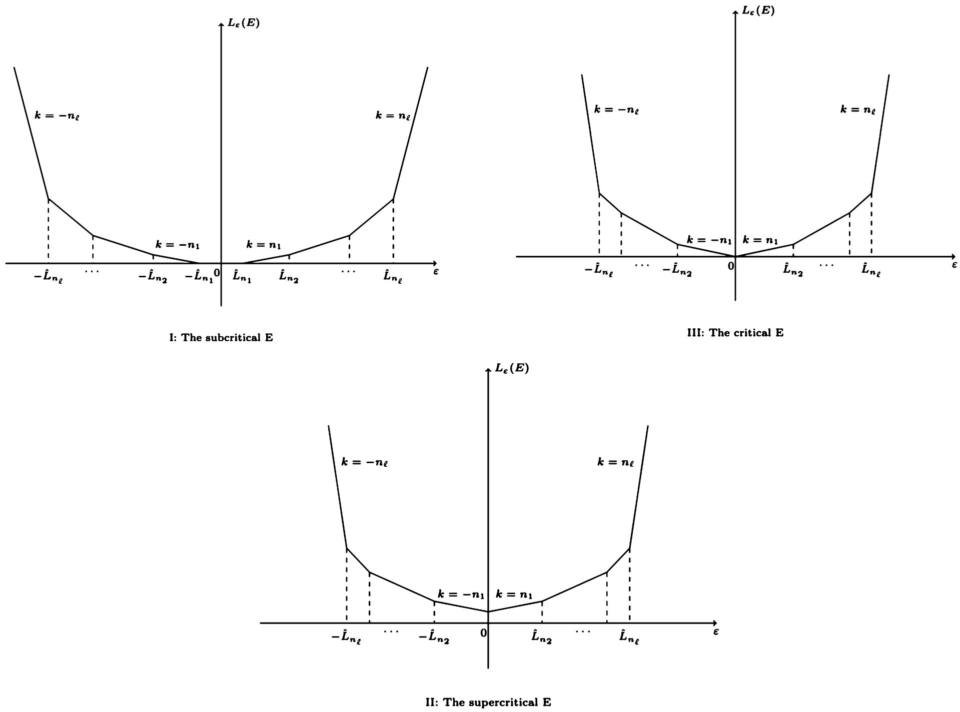

FIGURE 2
The complexified Lyapunov exponent.

where $\hat{L}_i^d(E), i=1,\dots,d$, are the Lyapunov exponents, as defined in (2.3)*, of the* $\mathrm{Sp}(2d,\mathbb{C})$ *transfer-matrix cocycle of the dual eigenvalue equation $\hat{H}_{V^d,\alpha,x}\Psi = E\Psi$, with $V^d(x) = D_d \star V$ and D_d being the Dirichlet kernel. Moreover,*

$$L_\epsilon(E) = L_0(E) - \sum_{\{i:\hat{L}_i(E)<2\pi|\epsilon|\}} \hat{L}_i(E) + 2\pi\big(\#\{i : \hat{L}_i(E) < 2\pi|\epsilon|\}\big)|\epsilon|$$

In fact, the theorem also holds for $V \in C_h^\omega(\mathbb{T},\mathbb{R})$ and $|\epsilon| < h$, where $C_h^\omega(\mathbb{T},\mathbb{R})$ is the space of bounded analytic functions f defined on a strip $\{|\Im z| < h\}$ with the norm $\|f\|_h = \sup_{|\Im z|<h} |f(z)|$. See Fig. 2 for an illustration of the three possible scenarios.

This means that for the trigonometric polynomials V the turning points b_i are given precisely by the Lyapunov exponents $\hat{L}_i(E)$ of the dual cocycle, and increases in the slopes are given by the 2π times their multiplicities; for analytic V, these objects are given by the limits of those quantities for successive trigonometric polynomial cutoffs of V. We call $\hat{L}_i(E)$ the dual Lyapunov exponents, the objects that play a role similar to that of zeros of an analytic function in the Jensen's formula. In particular, the acceleration $\omega(E)$ turns out to be precisely the number of vanishing dual Lyapunov exponents (an analogue of the winding number for an analytic function on $\mathbb{T}$).

Besides unraveling the mystery of the behavior of complexified Lyapunov exponents, this leads to a new understanding of the key statement of Avila's global theory, namely that for prevalent operators (0.1), almost all pairs of potentials and energies are acritical. Indeed, it immediately follows that

Theorem 4.2 ([47]). *Assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and V is analytic, then the energy $E \in \mathbb{R}$ is*

(1) outside the spectrum *if* $L(E) > 0$ *and* $\hat{L}_1(E) > 0$,

(2) supercritical *if* $L(E) > 0$ *and* $\hat{L}_1(E) = 0$,

(3) critical *if* $L(E) = 0$ *and* $\hat{L}_1(E) = 0$,

(4) subcritical *if* $L(E) = 0$ *and* $\hat{L}_1(E) > 0$.

Thus, in the regime $L(E) = 0$, criticality is in the locus of vanishing of an additional continuous [12] function $\hat{L}_1(E)$, implying the prevalence of the acriticality claim. Theorem 4.2, of course, also contains the statement of Theorem 1.1, with $\hat{L} := \hat{L}_1$, as well as the fact that Schrödinger cocycle is subcritical if and only if its dual Lyapunov exponents are all positive. It also leads to a number of other powerful spectral corollaries, both for the general analytic case and several particular models [47]. It also has exciting physics applications [100].

5. PRECISE ANALYSIS OF SMALL DENOMINATORS

One of the most fascinating features of the spectral theory of one-frequency quasiperiodic operators in the supercritical regime is its delicate dependence on the arithmetics, that can be analyzed to a remarkable depth, and in some cases completely. There were many exciting recent developments where the arithmetics has played a crucial role (e.g., [9,15,89]) but here we focus only on the analysis of small denominators in the proofs of point spectrum and related study of the eigenfunctions.

The main difficulty in proving point spectrum (or the phenomenon of Anderson localization, that is, pure point spectrum with exponentially decaying eigenfunctions) and analyzing the corresponding eigenfunctions of ergodic operators is in the fact that the eigenvalues are dense in the spectrum. Formal perturbative expansions of eigenfunctions and eigenvalues include the $(V(T^n x) - V(T^m x))^{-1}$ terms that, of course, get arbitrarily large. More generally, when we have *resonances*, that is, restrictions to boxes that are not too far away from each other that have eigenvalues that are too close (something that is bound to happen for ergodic operators), small denominators are created. Thus localization for ergodic and, in particular, quasiperiodic operators can be viewed as a small denominator problem.

Indeed, it has been traditionally approached in a perturbative way: through KAM-type schemes for large couplings [39,44,109], which all required Diophantine conditions on the frequency α. Small denominators are not simply a nuisance, but lead to actual change in the spectral behavior, since in the opposite regime of very Liouville frequencies (*too* small denominators), there is no localization even with the positivity of the Lyapunov exponent; and delocalization (which in this case means singular continuous spectrum) can be proved by perturbation of nearby periodic operators [20,54]. At the same time, for exponentially approximated frequencies that are neither far from nor close enough to rationals, there is nothing left to perturb about or to remove. Tackling those cannot be approached perturbatively, but requires a precise analysis, giving the problem a strong number-theoretic flavor.

It should be noted that the topology of the one-dimensional line is such that even occasional barriers make it difficult to pass through, strongly favoring localization in the presence of even small irregularities. For example, in the one-dimensional random case, localization holds for all couplings λ, when considering a family of potentials λV, and the same is expected but is apparently difficult to prove even for the underlying dynamics (X, μ, T) with very weak chaotic properties, such as a skew shift. It has even been conjectured by Kotani and Last that absolutely continuous spectrum is impossible for one-dimensional operators that are not almost periodic, but it has been disproved [6,111], and with a particularly simple construction in [119]. Those examples notwithstanding, the presence of metal–insulator transitions (that roughly correspond to transitions between the spectral types) as couplings change remains a distinctive feature of quasiperiodic operators.

The transitions in coupling between absolutely continuous and singular spectrum are fully determined by the vanishing/nonvanishing of the Lyapunov exponent. In the supercritical regime, absolutely continuous spectrum is impossible, but whether the spectrum is point or singular continuous is resolved in the competition between the depth of the small denominators—the strength of the resonances—and the Lyapunov growth.

Two types of resonances have played a special role in the spectral theory of quasiperiodic operators. *Frequency* resonances, when $|V(x) - V(x + k\alpha)|$ is small simply because $\|(x + k\alpha) - x)\|_{\mathbb{R}/\mathbb{Z}} = \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}$ is small, where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \inf_{\ell\in\mathbb{Z}} |x - \ell|$, were first exploited in [21] based on [54] to prove the absence of eigenvalues (and therefore singular continuous spectrum in the hyperbolic regime) for quasiperiodic operators with Liouville frequencies. Their strength is measured by the arithmetic parameter

$$\beta(\alpha) = \limsup_{k\to\infty} -\frac{\ln \|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{5.1}$$

that is equal to zero for Diophantine (thus a.e.) α. Frequency resonances are ubiquitous for all quasiperiodic potentials.

Another class of resonances, appearing for all *even* potentials, was discovered in [83], where it was shown that the arithmetic properties of the phase also play a role and may lead to singular continuous spectrum even for the Diophantine frequencies. Indeed, for even potentials, phases with almost symmetries, when $|V(x) - V(x + k\alpha|$ is small because $\|(x + k\alpha) - (-x)\|_{\mathbb{R}/\mathbb{Z}}$ is small, lead to resonances, regardless of the values of other parameters. The strength of *phase* resonances is measured by the arithmetic parameter

$$\delta(\alpha, \theta) = \limsup_{k\to\infty} -\frac{\ln \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|}. \tag{5.2}$$

Phase resonances are symmetry based and exist for all even functions V.

It was conjectured in [66] that for the almost Mathieu family no other resonances appear and the competition between the Lyapunov growth and combined exponential resonance strength resolves in a sharp way: there is a pure point spectrum for $L(E) >$

$\beta(\alpha) + \delta(\alpha, x)$ and a singular continuous spectrum in the regime $L(E) < \beta(\alpha) + \delta(\alpha, x)$. We note that for the special case of *α-rational* x, that is, such that $2x \in \mathbb{Z}\alpha + \mathbb{Z}$, we have $\delta(\alpha, x) = \beta(\alpha)$ so the resonances "double up" and the conjectured threshold becomes $2\beta(\alpha)$.

An early nonperturbative localization method was first developed in the 1990s for the almost Mathieu operator [84] and represented perhaps the first case of solving a traditionally KAM problem in a direct way, without an inductive procedure. It presented a (simple, but not sharp) technique to treat the *nonresonant* case, $\beta(\alpha) = \delta(\alpha) = 0$. Further breakthroughs came in [85] where the role of the Lyapunov exponents and corresponding deviations was first understood, allowing to achieve the nonresonant result up to the actual Lyapunov transition, and then in the work of Bourgain and collaborators [28,30] where robust nonperturbative methods were developed for general analytic potentials and more, leading to the proofs of localization for a.e. frequency throughout the supercritical regime. The ideas of [85] hold more generally, and have, in particular, led to very simple proofs of localization for the one-dimensional Anderson model [90]. Most importantly, however, their arithmetic nature has been crucial for further developments. For example, the fact that localization holds for α-rational x,[5] enabled Puig's proof [104] of the ten martini problem (that the spectrum is a Cantor set) for Diophantine α. The solution of the full ten martini problem [8,9] required, in particular, dealing with intermediate frequencies that are neither Diophantine nor Liouville, thus with the frequency resonances. A method to treat those has been devised in [9] leading to the proof of localization for $L(E) > \frac{16}{9}\beta$, but failing in the neighborhood of the actual transition. A sharp method to treat pure frequency resonances was developed in [77], and a sharp method to treat pure phase resonances in [78].

Therefore, the sharp arithmetic spectral transition conjecture of [66] has been established for single-type-resonances: for pure frequency resonances (that is, for the so-called α-Diophantine phases for which $\delta(\alpha, x) = 0$ so there are no exponential phase resonances) in [17,52,77],[6] and for pure phase resonances (that is, for Diophantine frequencies for which $\beta(\alpha) = 0$ so there are no exponential frequency resonances) in [78].

The methods to treat pure frequency and phase resonances in [77,78] are robust in a sense that weak exponential resonances of the other type can be added easily, but it is still an open problem to treat *combined* frequency and phase resonances in a sharp way. However, there were two very recent breakthroughs.

Namely, W. Liu has developed a way to sharply treat doubled resonances for the almost Mathieu operator, proving localization up to the conjectured threshold:

---

5 This was, in fact, established in [72].

6 In [17] the pure frequency part of the conjecture of [66] has been proved by a completely different method, namely through quantitative reducibility [117] and duality, but in a measure-theoretic in x sense, i.e., losing the control over the arithmetics of x. A recent breakthrough by Ge–You [50] where an arithmetic version of quantitative reducibility was developed has lead to a way to obtain sharp arithmetic in phase results through duality as well, enabling, in particular, an arithmetic duality-based proof of the frequency part of the conjecture [52], that works also for all Aubry duals (2.1) of operators (0.1).

Theorem 5.1 ([99]). *Operator $H_{2\lambda\cos,\alpha,x}$ with α-rational x has Anderson localization whenever $L(E) > 2\beta(\alpha)$ (or equivalently, $\lambda > e^{2\beta(\alpha)}$).*

In Liu's earlier work, this was established for $L(E) > 3\beta(\alpha)$ [98], but a significant new understanding of treatment of doubled resonances was necessary to go sharp, and it was achieved in [99]. Also α-rational phases x hold special importance for various questions because eigenvalues for such x are located at gap edges [104]. Puig's proof of the ten martini problem for the Diophantine case [104] was based precisely on localization for α-rational x. The original plan to prove the full ten martini problem was to establish localization for α-rational x and $L(E) > \beta(\alpha)$ [8]. Not surprisingly, it failed, prompting the resonance doubling-up conjecture in [9] that is now solved [99]. It should be noted that the singular-continuous part of the conjecture, namely singular-continuous spectrum for α-rational x and $L(E) < 2\beta(\alpha)$, is still open.

In a different direction, R. Han, F. Yang, and I [58] developed a sharp method to treat the third type of resonances: high barriers (that effectively play the role of *antiresonances*), and, moreover, *combinations* of frequency resonances and high barriers, in another popular quasiperiodic family originating in physics, the Maryland model.

Maryland model is a family

$$(M_{\lambda,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \lambda \tan\big(\pi(\theta + n\alpha)\big)u_n, \tag{5.3}$$

where $\lambda > 0$ is the coupling constant, irrational $\alpha \in \mathbb{T} = [0,1]$ is the frequency, and $\theta \in \mathbb{T}$ is the phase with $\theta \notin \Theta = \{\frac{1}{2} + \alpha\mathbb{Z} + \mathbb{Z}\}$.

It was originally proposed by Grempel, Fishman, and Prange [56] as a linear version of the quantum kicked rotor and has attracted continuing interest from the physics community, see, e.g., [26, 42, 45], due to its exactly solvable nature. It has explicit expression for the Lyapunov exponent, integrated density of states, and even (a little less explicit) for the eigenvalues and eigenfunctions. In particular, the Lyapunov exponent $L_\lambda(E)$ is an explicit function of λ, E not dependent on α. However, the implicit expressions for the eigenfunctions do not allow for easy conclusions about their behavior, which is expected to be quite interesting, with transfer matrices satisfying certain exact renormalization [41].

Phase resonances do not exist for the Maryland model, and as a result, for Diophantine (i.e., nonresonant) frequencies it has localization for *all* phases [87, 107]. However, it does have barriers, when the trajectory of a given phase approaches the singularity too early. Barriers compensate for the resonances, and therefore serve as what we call in [58] the *antiresonances*, providing the reason why for the Maryland model there are phases with localization even for the most Liouville frequencies [76]. Thus Maryland model features a combination of frequency resonances and phase antiresonances.

Maryland model was the first one where the spectral decomposition has been resolved completely, for *all* values of the parameters [76].[7] Let p_n/q_n be the continued fraction approximants of α. We note that the frequency resonance index $\beta(\alpha)$ defined in (5.1)

7 It also remains the only one with spectral transitions where this could be claimed.

also satisfies $\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}$. A new index, $\delta^M(\alpha, \theta)$, was introduced in [76] as

$$\delta^M(\alpha,\theta) := \limsup_{n\to\infty} \frac{\ln q_{n+1} + \ln \|q_n(\theta - \frac{1}{2})\|_{\mathbb{T}}}{q_n}. \tag{5.4}$$

We have

Theorem 5.2 ([76]). *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum on $\{E : L_\lambda(E) < \delta^M(\alpha,\theta)\}$, and pure point spectrum on $\{E : L_\lambda(E) > \delta^M(\alpha,\theta)\}$.*[8]

This provides complete spectral analysis, for all α, θ, but was established implicitly: through the combination of Cayley and Fourier transforms and the study of a resulting explicit cohomological equation, making sharp the previous work in [56,107]. The extension of the analysis from a.e. θ in [107] to *all* θ in [76] required accounting for the effect of the barriers, and Cayley transform allowed to do it, albeit in a highly implicit way. In particular, this proof did not allow the analysis of the structure of eigenfunctions.

The method of [85] was adapted to the Maryland model in [87] where the nonresonant situation was treated and localization for Diophantine α was shown, developing the initial framework to study the eigenfunctions in the much more difficult resonant situation.

In [58] we show that $\delta(\alpha,\theta)$ can be interpreted as the exponential strength of frequency resonances, $\beta(\alpha)$, combined with the (negative) exponential strength of phase antiresonances, defined as the positions of exponential smallness of the $\cos(\pi(\theta + k\alpha))$,[9] and develop the approach to sharply treat the "resonance tamed by an antiresonance" situation. In particular, we give a constructive proof of the localization part of Theorem 5.2 and obtain

Theorem 5.3 ([58]). *For any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and any θ, the spectrum on $\{E : L_\lambda(E) \geq \delta^M(\alpha,\theta)\}$ is pure point and for any eigenvalue $E \in \{L_\lambda(E) > \delta^M(\alpha,\theta)\}$ and any $\epsilon > 0$, the corresponding eigenfunction ϕ_E satisfies $|\phi_E(k)| < e^{-(L_\lambda(E)-\delta^M(\alpha,\theta)-\epsilon)|k|}$ for sufficiently large $|k|$.*

Theorem 5.3 provides the sharp upper envelope, and develops the key tools to study the fine behavior of the eigenfunctions, see Section 6.2. In fact, such a study is the most exciting outcome of the proofs of localization based on sharp analysis of resonances.

There are several other models where sharp arithmetic spectral transitions have been conjectured and partially established, most notably the extended Harper's model, where for the complete analysis one would need to develop tools to study the simultaneous presence of three different types of resonances: frequency, phase, and singularity-induced antiresonances. However, for a.e. phase we expect the arithmetic frequency transition to be universal in the class of general analytic potentials. As for the arithmetic transitions in phase, we expect the same results to hold for general even analytic potentials for a.e. frequency. We note that the singular continuous part up to the conjectured transition is already established, even in a far greater generality, in [17,71,78].

8 It follows from the explicit formula for $L_\lambda(E)$ that the equality can only happen for two values of E.

9 So exponential largeness of the tan.

Finally, there is a question of arithmetic interfaces, e.g., what happens for the almost Mathieu operators with $L(E) = \beta(\alpha) + \delta(\alpha, \theta)$? It turns out that (in the pure resonance situations) both pure point and singular continuous spectra are possible depending on the finer arithmetic properties of parameters [13, 86, 88]. So far we do not even have a good conjecture on where the arithmetic thresholds within the transition lines lie. Making a significant progress on this problem would require a development of polynomial (as contrasted with current exponential) methods to tackle resonances, a very important problem in its own right, as it could lead to universal hierarchical structures (see Section 6) on polynomial scales.

6. EXACT ASYMPTOTICS AND UNIVERSAL HIERARCHICAL STRUCTURE OF EIGENFUNCTIONS

A very captivating question and a longstanding theoretical challenge is to explain the self-similar hierarchical structure visually obvious in the Hofstadter's butterfly, as well as the hierarchical structure of eigenfunctions, as related to the arithmetics of parameters. Such structure was first predicted for the almost Mathieu operator in the work of Azbel in 1964 [22], some 12 years before Hofstadter [64], and before numerical experimentation was possible. Such self-similar behavior is present for spectra and eigenfunctions of all quasiperiodic operators.

While this does not describe or explain the self-similarity, a step in the right direction is to prove that the spectrum is a Cantor set. Mark Kac offered ten martinis in 1982 for the proof of the Cantor set part of Azbel's 1964 conjecture. It was dubbed the Ten Martini problem by Barry Simon, who advertised it in his lists of 15 mathematical physics problems [106] and later, mathematical physics problems for the XXI century [108]. Most substantial partial solutions were made by Bellissard, Simon, Sinai, Helffer, Sjöstrand, Choi, Eliott, Yui, and Last [25, 36, 62, 96, 109], between 1983 and 1993. J. Puig [104] solved it for Diophantine α by noticing that localization at $\theta = 0$ [73, 85] leads to gaps at corresponding (dense) eigenvalues. The final solution was given in [9]. Cantor spectrum is also prevalent for general one-frequency operators with analytic potential: in the subcritical regime [10], and, by very different methods, in the supercritical regime [53] (and it is conjectured [11] also in the critical regime, which is nongeneric in itself [5]). Moreover, even all gaps predicted by the gap labeling are open in the noncritical almost Mathieu case [10, 16], the statement that is also expected to be true in the critical case, and recently claimed in the physics literature [27] to follow directly from [69].

As for the understanding the hierarchical behavior of the eigenfunctions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [116], it has remained an important open challenge even at the physics level, although some indications existed in the perturbative regime [33, 62, 109, 120].

Sharp analysis of resonances and small denominators has led to the discovery of universal self-similar structures of eigenfunctions defined by the type of resonance. The universal nature of these structures manifests in two ways: there is the same universal function that depends only on the type of the resonance, that governs the behavior around each exponential frequency or phase resonance (upon (possibly) reflection and renormalization), and it is the same structure for all the parameters involved: any (Diophantine) frequency α, (any α-Diophantine phase θ) with $\beta(\alpha) < L$ ($\delta(\alpha, \theta) < L$), and any eigenvalue E. It has been discovered and proved for the almost Mathieu operator [77,78] but is expected to be universal also throughout the class of analytic potentials, and more,[10] that is to hold in the regime of pure resonances. For example, the same universal structure for frequency resonances has already been proved for the Maryland model [59], for a.e. phase, namely, phases without the exponential antiresonances, see also a result on the hierarchical structure in the semiclassical regime [93]. However, for phases whose trajectories approach the barrier too fast, the hierarchical structure of the eigenfunctions is very different, and the complete analysis is extremely delicate.

Generally, one can identify four types of (anti)resonances that lead to different universal structures:

- frequency
- phase (only even potentials)
- barriers (antiresonance)
- singularity (antiresonance for Jacobi matrices)

We describe the universal structures for phase and frequency resonances [77,78] in the following subsections, and the one for the barrier antiresonances will appear in [59].

We expect that when different types of resonances are present, there will be further different self-similar structures, universal for all corresponding parameters and different resonance positions. Describing these structures for different combinations of resonances is very challenging but seems to be potentially within reach. In particular, in [58] we developed the tools to fully describe the universal structures for the Maryland model for all parameters, that is for combinations of frequency resonances and barrier antiresonances. We expect it to be done in [59]. We also expect the latter structures to be universal in the class of monotone potentials with a simple pole.

To give a glimpse into the universality results, we present two of them in more detail.

6.1. Frequency resonances

In [77] we find explicit universal functions $f(k)$ and $g(k)$, depending only on the Lyapunov exponent and the position of k in the hierarchy defined by the denominators q_n of the continued fraction approximants of the flux α, that completely define the exponential behavior of, correspondingly, eigenfunctions and norms of the transfer matrices of the almost Mathieu operators, for all eigenvalues corresponding to α-Diophantine phase, see Theorem 6.1. This result holds for *all* frequency and coupling pairs in the frequency-

10 For example, C^2 cos-type potentials have been a popular object of study [43, 49, 51, 109, 115] and there are reasons to believe that they will feature the same structure, at least in the perturbative regime.

resonance localization regime. Since the behavior is fully determined by the frequency and does not depend on the phase, it is the same, eventually, around any starting point, so is also seen unfolding at different scales when magnified around local eigenfunction maxima, thus describing the exponential universality in the hierarchical structure.

Since we are interested in exponential growth/decay, the behavior of f and g becomes most interesting in case of frequencies with exponential rate of approximation by the rationals.

These functions allow describing *precise* asymptotics of *arbitrary* solutions of $H_{\lambda,\alpha,\theta}\varphi = E\varphi$ where E is an eigenvalue. The precise asymptotics of the norms of the transfer-matrices provides the first example of this sort for nonuniformly hyperbolic dynamics. Since those norms sometimes differ significantly from the reciprocals of the eigenfunctions, this leads to further interesting and unusual consequences, for example, exponential tangencies between contracted and expanded directions at the resonant sites.

Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, we define functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$ in the following way. Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. For any $\frac{q_n}{2} \leq k < \frac{q_{n+1}}{2}$, define $f(k), g(k)$ as follows:

Case 1. $q_{n+1}^{\frac{8}{9}} \geq \frac{q_n}{2}$ or $k \geq q_n$.
If $\ell q_n \leq k < (\ell+1)q_n$ with $\ell \geq 1$, set

$$f(k) = e^{-|k-\ell q_n|\ln|\lambda|}\bar{r}_\ell^n + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\bar{r}_{\ell+1}^n, \tag{6.1}$$

and

$$g(k) = e^{-|k-\ell q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_\ell^n} + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_{\ell+1}^n}, \tag{6.2}$$

where for $\ell \geq 1$,

$$\bar{r}_\ell^n = e^{-(\ln|\lambda| - \frac{\ln q_{n+1}}{q_n} + \frac{\ln \ell}{q_n})\ell q_n}.$$

Set also $\bar{r}_0^n = 1$ for convenience.
If $\frac{q_n}{2} \leq k < q_n$, set

$$f(k) = e^{-k\ln|\lambda|} + e^{-|k-q_n|\ln|\lambda|}\bar{r}_1^n, \tag{6.3}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{6.4}$$

Case 2. $q_{n+1}^{\frac{8}{9}} < \frac{q_n}{2}$ and $\frac{q_n}{2} \leq k \leq \min\{q_n, \frac{q_{n+1}}{2}\}$.
Set

$$f(k) = e^{-k\ln|\lambda|}, \tag{6.5}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{6.6}$$

Notice that f, g only depend on α and λ, but not on θ or E; $f(k)$ decays and $g(k)$ grows exponentially, globally, at varying rates that depend on the position of k in the hierarchy defined by the continued fraction expansion of α, see Figures 3 and 4.

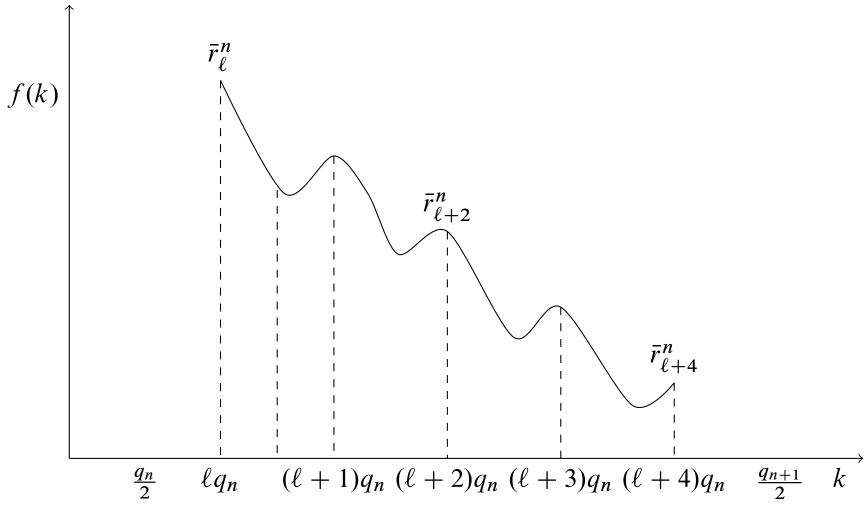

FIGURE 3
The universal behavior of eigenfunctions at scale n.

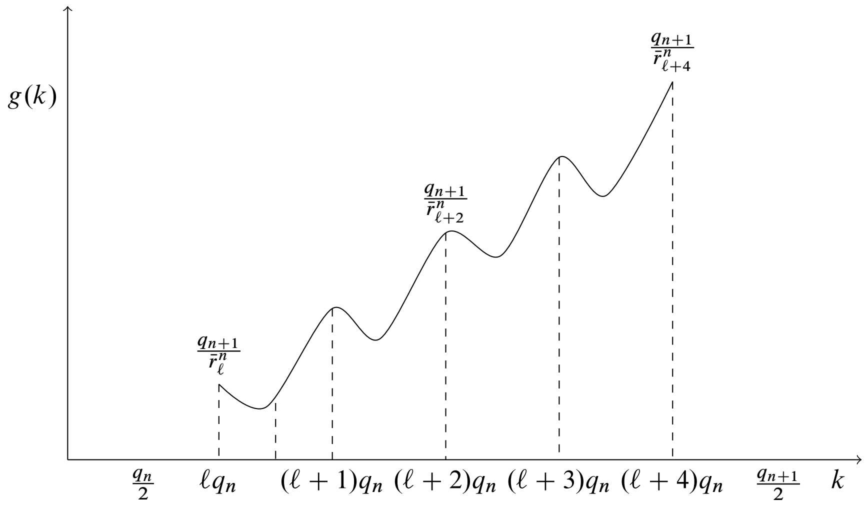

FIGURE 4
The universal behavior of transfer matrix norms at scale n.

It turns out that, in the entire regime $L(E) > \beta$, the exponential asymptotics of the eigenfunctions and norms of transfer matrices at the eigenvalues are completely determined by $f(k), g(k)$.

Theorem 6.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$. Suppose θ is Diophantine with respect to α, E is an eigenvalue of $H_{\lambda,\alpha,\theta}$, and ϕ is the eigenfunction. Let $U(k) = \left(\begin{smallmatrix}\phi(k)\\ \phi(k-1)\end{smallmatrix}\right)$. Then for any $\varepsilon > 0$, there exists K (depending on $\lambda, \alpha, \hat{C}, \varepsilon$) such that for any $|k| \geq K$, $U(k)$ and*

A_k[11] *satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leq \|U(k)\| \leq f(|k|)e^{\varepsilon|k|} \tag{6.7}$$

and

$$g(|k|)e^{-\varepsilon|k|} \leq \|A_k\| \leq g(|k|)e^{\varepsilon|k|}. \tag{6.8}$$

In fact, the theorem is formulated in **[77]** for *generalized eigenfunctions*, thus can also be used to establish pure point spectrum throughout the indicated regime. Certainly, there is nothing special about $k = 0$, so the behavior described in Theorem 6.1 happens around an arbitrary point $k = k_0$. This implies the self-similar nature of the eigenfunctions: $U(k)$ behave as described at scale q_n but, when looked at in windows of size $q_k, q_k \leq q_{n-1}$, will demonstrate the same universal behavior around appropriate local maxima/minima.

To further illustrate the above, let ϕ be an eigenfunction and $U(k) = \binom{\phi(k)}{\phi(k-1)}$. An immediate corollary of Theorem 6.1 is the universality of behavior at all appropriately defined nonresonant local maxima. We will say k_0 is a local j-maximum of ϕ if $\|U(k_0)\| \geq \|U(k)\|$ for $|k - k_0| \sim q_j$. Then, with an appropriate notion of nonresonance (see **[77]**), we have

Theorem 6.2 (**[77]**). *Given $\varepsilon > 0$, there exists $j(\varepsilon) < \infty$ such that if k_0 is a nonresonant local j-maximum for $j > j(\epsilon)$, then*

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{\|U(k_0 + s)\|}{\|U(k_0)\|} \leq f(|s|)e^{\varepsilon|s|}, \tag{6.9}$$

for $|s - k_o| \sim q_j$.

In case $\beta(\alpha) > 0$, Theorem 6.1 also guarantees an abundance (and a hierarchical structure) of local maxima of each eigenfunction.

Let k_0 be a global maximum. The self-similar hierarchical structure of local maxima can be described in the following way. We will say that a scale n_{j_0} is exponential if $\ln q_{n_{j_0}+1} > cq_{n_{j_0}}$. Then there is a *constant* scale $\hat{n}_0$, thus a constant $C := q_{\hat{n}_0+1}$, such that for any exponential scale n_j and any eigenfunction there are local n_j-maxima within distance C of $k_0 + sq_{n_{j_0}}$ for each $0 < |s| < e^{cq_{n_{j_0}}}$. Moreover, these are all the local n_{j_0}-maxima in $[k_0 - e^{cq_{n_{j_0}}}, k_0 + e^{cq_{n_{j_0}}}]$.

The exponential behavior of the eigenfunction in the local neighborhood (of size of order $q_{n_{j_0}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Note that only exponential behavior at the corresponding scale is determined by f and fluctuations of much smaller size are invisible.

Now, let $n_{j_1} < n_{j_0}$ be another exponential scale. Denoting "depth 1" local maximum located near $k_0 + a_{n_{j_0}} q_{n_{j_0}}$ by $b_{a_{n_{j_0}}}$, we then have a similar picture around $b_{a_{n_{j_0}}}$: there are local n_{j_1}-maxima in the vicinity of $b_{a_{n_{j_0}}} + sq_{n_{j_1}}$ for each $0 < |s| < e^{cq_{n_{j_1}}}$. Again, this describes all the local $q_{n_{j_1}}$-maxima within an exponentially large interval. And again, the exponential (for the n_{j_1} scale) behavior in the local neighborhood (of size of order $q_{n_{j_1}}$) of each such local maximum, normalized by the value at the local maximum, is given by f.

11 Products A_k are defined in (1.2).

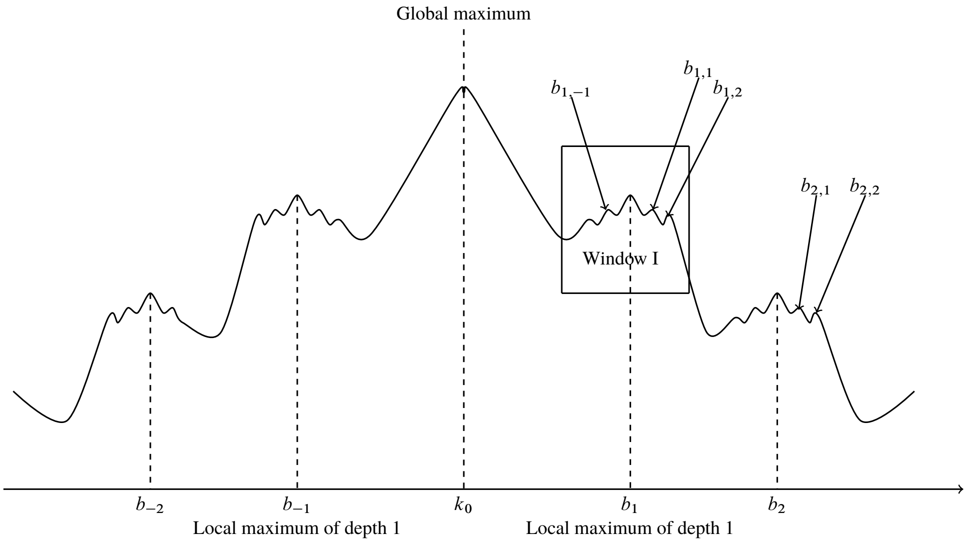

FIGURE 5
Universal self-similar structure of eigenfunctions

Denoting those "depth 2" local maxima located near $b_{a_{n_{j_0}}} + a_{n_{j_1}} q_{n_{j_1}}$ by $b_{a_{n_{j_0}}, a_{n_{j_1}}}$, we then get the same picture taking the magnifying glass another level deeper, and so on. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ with each "depth $s+1$" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ being in the corresponding vicinity of the "depth s" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}}$, and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, yet the depth of the hierarchy that can be so achieved is at least $j/2 - C$, Figure 5 schematically illustrates the structure of local maxima of depth one and two, and Figure 6 illustrates that the neighborhood of a local maximum appropriately magnified looks like a picture of the global maximum. See [77] for the exact statement.

6.2. Phase resonances

In [78] we found another universal structure, this time for phase resonances. Once again, we found (different) functions f that determine universal asymptotics of the eigenfunctions, also locally around the resonances, which features a self-similar hierarchical structure. In particular, we have Theorem just like Theorem 6.1 but with new f and for $\beta(\alpha) = 0$ and $L > \delta(\alpha, \theta)$ [78]. The behavior described in this theorem happens around an arbitrary point. This, coupled with effective control of parameters at the local maxima, allows uncover-

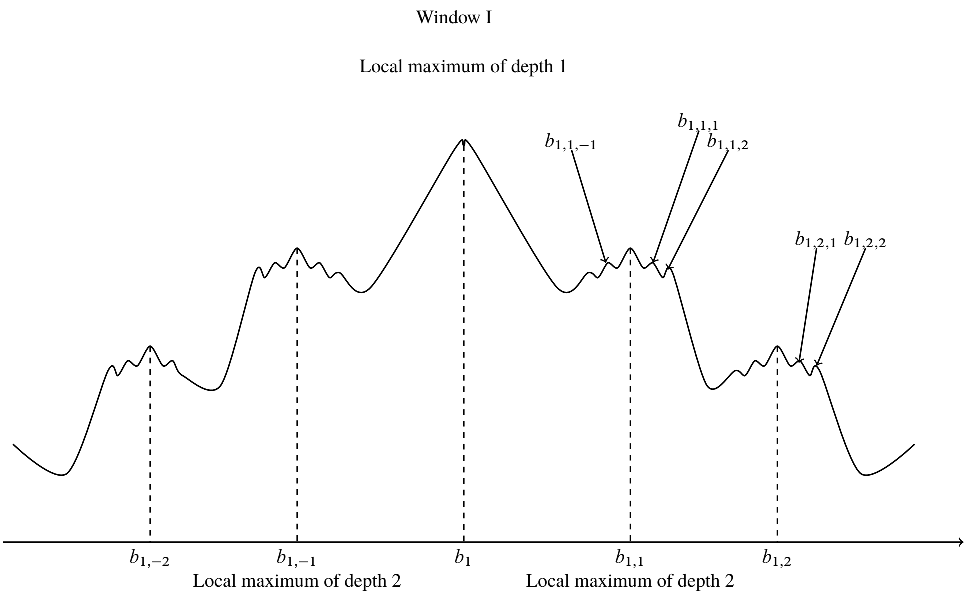

FIGURE 6
Universal self-similar structure of eigenfunctions, zoomed in

ing the self-similar nature of the eigenfunctions, but this time one needs not only the rescaling but also alternating reflections, leading to what we call the *reflective-hierarchical* structure.

Assume phase θ satisfies $0 < \delta(\alpha, \theta) < \ln \lambda$. Fix $0 < \varsigma < \delta(\alpha, \theta)$. Let k_0 be a global maximum of eigenfunction ϕ. Let K_i be the positions of exponential resonances of the phase $\theta' = \theta + k_0\alpha$ defined by

$$\|2\theta + (2k_0 + K_i)\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq e^{-\varsigma|K_i|}. \tag{6.10}$$

This means that $|v(\theta' + \ell\alpha) - v(\theta' + (K_i - \ell)\alpha)| \leq Ce^{-\varsigma|K_i|}$, uniformly in ℓ, or, in other words, the potential $v_n = v(\theta + n\alpha)$ is $e^{-\varsigma|K_i|}$-almost symmetric with respect to $(k_0 + K_i)/2$.

Since α is Diophantine, we have

$$|K_i| \geq ce^{c|K_{i-1}|}, \tag{6.11}$$

where c depends on ς and α through the Diophantine constants κ, τ. On the other hand, K_i is necessarily an infinite sequence. Let ϕ be an eigenfunction, and $U(k) = \binom{\phi(k)}{\phi(k-1)}$. We say k is a local K-maximum if $\|U(k)\| \geq \|U(k+s)\|$ for all $s - k \in [-K, K]$.

The informal description of the *reflective-hierarchical* structure of local maxima is the following. There exists a *constant* $\hat{K}$ such that there is a local cK_j-maximum b_j within distance $\hat{K}$ of each resonance K_j. The exponential behavior of the eigenfunction in the local cK_j-neighborhood of each such local maximum, normalized by the value at the local maximum, is given by the *reflection* of f.

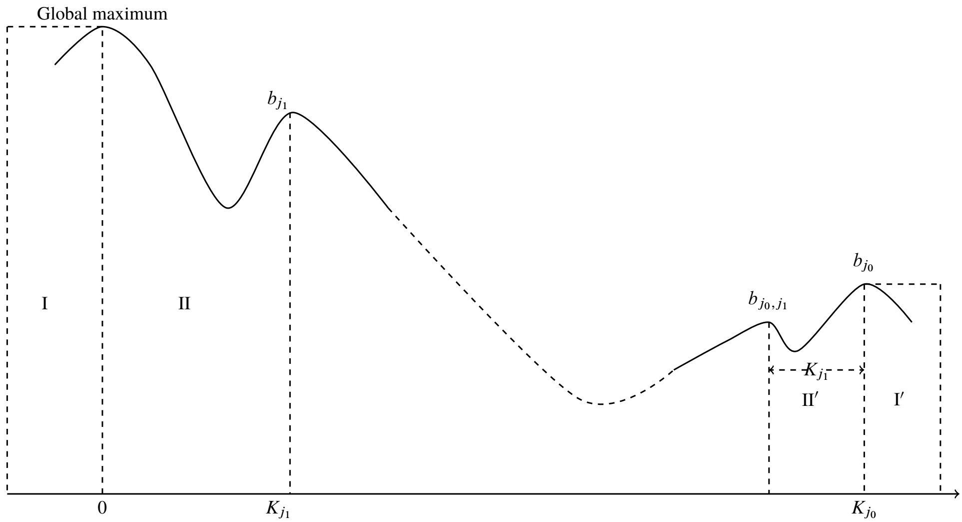

FIGURE 7
Reflective self-similarity of an eigenfunction.

Moreover, this describes the entire collection of local maxima of depth 1, that is, all K such that K is a cK-maximum. Then we have a similar picture in the vicinity of b_j: there are local cK_i-maxima $b_{j,i}$, $i < j$, within distance $\hat{K}^2$ of each $K_j - K_i$. The exponential (on the K_i scale) behavior of the eigenfunction in the local cK_i-neighborhood of each such local maximum, normalized by the value at the local maximum, is given by f.

Then we get the next level maxima $b_{j,i,s}$, $s < i$ in the $\hat{K}^3$-neighborhood of $K_j - K_i + K_s$ and reflected behavior around each, and so on, with reflections alternating with steps. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{j_0,j_1,\dots,j_s}$, with each "depth $s+1$" local maximum $b_{j_0,j_1,\dots,j_s}$ being in the corresponding vicinity of the "depth s" local maximum $b_{j_0,j_1,\dots,j_{s-1}} \approx k_0 + \sum_{i=0}^{s-1}(-1)^i K_{j_i}$ and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, with $b_{j_0,j_1,\dots,j_{s-1}}$ determined with $\hat{K}^s$ precision, thus it presents an accurate picture as long as $K_{j_s} \gg \hat{K}^s$.

Thus the behavior of $\phi(x)$ is described by the same universal f in each $\sim K_{j_s}$ window around the corresponding local maximum $b_{j_0,j_1,\dots,j_s}$ after alternating reflections. The positions of the local maxima in the hierarchy are determined up to errors that at all but possibly the last step are superlogarithmically small in K_{j_s}. We call such a structure *reflective hierarchy*.

Figure 7 depicts reflective self-similarity of an eigenfunction with global maximum at 0. The self-similarity is seen as follows: I′ is obtained from I by scaling the x-axis proportional to the ratio of the heights of the maxima in I and I′; II′ is obtained from II by scaling the x-axis proportional to the ratio of the heights of the maxima in II and II′. The behavior in the regions I′, II′ mirrors the behavior in regions I, II upon reflection and corresponding dilation.

6.3. Universality and extensions

The hierarchical structures of Sections 6.1 and 6.2 are expected to hold universally for most in the appropriate sense (albeit not all, as for the almost Mathieu) local maxima for general analytic potentials. Establishing this fully would require certain new ideas since so far even an arithmetic version of localization for the Diophantine case has not been established for the general analytic family, the current state-of-the-art result by Bourgain–Goldstein [30] being measure-theoretic in α.

The universality of the hierarchical structures of Sections 6.1 and 6.2 is twofold: not only it is the same universal function that governs the behavior around each exponential frequency or phase resonance (upon reflection and renormalization), it is the same structure for all the parameters involved: any (Diophantine) frequency α (any α-Diophantine phase θ) with $\beta(\alpha) < L$ ($\delta(\alpha, \theta) < L$), and any eigenvalue E. The universal reflective-hierarchical structure in Section 6.2 requires the evenness of the function defining the potential and, moreover, resonances of other types may also be present in general. However, we conjectured in [78] that for general even analytic potentials for a.e. frequency only finitely many other exponentially strong resonances will appear, thus the structure described in Section 6.2 will hold for the corresponding class.

The key elements of the technique developed for the treatment of arithmetic resonances are robust and have made it possible to approach other questions and, in particular, study delicate properties of the singular continuous regime. Among other things, it has allowed obtaining upper bounds on fractal dimensions of the spectral measures and quantum dynamics for the singular continuous almost Mathieu operator [79], as well as potentials defined by general trigonometric analytic functions [75], and determining also the *exact* exponent of the exponential decay rate in expectation for the two-point function [74], the first result of this kind for any model. These methods are also expected to be applicable to many other models.

FUNDING

This work was partially supported by NSF DMS-1901462, DMS-2052899, and Simons 681675. S.J. was a 2020–2021 Simons fellow.

REFERENCES

[1] S. Aubry and G. André, Analyticity breaking and Anderson localisation in incommensurate lattices. *Ann. Isr. Phys. Soc.* **3** (1980), 133–164.

[2] A. Avila, in preparation.

[3] A. Avila, The absolutely continuous spectrum of the almost Mathieu operator. 2008, arXiv:0810.2965.

[4] A. Avila, Almost reducibility and absolute continuity I. 2010, arXiv:1006.0704.

[5] A. Avila, Global theory of one-frequency Schrödinger operators. *Acta Math.* **215** (2015), no. 1, 1–54.

[6] A. Avila, On the Kotani–Last and Schrödinger conjectures. *J. Amer. Math. Soc.* **28** (2015), 579–616.

[7] A. Avila, B. Fayad, and R. Krikorian, A KAM Scheme for $\mathrm{SL}(2,\mathbb{R})$ Cocycles with Liouvillean Frequencies. *Geom. Funct. Anal.* **21** (2011), no. 5, 1001–1019.

[8] A. Avila and S. Jitomirskaya, Solving the ten martini problem. *Lecture Notes in Phys.* **690** (2006), 5–16.

[9] A. Avila and S. Jitomirskaya, The Ten Martini Problem. *Ann. of Math. (2)* **170** (2009), no. 1, 303–342.

[10] A. Avila and S. Jitomirskaya, Almost localization and almost reducibility. *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 1, 93–131.

[11] A. Avila, S. Jitomirskaya, and C. Marx, Spectral theory of extended Harper's model and a question by Erdős and Szekeres. *Invent. Math.* **210** (2017), no. 1, 283–339.

[12] A. Avila, S. Jitomirskaya, and C. Sadel, Complex one-frequency cocycles. *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 9, 1915–1935.

[13] A. Avila, S. Jitomirskaya, and Q. Zhou, Second phase transition line. *Math. Ann.* **370** (2018), 271–285.

[14] A. Avila and R. Krikorian, Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles. *Ann. of Math.* **164** (2006), 911–940.

[15] A. Avila, Y. Last, M. Shamis, and Q. Zhou, On the abominable properties of the almost Mathieu operator with well approximated frequencies. 2021, arXiv:2110.07974.

[16] A. Avila, J. You, and Q. Zhou, in preparation.

[17] A. Avila, J. You, and Q. Zhou, Sharp phase transitions for the almost Mathieu operator. *Duke Math. J.* **166** (2017), no. 14, 2697–2718.

[18] J. E. Avron, Topological quantum states and the 2016 Nobel prize. *Bull. IAMP* (2017), 3–22.

[19] J. Avron, D. Osadchy, and R. Seiler, A Topological Look at the Quantum Hall Effect. *Phys. Today* **56** (2003), 8–38.

[20] J. Avron and B. Simon, Singular continuous spectrum for a class of almost periodic Jacobi matrices. *Bull. Amer. Math. Soc. (N.S.)* **6** (1982), 81–85.

[21] J. Avron and B. Simon, Almost periodic operators, II. The density of states. *Duke Math. J.* **50** (1983), 369–391.

[22] M. Y. Azbel, Energy spectrum of a conduction electron in a magnetic field. *Sov. Phys. JETP* **19** (1964), 634–645.

[23] S. Becker, R. Han, and S. Jitomirskaya, Cantor spectrum of graphene in magnetic fields. *Invent. Math.* **2** (2019), no. 18, 979–1041.

[24] J. Bellissard, R. Lima, and D. Testard, A metal–insulator transition for the almost Mathieu model. *Comm. Math. Phys.* **88** (1983), no. 2, 207–234.

[25] J. Bellissard, B. Simon, Cantor spectrum for the almost Mathieu equation. *J. Funct. Anal.* **48**, (1982), 408–419.

[26] M. Berry, Incommensurability in an exactly-soluble quantal and classical model for a kicked rotator. *Phys. D* **10** (1984), 369–378.

[27] D. S. Borgnia and R.-J. Slagerhttps, The Dry Ten Martini Problem at Criticality. 2021, arXiv:2112.06869.

[28] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*. Ann. of Math. Stud. 158, Princeton University Press, Princeton, NJ, 2005.

[29] J. Bourgain, Positivity and continuity of the Lyapunov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. *J. Anal. Math.* **96** (2005), 313–355.

[30] J. Bourgain and M. Goldstein, On nonperturbative localization with quasiperiodic potential. *Ann. of Math.* **152** (2000), 835–879.

[31] J. Bourgain and S. Jitomirskaya, Absolutely continuous spectrum for 1D quasiperiodic operators. *Invent. Math.* **148** (2002), 453–463.

[32] J. Bourgain and S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Stat. Phys.* **108** (2002), 1203–1218. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[33] V. Buslaev and A. Fedotov, The monodromization and Harper equation. *Sémin. Équ. Dériv. Partielles* exp. no. **21** (1993–1994), 1–21.

[34] C. Cedzich, J. Fillman, and D. C. Ong, Almost everything about the unitary almost Mathieu operator. 2021, arXiv:2112.03216.

[35] H. Cheng, L. Ge, J. You, and Q. Zhou, Global rigidity for ultra-differentiable quasiperiodic cocycles and its spectral applications. 2021, arXiv:2101.11150.

[36] M. D. Choi, G. A. Elliott, and N. Yui, Gauss polynomials and the rotation algebra. *Invent. Math.* **99** (1990), 225–246.

[37] E. Dinaburg and Ya. Sinai, The one-dimensional Schrödinger equation with a quasi-periodic potential. *Funct. Anal. Appl.* **9** (1975), 279–289.

[38] P. Duarte and S. Klein, *Lyapunov exponents of linear cocycles: continuity via large deviations*. Atlantis Ser. Dyn. Syst. 3, Atlantis Press, 2016.

[39] L. H. Eliasson, Reducibility and point spectrum for linear quasi-periodic skew products. *Doc. Math.* Extra volume ICM 1998 II, 779–787.

[40] A. Fedotov and F. Klopp, Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case. *Comm. Math. Phys.* **227** (2002), 1–92.

[41] A. Fedotov and F. Sandomirskiy, An exact renormalization formula for the Maryland model. *Comm. Math. Phys.* **334** (2015), no. 2, 1083–1099.

[42] S. Fishman, Anderson localization and quantum chaos maps. *Scholarpedia* **5** (2010), no. 8, 9816.

[43] Y. Forman and T. VandenBoom, Localization and Cantor spectrum for quasiperiodic discrete Schrödinger operators with asymmetric, smooth, cosine-like sampling functions. 2021, arXiv:2107.05461.

[44] J. Fröhlich, T. Spencer, and P. Wittwer, Localization for a class of one dimensional quasi-periodic Schrödinger operators. *Comm. Math. Phys.* **132** (1990), 5–25.

[45] S. Ganeshan, K. Kechedzhi, and S. Das Sarma, Critical integer quantum hall topology and the integrable Maryland model as a topological quantum critical point. *Phys. Rev. B* **90** (2014), no. 4, 041405.

[46] L. Ge, in preparation.

[47] L. Ge, S. Jitomirskaya, J. You, and Q. Zhou, Multiplicative Jensen's formula and quantitative global theory of one-frequency Schrödinger operators, in preparation.

[48] L. Ge, Y. Wang, J. You, and X. Zhao, Transition space for the continuity of the Lyapunov exponent of quasiperiodic Schrödinger cocycles. 2021, arXiv:2102.05175.

[49] L. Ge, J. Xu, and J. Wang, The Hölder continuity of Lyapunov exponents for a class of cos-type quasiperiodic Schrödinger cocycles. 2020, arXiv:2006.03381.

[50] L. Ge and J. You, Arithmetic version of Anderson localization via reducibility. *Geom. Funct. Anal.* **30** (2020), no. 5, 1370–1401.

[51] L. Ge, J. You, and X. Zhao, Arithmetic version of Anderson localization for quasiperiodic Schrödinger operators with even cosine type potentials. 2021, arXiv:2107.08547.

[52] L. Ge, J. You, and X. Zhao, Arithmetic version of the frequency transition conjecture: new proof and generalization. *Peking J. Math.* (2021). https://doi.org/10.1007/s42543-021-00040-y.

[53] M. Goldstein and W. Schlag, On resonances and the formation of gaps in the spectrum of quasi-periodic Schroedinger equations. *Ann. of Math.* **173** (2011), 337–475.

[54] A. Y. Gordon, The point spectrum of the one-dimensional Schrödinger operator. *Uspekhi Mat. Nauk* **31** (1976), 257–258.

[55] A. Y. Gordon, S. Jitomirskaya, Y. Last, and B. Simon, Duality and singular continuous spectrum in the almost Mathieu equation. *Acta Math.* **178** (1997), 169–183.

[56] D. Grempel, S. Fishman, and R. Prange, Localization in an incommensurate potential: an exactly solvable model. *Phys. Rev. Lett.* **49** (1982), no. 11, 833.

[57] R. Han and S. Jitomirskaya, Full measure reducibility and localization for quasiperiodic Jacobi operators: a topological criterion. *Adv. Math.* **319** (2017), 224–250.

[58] R. Han, S. Jitomirskaya, and F. Yang, Anti-resonances and sharp analysis of Maryland localization for all parameters, 2022, arXiv:2205.04021

[59] R. Han, S. Jitomirskaya, and F. Yang, Universal hierarchical structure of eigenfunctions in the Maryland model, in preparation.
[60] A. Haro and J. Puig, A Thouless formula and Aubry duality for long-range Schrödinger skew-products. *Nonlinearity* **26** (2013), 1163–1187.
[61] P. G. Harper, Single band motion of conducting electrons in a uniform magnetic field. *Proc. Phys. Soc. A* **68** (1955), 874–878.
[62] B. Helffer and J. Sjöstrand, Semi-classical analysis for Harper's equation. III. Cantor structure of the spectrum. *Mém. Soc. Math. Fr. (N.S.)* **39** (1989), 1–139.
[63] M. R. Herman, Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractere local d'un théoreme d'Arnold et de Moser sur le tore de dimension 2. *Comment. Math. Helv.* **58** (1983), 453–502.
[64] D. R. Hofstadter, Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields. *Phys. Rev. B* **14** (1976), 22–39.
[65] X. Hou and J. You, Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems. *Invent. Math.* **190** (2012), no. 1, 209–260.
[66] S. Jitomirskaya, Almost everything about the almost Mathieu operator. II. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, pp. 373–382, International Press of Boston, 1994.
[67] S. Jitomirskaya, Ergodic Schrödinger operators (on one foot), Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, pp. 613–647, Proc. Sympos. Pure Math. 76, Part 2, Amer. Math. Soc., Providence, RI, 2007.
[68] S. Jitomirskaya, Critical phenomena, arithmetic phase transitions, and universality: some recent results on the almost Mathieu operator. *Curr. Dev. Math.* **2019** (2019), no. 1, 1–42.
[69] S. Jitomirskaya, On point spectrum at critical coupling. *Adv. Math.* **392** (2021), 6 pp.
[70] S. Jitomirskaya and I. Kachkovskiy, $L2$-reducibility and localization for quasiperiodic operators. *Math. Res. Lett.* **23** (2016), no. 2, 431–444.
[71] S. Jitomirskaya and S. Kocic, Spectral theory of Schrödinger operators over circle diffeomorphisms, *Int. Math. Res. Not.***2** (2021), doi = 10.1093/imrn/rnaa362
[72] S. Jitomirskaya, D. A. Koslover, and M. S. Schulteis, Localization for a family of one-dimensional quasiperiodic operators of magnetic origin. *Ann. Henri Poincaré* **6** (2005), 103–124.
[73] S. Jitomirskaya, D. A. Koslover, and M. S. Schulteis, Continuity of the Lyapunov Exponent for analytic quasiperiodic cocycles. *Ergodic Theory Dynam. Systems* **29** (2009), 1881–1905.
[74] S. Jitomirskaya, H. Krüeger, and W. Liu, Exact dynamical decay rate for the almost Mathieu operator. *Math. Res. Lett.* **27** (2020), no. 3, 789–808.
[75] S. Jitomirskaya and W. Liu, in preparation.
[76] S. Jitomirskaya and W. Liu, Arithmetic spectral transitions for the Maryland model. *Comm. Pure Appl. Math.* **70** (2017), no. 6, 1025–1051.

[77] S. Jitomirskaya and W. Liu, Universal hierarchical structure of quasiperiodic eigenfunctions. *Ann. of Math. (2)* **187** (2018), 721–776.

[78] S. Jitomirskaya and W. Liu, Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase. 2018, arXiv:1802.00781.

[79] S. Jitomirskaya, W. Liu, and S. Tcheremchantsev, Upper bounds on the fractal spectral dimensions and singular continuous spectrum near the arithmetic transition, in preparation.

[80] S. Jitomirskaya, W. Liu, and S. Zhang, Arithmetic spectral transitions: A competition between hyperbolicity and the arithmetics of small denominators IAS/Park City Math. Ser.: Harmon. Anal. Appl. **27**, 10.1090/pcms/027/02 pp. 35–72, 2020

[81] S. Jitomirskaya and C. Marx, Analytic quasi-periodic cocycles with singularities and the Lyapunov Exponent of Extended Harper's Model. *Comm. Math. Phys.* **316** (2012), 237–267.

[82] S. Jitomirskaya and C. Marx, Dynamics and spectral theory of quasi-periodic Schrödinger-type operators. *Ergodic Theory Dynam. Systems* **37** (2017), 2353–2393.

[83] S. Jitomirskaya and B. Simon, Operators with singular continuous spectrum: III. Almost periodic Schrödinger operators. *Comm. Math. Phys.* **165** (1994), 201–205.

[84] S. Y. Jitomirskaya, Anderson localization for the almost Mathieu equation: a nonperturbative proof. *Comm. Math. Phys.* **165** (1994), 49–57.

[85] S. Y. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator. *Ann. of Math. (2)* **150** (1999), 1159–1175.

[86] S. Jitomirskaya and F. Yang, Phase transition in phase: the interface. 2021, preprint.

[87] S. Jitomirskaya and F. Yang, Pure point spectrum for the Maryland model: a constructive proof. *Ergodic Theory Dynam. Systems* **41** (2021), no. 1, 283–294.

[88] S. Jitomirskaya, F. Yang, and Q. Zhou, Second phase transition line: a constructive proof, in preparation.

[89] S. Jitomirskaya and S. Zhang, Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schödinger operators. *J. Eur. Math. Soc. (JEMS)* (2022). https://doi.org/10.4171/jems/1139.

[90] S. Jitomirskaya and X. Zhu, Large deviations of the Lyapunov exponent and localization for the 1D Anderson model. *Comm. Math. Phys.* **370** (2019), 311–324.

[91] R. A. Johnson, Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients. *J. Differential Equations* **61** (1986), 54–78.

[92] S. Klein, Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function. *J. Funct. Anal.* **218** (2005), 255–292.

[93] F. Klopp and A. A. Fedotov, On the hierarchical behavior of solutions of the Maryland equation in the semiclassical approximation. *Math. Notes* **108** (2020), no. 5, 906–910.

[94] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary one-dimensional Schrödinger operators. In *Stochastic Analysis*, pp. 225–248, N.-Holl. Math. Libr. 32, Elsevier, 1984.
[95] H. Kunz and B. Souillard, Sur le spectre des opérateurs aux différences finies aléatoires. *Comm. Math. Phys.* **78** (1980), 201–246.
[96] Last, Y.: Zero measure spectrum for the almost Mathieu operator. *Commun. Math Phys.* **164**(1994), 421–432 .
[97] Y. Last and B. Simon, Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators. *Invent. Math.* **135** (1999), 329–367.
[98] W. Liu, Almost Mathieu operators with completely resonant phases. *Ergodic Theory Dynam. Systems* **40** (2020), 1875–1893.
[99] W. Liu, Small denominators and large numerators of quasiperiodic Schrödinger operators. arXiv:2205.04648
[100] Y. Liu, Q. Zhou, and S. Chen, Localization transition, spectrum structure, and winding numbers for one-dimensional non-Hermitian quasicrystals. *Phys. Rev. B* **104** (2021), no. 2, 024201.
[101] V. A. Mandelshtam and S. Ya. Zhitomirskaya, 1D-quasiperiodic operators. Latent symmetries. *Comm. Math. Phys.* **139** (1991), 589–604.
[102] L. A. Pastur, Spectral properties of disordered systems in the one-body approximation. *Comm. Math. Phys.* **75** (1980), 179–196.
[103] R. Peierls, Zur Theorie des Diamagnetismus von Leitungselektronen. *Z. Phys. A: Hadrons Nucl.* **80** (1933), 763–791.
[104] J. Puig, Cantor spectrum for the almost Mathieu operator. *Comm. Math. Phys.* **244** (2004), 297–309.
[105] B. Simon, Almost periodic Schrödinger operators: A review. *Adv. in Appl. Math.* **3** (1982), 463–490.
[106] B. Simon, Fifteen problems in mathematical physics. *Oberwolfach Anniversary Volume* (1984), 423–454.
[107] B. Simon, Almost periodic Schrödinger operators. IV. The Maryland model. *Ann. Physics* **159** (1985), no. 1, 157–183.
[108] B. Simon, In *Schrödinger operators in the twenty-first century*, pp. 283–288, Math. Physics 2000, Imp. Coll. Press, London, 2000.
[109] Ya. Sinai, Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential. *J. Stat. Phys.* **46** (1987), 861–909.
[110] E. Sorets and T. Spencer, Positive Lyapunov exponents for Schrödinger operators with quasi-periodic potential. *CMP* **142** (1991), 543–566.
[111] A. Volberg and P. Yuditskii, Kotani–Last problem and Hardy spaces on surfaces of Widom type. *Invent. Math.* **197** (2014), 683–740.
[112] Y. Wang, X. Xia, J. You, Z. Zheng, and Q. Zhou, Exact mobility edges for 1D quasiperiodic models. 2021, arXiv:2110.00962.

[113] Y. Wang and J. You, Examples of discontinuity of Lyapunov exponent in smooth quasi-periodic cocycles. *Duke Math. J.* **1** (2013), no. 62, 2363–2412.

[114] Y. Wang and J. You, Quasi-Periodic Schrödinger Cocycles with Positive Lyapunov Exponent are not Open in the Smooth Topology. *Comm. Math. Phys.* **362** (2018), no. 9, 801–826.

[115] Y. Wang and Z. Zhang, Cantor spectrum for a class of C^2 quasiperiodic Schrödinger operators. *Int. Math. Res. Not.* **2017** (2017), no. 8, 2300–2366.

[116] P. B. Wiegmann and A. V. Zabrodin, Quantum group and magnetic translations Bethe ansatz for the Azbel–Hofstadter problem. *Nucl. Phys. B* **422** (1994), 495–514.

[117] J. You, Quantitative almost reducibility and applications. In *Proceedings of ICM 2018*, pp. 2131–2154, World Scientific Publishing Co Pte Ltd, Singapore, 2019.

[118] J. You and Q. Zhou, Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications. *Comm. Math. Phys.* **323** (2013), no. 3, 975–1005.

[119] J. You and Q. Zhou, Simple Counter-Examples to Kotani–Last Conjecture Via Reducibility. *Int. Math. Res. Not.* **2015** (2015), no. 19, 9450–9455.

[120] S. Zhitomirskaya, Singular spectral properties of a one dimensional discrete Schrödinger operator with quasiperiodic potential. *Adv. Sov. Math.* **3** (1991), 215–254.

SVETLANA JITOMIRSKAYA

Georgia Institute of Technology, Atlanta GA 30332, USA; and UCI Department of Mathematics, Irvine CA 92617, USA, szhitomi@uci.edu

Critical phenomena, arithmetic phase transitions, and universality: some recent results on the almost Mathieu operator.

S. Jitomirskaya[1]

Abstract. This is an expanded version of the notes of lectures given at the conference "Current Developments in Mathematics 2019" held at Harvard University on November 22–23, 2019. We present an overview of some recent developments.

Contents

[1]Department of Mathematics, UC Irvine, USA

1 Introduction

Consider a two-dimensional discrete Laplacian: an operator on $\ell^2(\mathbb{Z}^2)$ of the form

$$(H\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + \psi_{m,n-1} + \psi_{m,n+1} \tag{1}$$

The Fourier transform makes it unitarily equivalent to multiplication by $2(\cos 2\pi x + \cos 2\pi y)$ on $L^2(\mathbb{T}^2)$ where $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. Thus its spectrum[2] is the segment $[-4, 4]$.

In physics this is a tight binding model of a single electron confined to a 2D crystal layer. What happens if we put this crystal in a uniform magnetic field with flux orthogonal to the lattice plane? Of course, we have a freedom of gauge choice, but all the resulting operators are unitarily equivalent, so we may as well choose one, the so called Landau gauge, leading to the discrete magnetic Laplacian[3] operator

$$(H(\alpha)\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + e^{-i\alpha m}\psi_{m,n-1} + e^{i\alpha m}\psi_{m,n+1} \tag{2}$$

Even though incorporating those phase factors may seem innocent enough, basic quantum mechanics teaches us that magnetic fields may have a profound effect on allowed energies. In the continuum model, subjecting the electron plane to a perpendicular magnetic field of flux α changes the standard Laplacian into a direct integral of shifted harmonic oscillators, and thus the $[0, \infty)$ spectrum of the Laplacian turns into a discrete set of infinitely degenerate Landau levels, at $c|\alpha|(n + 1/2), n \in \mathbb{N}$. It turns out that in the discrete setting, the situation is even more dramatic and also much more rich and interesting. For any irrational α, the spectrum of $H(\alpha)$ is a Cantor

[2]See Section 2 for a quick reminder on the basics of spectral theory and ergodic operators

[3]The name "discrete magnetic Laplacian" first appeared in [82]

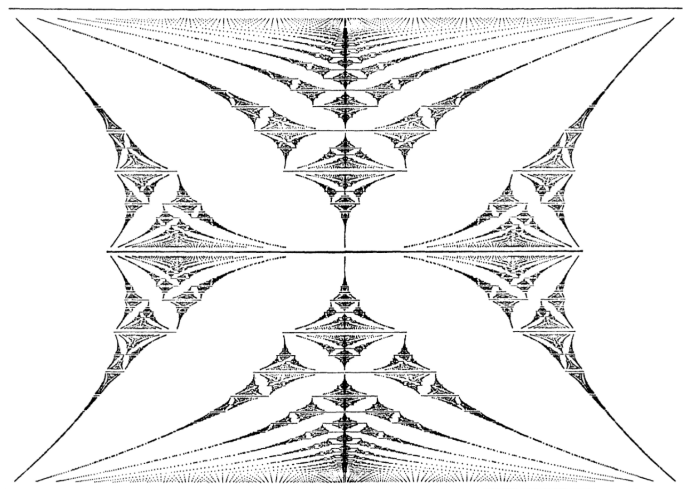

Figure 1: This picture is a plot of spectra of $H(\alpha)$ for 50 rational values of α [48]. The fluxes $\alpha = p/q$ are listed on the vertical line, and the corresponding horisontal sections are spectra of $H(\alpha)$.

set of measure zero, and the spectra for rational α, plotted together, form a beautiful self-similar structure, shown on Fig.1, called Hofstadter's butterfly.

The operator $H(\alpha)$, to the best of our knowledge, was introduced by Peierls in [80], and later studied by his student Harper. The first predictions of Cantor spectrum with arithmetic, continued-fraction based hierarchy of both the spectrum and eigenfunctions was made by Mark Azbel [21], remarkably, before any numerics was even possible. Yet, the model got a particular prominence only after Hofstadter's numerical discovery [48].

It was noticed already by Peierls in [80] that, similarly to the described above Landau gauge solutions for free electrons in a uniform magnetic field, the Landau gauge in the discrete setting, as in (2), also makes the Hamiltonian separable and turns it into the direct integral in θ of operators $H_{\alpha,\theta} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, of the form

$$(H_{\alpha,\theta}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\cos 2\pi(\alpha n + \theta)\phi(n). \qquad (3)$$

In this sense, $H_{\alpha,\theta}$ can be viewed as the tight-binding analogue of the harmonic oscillator. Here α is a magnetic flux per unit cell, and θ is a phase parameter characterizing plane waves in the direction perpendicular to the vector-potential, so has no meaning to the physics of the original 2D operators. Usually, one introduces also another parameter λ, characterizing the anisotropy of the lattice: it is the the ratio between the length of a unit cell in the direction of the vector potential and its length in the transversal direction, leading to the 2D operator

$$(H(\alpha,\lambda)\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + \lambda e^{-i\alpha m}\psi_{m,n-1} + \lambda e^{i\alpha m}\psi_{m,n+1} \tag{4}$$

and the family

$$(H_{\alpha,\theta,\lambda}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\lambda\cos 2\pi(\alpha n+\theta)\phi(n). \tag{5}$$

In physics literature, this family has appeared under the names Harper's, Azbel-Hofstadter, and Aubry-Andre model (with the first two names also used for the discrete magnetic Laplacian $H(\alpha,\lambda)$) and often restricted to the isotropic case $\lambda = 1$. In mathematics, the name almost Mathieu operator is used universally, so we also use it for these lectures. This name was originally introduced by Barry Simon [83] in analogy with the Mathieu equation $-f'' + 2\lambda\cos x f(x) = Ef(x)$. For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $H_{\alpha,\theta,\lambda}$ is an ergodic (and minimal) family, so (see Sections 2.3,2.4) the spectra $\sigma(H_{\alpha,\theta,\lambda})$ do not depend on θ and coincide with the spectrum $\sigma_{\alpha,\lambda}$ of the 2D operator $H(\alpha,\lambda)$. In general (which is only relevant for rational α), we have $\sigma_{\alpha,\lambda} = \cup_\theta \sigma(H_{\alpha,\theta,\lambda})$, and this is what the Hofstadter butterfly represents.

These lectures are devoted to some recent (roughly last 3 years) advances on this model. They are by no means comprehensive, neither historically, as we only mention past papers directly relevant to the presented results, nor even in terms of very recent advances, as it is a fast developing field with many exciting developments even in the last few years.

In physics, this model is the theoretical underpinning of the Quantum Hall Effect (QHE), as proposed by D.J. Thouless in 1983, and is therefore directly related to two Nobel prizes: von Klitzig (1998, for his experimental discovery of the Integer QHE) and Thouless (2016, for the theory behind the QHE and related topological insulators). Thouless theory is illustrated by Fig. 2, where Chern numbers corresponding to each gap are produced using the equations in [92], and color-coded, with warmer colors corresponding to positive numbers, and colder colors to negative ones.

The model also has very strong relationship to the theory of graphene (Geim and Novoselov, Nobel prize 2010), a robust 2D magnetic material

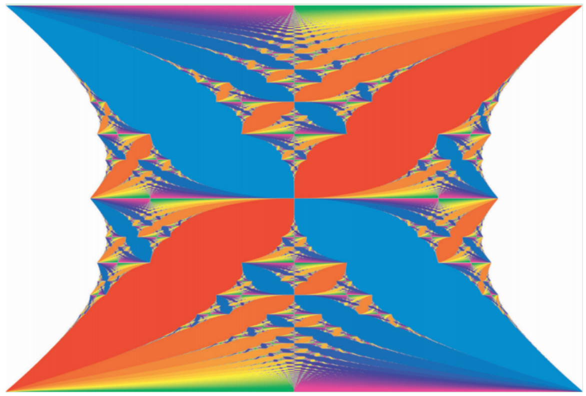

Figure 2: This picture is produced by Avron-Osadchy-Seiler [19].

whose spectra also form similar butterflies, and quasicrystals (Schechtman, Nobel prize 2011), as it is a standard model of a 1D quasicrystal. To make it a total of five Nobel prizes, one can also argue a weak relationship to the Anderson localization Nobel prize (Anderson, 1977), for Anderson localization is one property of this family for certain parameters, and, more importantly, it features the metal-insulator transition (something only seen, but prominently yet mysteriously so, in 3D or higher, for the random model). Then, one can also add a 2014 Fields medal and the 2020 Heineman prize to the list!

One of the most interesting features of the almost Mathieu family is sharp phase transitions in its several parameters, for various properties. The system, in particular, has distinct behaviors for $\lambda < 1$ and $\lambda > 1$. These two regimes have traditionally been approached perturbatively, by different KAM-type schemes, and then non-perturbative methods have been developed [51, 53], allowing to obtain the a.e. results up to the phase transition value $\lambda = 1$. Since then, even sharper localization [7, 60, 61] and reducibility [96] techniques have been developed, allowing to treat various delicate questions on both $\lambda > 1$ and $\lambda < 1$ sides. None of these methods work for the actual transition point $\lambda = 1$, and the operator at the critical value remains least understood. Yet it corresponds to the isotropic model, so is the most important operator in the one-parameter family from the physics viewpoint. From the dynamical systems point of view, the critical case is also special: the transfer-matrix cocycle for energies on the spectrum is critical in the sense of Avila's global theory (see Sections 2.5, 2.7), and thus non-amenable to either supercritical (localization) or subcritical (reducibility) methods. The global theory tells us that critical cocycles are rare in many ways, so it is almost tempting to ignore them in a large mathematical picture. Yet, as models coming from physics tend to be entirely critical on their spectra in this sense, one can actually argue that it is their study that is the most important.

After the preliminaries, we start with two very recent results on the critical case: the singular-continuous nature of the spectrum and Hausdorff dimension of the spectrum as a set, both subject to long-standing conjectures. Our solution of both conjectures is based on exploring certain hidden singularity of the model. The developed technique allowed also to obtain sharp estimate on the Hausdorff dimension of the spectrum for another interesting model, quantum graph graphene, where singularity is also present. The study of the Hausdorff dimension of course only makes sense once we know the spectrum has measure zero. This was proved by Last [73] for a.e. irrational α, but remarkably resisted treatment for the remaining zero measure set, that included the golden mean, the most popular irrational number in the physics community. Barry Simon listed the problem to obtain the result for the remaining parameters in his list of mathematical problems for the XXI century [86]. It was solved by Avila-Krikorian [10] who were able to treat Diophantine α using deep dynamical methods (for $\lambda \neq 1$ the solution was given in [57]). Our proof of the Hausdorff dimension estimate [58] (joint with Igor Krasovsky) allows also to give a very simple proof of this theorem, simultaneously for all irrational α.

Another very interesting feature of the almost Mathieu family is that, while α is a parameter coming from physics, the system behaves differently depending on whether α is rational or irrational. While this aspect was well understood already in the 60s, and the metal-insulator transition at $\lambda = 1$ was discovered by the physicists, Aubry and Andre [1], the physicists missed further dependence on the arithmetics within the class of irrational numbers. In mathematics, it was soon understood by Avron and Simon [17], based on Gordon [40], that within the super-critical regime the arithmetics of α plays a role, and later, in [67], that so does the arithmetics of θ. In [50] we conjectured that there is the second sharp transition governed by the arithmetics of the continued fraction expansion of α and the exponential rate of phase-resonances. The recent proof of this conjecture, joint with Wencai Liu, for both the frequency and phase cases, is discussed in Sec. 6.

A very captivating question and a longstanding theoretical challenge is to explain the self-similar hierarchical structure visually obvious in the Hofstadter's butterfly, as well as the hierarchical structure of eigenfunctions, as related to the continued fraction expansion of the magnetic flux. Such structure was first predicted in the work of Azbel in 1964 [21], some 12 years before Hofstadter [48] and before numerical experimentation was possible.

The simplest mathematical feature of the spectrum for irrational α one observes in the Hofstadter's picture, is that it is a Cantor set. Mark Kac offered ten martinis in 1982 for the proof of Azbel's 1964 Cantor set conjecture. It was dubbed the Ten Martini problem by Barry Simon, who advertised it in his lists of 15 mathematical physics problems [85] and later, mathematical physics problems for the XXI century [86]. Most substantial partial solutions were made by Bellissard, Simon, Sinai, Helffer, Sjöstrand, Choi, Eliott, Yui, and Last, between 1983-1994. J. Puig [81] solved it for Diophantine α by noticing that localization at $\theta = 0$ [53] leads to gaps at corresponding (dense) energies. Final solution was given in [7]. Cantor spectrum is also generic for general one-frequency operators with analytic potential: in the subcritical regime [8], and, by very different methods, in the supercritical regime [39] (and it is conjectured [9] also in the critical regime, which is actually non-generic in itself [5]). Moreover, even all gaps predicted by the gap labeling are open in the non-critical almost Mathieu case [8, 15]. Ten Martini and its dry version were very important challenges in themselves, even though these results, while strongly indicate, do not describe or explain the hierarchical structure, and the problem of its description/explanation remains open, even in physics. As for the understanding the hierarchical behavior of the eigenfunctions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [94], it has also remained an important open challenge even at the physics level. Certain results indicating the hierarchical structure in the corresponding semi-classical/perturbative regimes were previously obtained in the works of Sinai, Helffer-Sjostrand, and Buslaev-Fedotov (see [30, 46, 88], and also [99] for a different model).

In Secs. 7,8 we present the solution of the latter problem in the exponential regime. We describe the universal self-similar exponential structure of eigenfunctions throughout the entire localization region. In particular, we determine explicit universal functions $f(k)$ and $g(k)$, depending only on the Lyapunov exponent and the position of k in the hierarchy defined by the denominators q_n of the continued fraction approximants of the flux α, that completely define the exponential behavior of, correspondingly, eigenfunctions and norms of the transfer matrices of the almost Mathieu operators, for all eigenvalues corresponding to a.e. phase, see Theorem 8.1. Our result holds for *all* frequency and coupling pairs in the localization regime. Since the behavior is fully determined by the frequency and does not depend on the phase, it is the same, eventually, around any starting point, so is also seen unfolding at different scales when magnified around local eigenfunction maxima, thus describing the exponential universality in the hierarchical structure, see, for example, Theorems 7.2, 7.4.

Moreover, our proof of the phase part of the arithmetical spectral transition conjecture uncovers a universal structure of the eigenfunctions throughout the corresponding pure point spectrum regime, Theorem 8.1, which, in presence of exponentially strong resonances, demonstrates a new phenomenon that we call a *reflective hierarchy*, when the eigenfunctions feature self-similarity upon proper reflections (Theorem 8.2). This phenomenon was not even previously described in the (vast) physics literature. This joint work with Wencai Liu will also be presented in Sections 7,8.

In the next section we list the basic definitions/necessary facts. Sections 3-5 are devoted to the critical almost Mathieu operator, and Sections 6-8 to sharp arithmetic spectral transitions and universal structure of eigenfunctions in the (supercritical) regime of localization.

2 The basics

2.1 The spectrum

The spectrum of a bounded linear operator H on a Hilbert space $\mathcal{H}$, denoted $\sigma(H)$, is the set of energies E for which $H - E$ does not have a bounded inverse. If $\mathcal{H}$ is finite-dimensional, it clearly coincides with the set of the eigenvalues. For an infinite-dimensional space, however, there are more ways not to be invertible than to have a kernel.

Example: Let (X, μ) be a measure space. Given bounded $f : X \to \mathbb{R}$, define the multiplication operator H_f by

$$H_f : L^2(X, \mu) \to L^2(X, \mu),\ H_f(g) = fg.$$

Then the formal inverse of $H_f - E$ is, of course, $H_{\frac{1}{f-E}}$, and it is easy to show that $\sigma(H_f)$ is the μ-essential range of f, that is $\{E : \mu\{x : |f(x) - E| < \epsilon\}) > 0, \text{any } \ \epsilon > 0.\}$

Note that the specrum is a unitary invariant, and it turns out that the example above is in this sense all there is:

Spectral theorem: Every self-adjoint $A : \mathcal{H} \to \mathcal{H}$ is unitarily equivalent to H_f for some f, X, μ.

It should be noted that no uniqueness of either of f, X or μ is claimed (or holds) here; in fact the more standard statement is with and f fixed as x, X being a direct sum of copies of $\mathbb{R}$.

Example 1: If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a self-adjoint matrix with distinct eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, one can take $X = \mathbb{R}$, μ any measure that lives on $\cup_{i=1}^n \lambda_i$ and gives non-zero weight to each λ_i, and $f = x$. Then $L^2(X, \mu)$ is just $\mathbb{R}^n$ and the spectral theorem boils down to the diagonalization theorem for self-adjont matrices. In case of higher dimensional eigenspaces, one can take X equal to the union of k copies of $\mathbb{R}$, with k equal to the largest multiplicity of an eigenvalue, and modify the μ accordingly, keeping $f = x$.

Example 2: By Fourier transforming $\ell^2(\mathbb{Z}^2)$ into $L^2(\mathbb{T}^2)$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the discrete 2D Laplacian

$$(H\psi)_{m,n} = \psi_{m-1,n} + \psi_{m+1,n} + \psi_{m,n-1} + \psi_{m,n+1}$$

is unitarily equivalent to $H_{2\cos x + 2\cos y}$ on $L^2(\mathbb{T}^2)$, so $\sigma(H) = [-4, 4]$.

2.2 Spectral measure of a self-adjoint operator

Let H be a self-adjoint operator on a Hilbert space $\mathcal{H}$. The time evolution of a wave function is described in the Schrödinger picture of quantum mechanics by

$$i\frac{\partial \psi}{\partial t} = H\psi.$$

The solution with initial condition $\psi(0) = \psi_0$ is then given by

$$\psi(t) = e^{-itH}\psi_0.$$

Another version of the spectral theorem says that for any $\psi_0 \in \mathcal{H}$, there is a unique finite measure μ_{ψ_0} (called the spectral measure of $\psi_0 \in \mathcal{H}$) such that

$$(e^{-itH}\psi_0, \psi_0) = \int_{\mathbb{R}} e^{-it\lambda} d\mu_{\psi_0}(\lambda). \tag{6}$$

2.3 Spectral decompositions

Every finite measure on $\mathbb{R}$ is uniquely decomposed into three mutually singular parts

$$\mu = \mu_{pp} + \mu_{sc} + \mu_{ac},$$

where *pp* stands for pure point, the atomic part of the measure, *ac* stands for absoltely continuous with respect to Lebesgue measure, and *sc* stands for singular continuous, that is all the rest: the part that is singular (with respect to Lebesgue), yet continuous (has no atoms).
Define

$$\mathcal{H}_\gamma = \{\phi \in \mathcal{H} : \mu_\phi \text{ is } \gamma\}$$

where $\gamma \in \{pp, sc, ac\}$. Then we have $\mathcal{H} = \mathcal{H}_{pp} \bigoplus \mathcal{H}_{sc} \bigoplus \mathcal{H}_{ac}$.

H preserves each $\mathcal{H}_\gamma$, so we can define: $\sigma_\gamma(H) = \sigma(H|_{\mathcal{H}_\gamma})$, $\gamma \in \{pp, sc, ac\}$. The set $\sigma_{pp}(H)$ admits a direct characterization as the closure of the set of all eigenvalues

$$\sigma_{pp}(H) = \overline{\sigma_p(H)},$$

where

$$\sigma_p(H) = \{\lambda : \text{ there exists a nonzero vector } \psi \in \mathcal{H} \text{ such that } H\psi = \lambda\psi\}.$$

2.4 Ergodic operators

We are going to study Schrödinger operators with potentials related to dynamical systems. Let $H = \Delta + V$ be defined by

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n) \tag{7}$$

on a Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$. Here $V : \mathbb{Z} \to \mathbb{R}$ is the potential. Let (Ω, P) be a probability space. A measure-preserving bijection $T : \Omega \to \Omega$ is called ergodic, if any T-invariant measurable set $A \subset \Omega$ has either $P(A) = 1$ or $P(A) = 0$. By a dynamically defined potential we understand a family $V_\omega(n) = v(T^n\omega), \omega \in \Omega$, where $v : \Omega \to \mathbb{R}$ is a measurable function. The corresponding family of operators $H_\omega = \Delta + V_\omega$ is called an ergodic family. More precisely,

$$(H_\omega u)(n) = u(n+1) + u(n-1) + v(T^n\omega)u(n). \tag{8}$$

Theorem 2.1 (Pastur [79]; Kunz-Souillard [72])**.** *There exists a full measure set Ω_0 and $\sum$, $\sum_{pp}$, $\sum_{sc}$, $\sum_{ac} \subset \mathbb{R}$ such that for all $\omega \in \Omega_0$, we have $\sigma(H_\omega) = \sum$, and $\sigma_\gamma(H_\omega) = \sum_\gamma$, $\gamma = pp, sc, ac$.*

Theorem 2.2 (Avron-Simon [18], Last-Simon [76])**.** *If T is minimal, then $\sigma(H_\omega) = \sum$, and $\sigma_{ac}(H_\omega) = \sum_{ac}$ for all $\omega \in \Omega$.*

Theorem 2.2 does not hold for $\sigma_\gamma(H_\omega)$ with $\gamma \in \{sc, pp\}$ [67], but it is an interesting and difficult open problem whether it holds for $\sigma_{sing}(H_\omega)$.

2.5 Cocycles and Lyapunov exponents

By an $SL(2,\mathbb{R})$ cocycle, we mean a pair (T, A), where $T : \Omega \to \Omega$ is ergodic, A is a measurable 2×2 matrix valued function on Ω and $\det A = 1$.

We can regard it as a dynamical system on $\Omega \times \mathbb{R}^2$ with

$$(T, A) : (x, f) \longmapsto (Tx, A(x)f),\ (x, f) \in \Omega \times \mathbb{R}^2.$$

For $k > 0$, we define the k-step transfer matrix as

$$A_k(x) = \prod_{l=k}^{1} A(T^{l-1}x). \tag{9}$$

For $k < 0$, define

$$A_k(x) = A_{-k}^{-1}(T^k x). \tag{10}$$

Denote $A_0 = I$, where I is the 2×2 identity matrix. Then $f_k(x) = \ln ||A_k(x)||$ is a subadditive ergodic process. The (non-negative) Lyapunov exponent for the cocycle (α, A) is given by

$$L(T, A) = \inf_n \frac{\int_\Omega \ln \|A_n(x)\| dx}{n} = \lim_n \frac{\int_\Omega \ln \|A_n(x)\| dx}{n} \xlongequal{\text{a.e. } x} \lim_{n\to\infty} \frac{\ln \|A_n(x)\| dx}{n}. \tag{11}$$

with both the second and the third equality in (11) guaranteed by Kingman's subadditive ergodic theorem. Cocycles with positive Lyapunov exponent are called hyperbolic. Here one should distinguish uniform hyperbolicity where there exists a continuous splitting of $\mathbb{R}^2$ into expanding and contracting directions, and nonuniform hyperbolicity, where $L > 0$ but such splitting does not exist. Nevertheless, we have

Theorem 2.3 (Oseledets)**.** *Suppose $L(T, A) > 0$. Then, for almost every $x \in \Omega$, there exist solutions $v^+, v^- \in \mathbb{C}^2$ such that $||A_k(x)v^{\pm}||$ decays exponentially at $\pm\infty$, respectively, at the rate $-L(T, A)$. Moreover, for every vector w which is linearly independent with v^+ (resp., v^-), $||A_k(x)w||$ grows exponentially at $+\infty$ (resp., $-\infty$) at the rate $L(T, A)$.*

Suppose u is an eigensolution of $H_x u = Eu$, where H_x is given by (8). Then

$$\begin{bmatrix} u(n+m) \\ u(n+m-1) \end{bmatrix} = A_n(T^m x) \begin{bmatrix} u(m) \\ u(m-1) \end{bmatrix}, \tag{12}$$

where $A_n(x)$ is the n-step transfer matrix of $(T, A_E(x))$ and

$$A_E(x) = \begin{bmatrix} TE - v(x) & -1 \\ 1 & 0 \end{bmatrix}.$$

Such $(T, A_E(x))$ are called Schrödinger cocycles. Denote by $L(E)$ the Lyapunov exponent of a Schrödinger cocycle (we omit the dependence on T and v). It turns out that (at least for uniquely ergodic dynamics) the resolvent set of H_x is precisely the set of E such that the Schrödinger cocycle $(T, A_E(x))$ is uniformly hyperbolic. The set $\sigma \cap \{L(E) > 0\}$ is therefore the set of non-uniform hyperbolicity for the one-parameter family of cocycles $(T, A_E(x))_{E \in \mathbb{R}}$, and is our main interest. Then Oseledets theorem can be reformulated as

Theorem 2.4. *Suppose that $L(E) > 0$. Then, for every $x \in \Omega_E$ (where Ω_E has full measure), there exist solutions ϕ^+, ϕ^- of $H_x\phi = E\phi$ such that $\phi^{\pm}$ decays exponentially at $\pm\infty$, respectively, at the rate $-L(E)$. Moreover, every solution which is linearly independent of ϕ^+ (resp., ϕ^-) grows exponentially at $+\infty$ (resp., $-\infty$) at the rate $L(E)$.*

It turns out that the set where the Lyapunov exponent vanishes fully determines the absolutely continuous spectrum.

Theorem 2.5 (Ishii-Pastur-Kotani)**.** *$\sigma_{ac}(H_x) = \overline{\{E \in \mathbb{R} : L(E) = 0\}}^{ess}$ for almost every $x \in \Omega$.*

The inclusion "$\subseteq$" was proved by Ishii and Pastur [49, 79]. The other, a lot more difficult, inclusion was proved by Kotani [71, 84].

2.6 Continuity of the Lyapunov exponent

Lyapunov exponent $L(\alpha, A) := L(R_\alpha, A)$ is generally not a very nice function of its parameters. It can be a discontinuous function of α at $\alpha \in \mathbb{Q}$ (almost Mathieu cocycle is one example), is generally discontinuous in A in C^0 and can be discontinuous in A even in C^∞ [93]. It is a remarkable fact, enabling much of the related theory, that it is continuous in the analytic category

Theorem 2.6. *[29, 56] $L(\beta + .,.) : \mathbb{T} \times \mathcal{C}^\omega(\mathbb{T}, SL(2,\mathbb{R})) \to \mathbb{R}$ is jointly continuous at irrational β .*

For the almost Mathieu operator, it leads to

Theorem 2.7. *[29] For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\lambda \in \mathbb{R}$ and $E \in \sigma(H_{\lambda,\alpha,\theta})$, one has $L_{\lambda,\alpha}(E) = \max\{\ln|\lambda|, 0\}$.*

2.7 Implications of Avila's global theory

Continuity of the Lyapunov exponent in the analytic category [29, 56] makes it possible to make conclusions from the study of its behavior for complexified cocycles, and Avila [5] discovered a remarkable related structure. Analytic cocycles $A(x)$ can be classified depending on the behavior of the Lyapunov exponent L^ϵ of the complexified cocycle $A(x + i\epsilon)$. Namely, we distinguish three cases:

Subcritical: $L^\epsilon = 0, \epsilon < \delta,\ \delta > 0.$

Supercritical: $L^0 > 0$

Critical: Otherwise, that is $L^0 = 0, L^\epsilon > 0, \epsilon > 0.$

Avila observed that, for a given cocycle, L^ϵ is a convex function of ϵ, and proved that it has quantized derivative in ϵ. This has enabled the global theory [5], where Avila shows, in particular, that prevalent potentials are acritical, that is have no critical transfer-matrix cocycles for energies in their spectrum. The almost reducibility conjecture [5, 8] states that subcritical cocycles are almost reducible, that is have constant cocycles in the closure of their analytic conjugacy class. It was solved by Avila for the Liouville case in [3] and the solution for the Diophantine case has been announced [4]. Both almost reducible and supercritical cocycles are well studied and their basic spectral theory is understood.

For the almost Mathieu cocycle, quantization of acceleration allows to exactly compute $L^\epsilon(E)$ for E in the spectrum, leading to

Subcritical, i.e. $\lambda < 1$**:** In this case, $L^\epsilon(E) = 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$ and $\epsilon \leq \frac{-\ln\lambda}{2\pi}$. $H_{\lambda,\alpha,\theta}$ has purely ac spectrum [8, 16].

Critical, i.e. $\lambda = 1$**:** In this case, for $E \in \sigma(H_{\lambda,\alpha,\theta})$ the cocycle is critical

Supercritical, i.e. $\lambda > 1$**:** $L(E) = \ln\lambda > 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$.

We now quickly review the basics of continued fraction approximations.

2.8 Continued fraction expansion

Define, as usual, for $0 \leq \alpha < 1$,

$$a_0 = 0, \alpha_0 = \alpha,$$

and, inductively for $k > 0$,

$$a_k = [\alpha_{k-1}^{-1}], \alpha_k = \alpha_{k-1}^{-1} - a_k.$$

We define

$$\begin{aligned} p_0 &= 0, \quad q_0 = 1, \\ p_1 &= 1, \quad q_1 = a_1, \end{aligned}$$

and inductively,

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2}, \\ q_k &= a_k q_{k-1} + q_{k-2}. \end{aligned}$$

Recall that $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of denominators of best rational approximants to irrational number α, since it satisifies

$$\text{for any } 1 \leq k < q_{n+1}, \|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq ||q_n\alpha||_{\mathbb{R}/\mathbb{Z}}. \tag{13}$$

Moreover, we also have the following estimate,

$$\frac{1}{2q_{n+1}} \leq \Delta_n \triangleq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{q_{n+1}}. \tag{14}$$

- α is called Diophantine if there exists $\kappa, \nu > 0$ such that $||k\alpha|| \geq \frac{\nu}{|k|^\kappa}$ for any $k \neq 0$, where $||x|| = \min_{k\in\mathbb{Z}} |x-k|$.

- α is called Liouville if

$$\beta(\alpha) = \limsup_{k\to\infty} \frac{-\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n} > 0 \tag{15}$$

- α is called weakly Diophantine if $\beta(\alpha) = 0$.

Clearly, Diophantine implies weakly Diophantine. By Borel-Cantelli lemma, Diophantine α form a set of full Lebesgue measure.

3 Do critical almost Mathieu operators ever have eigenvalues?

The critical almost Mathieu operator $H_{\alpha,\theta}$ given by

$$(H_{\alpha,\theta}\phi)(n) = \phi(n-1) + \phi(n+1) + 2\cos 2\pi(\alpha n + \theta)\phi(n), \tag{16}$$

has been long (albeit not from the very beginning [83][4]) conjectured to have purely singular continuous spectrum for every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and every θ. Since the spectrum (which is θ-independent for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ [18]) has Lebesgue measure zero [10], the problem boils down to the proof of absence of eigenvalues, see e.g. problem 7 in [52]. This simple question has a surprisingly rich (and dramatic) history.

Aside from the results on topologically generic absence of point spectrum [17, 67] that hold in a far greater generality, all the proofs were, in one way or another, based on the Aubry duality [1], a Fourier-type transform for which the family $\{H_{\alpha,\theta}\}_\theta$ is a fixed point. One manifestation of the Aubry duality is: if $u \in \ell^2(\mathbb{Z})$ solves the eigenvalue equation $H_{\alpha,\theta}u = Eu$, then $v_n^x := e^{2\pi i n\theta}\hat{u}(x+n\alpha)$ solves

$$H_{\alpha,x}v^x = Ev^x \tag{17}$$

for a.e. x, where $\hat{u}(x) = \sum e^{2\pi i n x}u_n$ is the Fourier transform of u. This led Delyon [35] to prove that there are no ℓ^1 solutions of $H_{\alpha,\theta}u = Eu$, for

[4]It is the paper where the name *almost Mathieu* was introduced.

otherwise (17) would hold also for $x = \theta$, leading to a contradiction. Thus any potential eigenfunctions must be decaying slowly. Chojnacki [32] used duality-based C^*-algebraic methods to prove the existence of some continuous component, but without ruling out the point spectrum. [41] gave a duality-based argument for no point spectrum for a.e. θ, but it had a gap, as it was based on the validity of Deift-Simon's [33] theorem on a.e. mutual singularity of singular spectral measures, which is only proved in [33] in the hyperbolic case, and is still open in the regime of zero Lyapunov exponents. Avila and Krikorian (see [6]) used convergence of renormalization [11] and non-perturbative reducibility [29] to show that for every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, eigenvalues may only occur for countably many θ. Then Avila [6] found a simple proof of the latter fact, also characterizing this potentially exceptional set of phases explicitly: these are phases θ that are α-rational, i.e. $2\theta + k\alpha \in \mathbb{Z}$, for some k. The argument of [6] was incorporated in [9], where it was developed to prove a.e. absence of point spectrum for the extended Harper's model (EHM) in the entire critical region (the EHM result was later further improved by Han [42]). The proof in [6, 9] has as a starting point the dynamical formulation of the Aubry duality: if v_n^x solves the eigenvalue equation $H_{\alpha,x}v = Ev$, then so does its complex conjugate $\bar{v}_n^x$, and this can be used to construct an L^2-reducibility of the transfer-matrix cocycles to the rotation by θ, given independence of v and $\bar{v}$. Unfortunately those vectors are always linearly dependent if θ is α-rational. Thus the argument hopelessly breaks down for $2\theta + k\alpha \in \mathbb{Z}$.

Moreover, it was noted in [9] that in the bulk of the critical region, for α-rational phases θ, the extended Harper's operator actually does have eigenvalues. Also, supercritical almost Mathieu with Diophantine α, has eigenvalues (with exponentially decaying eigenfunctions) for α-rational phases as well [56]. All this increased the uncertainty about whether eigenvalues may exist for the α-rational phases also for the critical almost Mathieu.

We will present the fully self-contained proof of

Theorem 3.1. *[54] $H_{\alpha,\theta}$ does not have eigenvalues for any α, θ (and thus has purely singular-continuous spectrum for all $\alpha \notin \mathbb{Q}$).*

In our proof we replace the Aubry duality by a new transform, inspired by the chiral gauge transform of [58].

4 Proof of Theorem 3.1.

Given $u \in \ell^2(\mathbb{Z})$, set

$$u(x) = \sum_{n=-\infty}^{\infty} u_n e^{\pi i n(\theta + n\alpha - 2x)} \tag{18}$$

and

$$u_n^x = u(x + n\alpha) e^{\pi i n(x + \frac{n\alpha - 3\theta}{2})} \tag{19}$$

where u_n^x is defined for a.e. x.

Let $\tilde{H}_\alpha^x : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}, x \in \mathbb{R}/2\mathbb{Z}$, be given by

$$(\tilde{H}_\alpha^x v)_n = 2\cos \pi(x + n\alpha) v_{n-1} + 2\cos \pi(x + (n+1)\alpha) v_{n+1} \tag{20}$$

Lemma 4.1. *If $u \in \ell^2(\mathbb{Z})$ solves $H_{\alpha,\theta} u = Eu$, then $u^x \in \mathbb{R}^{\mathbb{Z}}$ is a formal solution of the difference equation*

$$\tilde{H}_\alpha^{x + \frac{\theta - \alpha}{2}} u^x = E u^x \tag{21}$$

for a.e. x.

Proof. If $(Tu)_n := u_{n+1} + u_{n-1}$, and $(Su)_n := \cos 2\pi(\theta + n\alpha) u_n$, we obtain $(Tu)(x) = u(x - \alpha) e^{\pi i(\theta + \alpha - 2x)} + u(x + \alpha) e^{\pi i(-\theta + \alpha + 2x)}$ and $(Su)(x) = u(x - \alpha) e^{2\pi i\theta} + u(x + \alpha) e^{-2\pi i\theta}$, leading, by a straightforward computation, to $((T + S)u)^x = \tilde{H}_\alpha^{x + \frac{\theta - \alpha}{2}} u^x$. □

We note that the family $\{\tilde{H}_\alpha^x\}_{x \in \mathbb{R}/2\mathbb{Z}}$ is self-dual with respect to the Aubry-type duality. Namely, the following holds. For $x \in \mathbb{R}/2\mathbb{Z}, v \in \ell^2(\mathbb{Z})$ for a.e. β, we can define $w^\beta \in \mathbb{R}^{\mathbb{Z}}$ by

$$w_n^\beta = \hat{v}(\frac{\beta + n\alpha}{2}) e^{\pi i n(x + \frac{\alpha}{2})}. \tag{22}$$

Lemma 4.2. *If $v \in \ell^2(\mathbb{Z})$ solves $\tilde{H}_\alpha^x v = Ev$, then, for a.e. β, $w^\beta \in \mathbb{R}^{\mathbb{Z}}$ is a formal solution of the difference equation*

$$\tilde{H}_\alpha^{\beta - \frac{\alpha}{2}} w^\beta = E w^\beta. \tag{23}$$

Proof. A similar direct computation. □

Let now $u \in \ell^2(\mathbb{Z})$ with $\|u\|_2 = 1$ be a solution of $H_{\alpha,\theta} u = Eu$. By Lemma 4.1, (21) holds, which implies that we also have, for a.e. x,

$$\tilde{H}_{\alpha}^{x+\frac{\theta-\alpha}{2}}\bar{u}^{x} = E\bar{u}^{x} \tag{24}$$

thus the Wronskian of u^x and $\bar{u}^x$ is constant in n. That is

$$\cos\pi(x+n\alpha)\mathrm{Im}\,(u(x+n\alpha)\bar{u}(x+(n-1)\alpha)e^{\pi i(x+n\alpha+ia(\alpha,\theta))}) = c(x) \tag{25}$$

for some $c(x)$, all n and a.e. x. Here and below $a(\alpha,\theta)$ stands for (an explicit) real-valued function that does not depend on n, x. Its exact form is not important. $a(\alpha,\theta)$ may stand for different such functions in different expressions.

By ergodicity, this implies that, for a.e. x and some constant c,

$$\cos\pi x(u(x)\bar{u}(x-\alpha)e^{\pi ix+ia(\alpha,\theta)} - u(x-\alpha)\bar{u}(x)e^{-\pi ix-ia(\alpha,\theta)}) = c. \tag{26}$$

It follows by Cauchy-Schwarz that $u(x)\bar{u}(x-\alpha)e^{\pi ix+ia(\alpha,\theta)} \in L^1$, which implies that $c=0$. We note that a similar argument was used by R. Han in [42]. Thus we have

$$u(x)\bar{u}(x-\alpha)e^{\pi ix+ia(\alpha,\theta)} - u(x-\alpha)\bar{u}(x)e^{-\pi ix-ia(\alpha,\theta)} = 0 \tag{27}$$

for a.e. x.

Lemma 4.3. *For a.e. x, we have $u(x)\neq 0$.*

Proof. Indeed, otherwise, by the ergodic theorem, there would exist (in fact, a full measure of, but it is not important) x such that u_n^x solves (21) and $u_n^x = 0$ for infinitely many n (in fact, only four such n suffice for the argument). Let $n_i < n_{i+1}-1, i\in\mathbb{Z}$, be the labeling of zeros of such u_n^x. Clearly, if $v\in\mathbb{R}^{\mathbb{Z}}$ is a solution of (20) with $v_n = v_m = 0$, we have that $v_{[n,m]}\in\ell^2(\mathbb{Z})$ defined by $(v_{[n,m]})_k = \begin{cases} v_k, k\in[n+1,m-1] \\ 0, \text{otherwise}\end{cases}$ is also a solution of (21). Set $v^{x,i} := u^x_{[n_i,n_{i+1}]}$.

Clearly, for any $I\subset\mathbb{Z}$ the collection $\{v^{x,i}\}_{i\in I}$ is linearly independent in $\ell^2(\mathbb{Z})$. This implies that the corresponding Aubry dual collection $\{w^{x,i,\beta}\}_{i\in I}$ constructed by (22) from $\{v^{x,i}\}_{i\in I}$, is linearly independent in $\mathbb{R}^{\mathbb{Z}}$. Thus, by Lemmas 4.1,4.2 we obtain, for a.e. β, infinitely many linearly independent

$w^{x,i,\beta} \in \mathbb{R}^{\mathbb{Z}}$, that all solve (23). This is in contradiction with the fact that the space of solutions of (23) is two-dimensional for a.e. β. $\square$

Therefore we can define for a.e. x, a unimodular measurable function on $\mathbb{R}/2\mathbb{Z}$

$$\phi(x) := \frac{u(x)}{\bar{u}(x)} e^{\pi i x + i a(\alpha,\theta)} \tag{28}$$

By (5.2), (28) we have that, for a.e. x,

$$\phi(x) = \phi(x-\alpha) e^{-2\pi i x + i a(\alpha,\theta)}, \tag{29}$$

and expanding $\phi(x)$ into the Fourier series, $\phi(x) = \sum_{k=-\infty}^{\infty} a_k e^{\pi i k x}$, we obtain $|a_{k+2}| = |a_k|$, a contradiction. $\square$

5 Thouless' Hausdorff dimension conjecture

The spectrum of $H_{\alpha,\theta}$ for irrational α is a θ-independent[5] fractal, beautifully depicted via the Hofstadter butterfly [48]. There have been many numerical and heuristic studies of its fractal dimension in physics literature (e.g., [38, 69, 89, 95]). A conjecture attributed to Thouless (e.g., [95]), and appearing already in the early 1980's, is that the dimension is equal to 1/2. It has been rethought after rigorous and numerical studies demonstrated that the Hausdorff dimension can be less than 1/2 (and even be zero) for some α [12, 75, 95], while packing/box counting dimension can be higher (even equal to one) for some (in fact, of the same!) α [68]. However, all these are Lebesgue measure zero sets of α, and the conjecture may still hold, in some sense. There is also a conjecture attributed to J. Bellissard (e.g., [45, 75]) that the dimension of the spectrum is a property that only depends on the tail in the continued fraction expansion of α and thus should be the same for a.e. α (by the properties of the Gauss map). We discuss the history of rigorous results on the dimension in more detail below.

In the past few years, there was an increased interest in the dimension of the spectrum of the critical almost Mathieu operator, leading to a number of other rigorous results mentioned above. Those include zero Hausdorff dimension for a subset of Liouville α by Last and Shamis [75], also extended to all weakly Liouville[6] α by Avila, Last, Shamis, Zhou [12]; the full packing (and therefore box counting) dimension for weakly Liouville α [68], and

[5] Also for any $\lambda \neq 0$.

[6] We say α is weakly Liouville if $\beta(\alpha) := -\limsup \frac{\ln \|n\alpha\|}{n} > 0$, where $\|\theta\| = \operatorname{dist}(\theta, \mathbb{Z})$.

existence of a dense positive Hausdorff dimension set of Diophantine α with positive Hausdorff dimension of the spectrum by Helffer, Liu, Qu, and Zhou [45]. All those results, as well as heuristics by Wilkinson-Austin [95] and, of course, numerics, hold for measure zero sets of α. Recently, B. Simon listed the problem to determine the Hausdorff dimension of the spectrum of the critical almost Mathieu on his new list of hard unsolved problems [87].

The equality in the original conjecture can be viewed as two inequalities. In a joint work with Igor Krasovsky [58] we prove one of those for *all* irrational α. This is also the first result on the fractal dimension that holds for more than a measure zero set of α. Denote the spectrum of an operator K by $\sigma(K)$, the Lebesgue measure of a set A by $|A|$, and its Hausdorff dimension by $\dim_{\mathrm{H}}(A)$. We have

Theorem 5.1. *[58] For any irrational α and real θ,* $\dim_{\mathrm{H}}(\sigma(H_{\alpha,\theta})) \leq 1/2$.

Of course, it only makes sense to discuss upper bounds on the Hausdorff dimension of a set on the real line once its Lebesgue measure is shown to be zero. The Aubry-Andre conjecture stated that the measure of the spectrum of $H_{\alpha,\theta,\lambda}$ is equal to $4|1-|\lambda||$, so to 0 if $\lambda = 1$, for any irrational α. This conjecture was popularized by B. Simon, first in his list of 15 problems in mathematical physics [85] and then, after it was proved by Last for a.e. α [73, 74], again as Problem 5 in [86], which was to prove this conjecture for the remaining measure zero set of α, namely, for α of bounded type.[7] The arguments of [73, 74] did not work for this set, and even though the semi-classical analysis of Hellfer-Sjöstrand [46] applied to some of this set for $H_{\alpha,\theta}$, it did not apply to other such α, including, most notably, the golden mean — the subject of most numerical investigations. For the non-critical case, the proof for all α of bounded type was given in [58], but the critical "bounded-type" case remained difficult to crack. This remaining problem for zero measure of the spectrum of $H_{\alpha,\theta}$ was finally solved by Avila-Krikorian [10], who employed a deep dynamical argument. We note that the argument of [10] worked not for all α, but for a full measure subset of Diophantine α. Here we give a very simple argument that recovers this theorem and thus gives an elementary solution to Problem 5 of [86]. Moreover, our argument works simultaneously for all irrational α.

Theorem 5.2. *For any irrational α and real θ,* $|\sigma(H_{\alpha,\theta})| = 0$.

[7] That is α with all coefficients in the continued fraction expansion bounded by some M.

The proofs are based on two key ingredients. We introduce what we call the chiral gauge transform and show that the direct sum in θ of operators $H_{2\alpha,\theta}$ is isospectral with the direct sum in θ of $\widehat{H}_{\alpha,\theta}$ given by

$$(\widehat{H}_{\alpha,\theta}\phi)(n) = 2\sin 2\pi(\alpha(n-1)+\theta)\phi(n-1) + 2\sin 2\pi(\alpha n+\theta)\phi(n+1). \tag{30}$$

This representation of the almost Mathieu operator corresponds to choosing the chiral gauge for the perpendicular magnetic field applied to the electron on the square lattice,

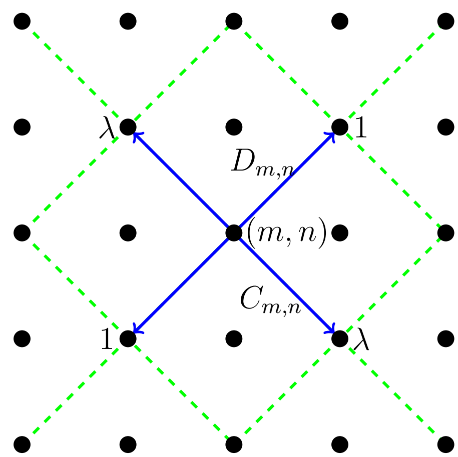

Figure 3

Any choice of gauge such that

$$C_{m,n} + D_{m+1,n-1} - C_{m+1,n-1} - D_{m,n} = 2\pi \cdot 2\alpha, \tag{31}$$

leads to an operator on $\ell^2(\mathbb{Z}^2)$

$$(H_{C,D}\psi)_{m,n} = e^{iC_{m,n}}\psi_{m+1,n-1} + e^{iD_{m,n}}\psi_{m+1,n+1} + e^{-iC_{m-1,n+1}}\psi_{m-1,n+1} + e^{-iD_{m-1,n-1}}\psi_{m-1,n-1} \tag{32}$$

which represents the Hamiltonian of an electron in a uniform perpendicular magnetic field with flux $2\pi\alpha$. Here $4\pi\alpha$ is the total flux through each doubled cell.

The chiral gauge that corresponds to (30) is given by

$$\begin{cases} C_{m,n} \equiv 0 \\ D_{m,n} = 4\pi m\alpha \end{cases}$$

It was previously discussed non-rigorously in [78, 94]. The advantage of (30) is that it is a *singular* Jacobi matrix, that is one with off-diagonal elements not bounded away from zero, so that the matrix quasi-separates into blocks.

This alone is already sufficient to conclude Theorem 5.2 because $\hat{H}$ is represented by a matrix with off-diagonal terms nearly vanishing along a subsequence. Singular Jacobi matrices are trace-class perturbations of direct sums of finite blocks, thus never have absolutely continuous spectrum. Therefore, by Kotani theory (that does extend to the singular case), and the fact that the Lyapunov exponent is zero on the spectrum, as easily follows from the formula for the invariance of the IDS under the gauge transform and a Thouless-type formula for the Lyapunov exponent, the measure of the spectrum must be zero.

The second key ingredient is a general result on almost Lipshitz continuity of spectra for *singular* quasiperiodic Jacobi matrices. The modulus of continuity statements have, in fact, been central in previous literature. We consider a general class of quasiperiodic C^1 Jacobi matrices, that is operators on $\ell^2(\mathbb{Z})$ given by

$$(H_{v,b,\alpha,\theta}\phi)(n) = b(\theta+(n-1)\alpha)\phi(n-1)+b(\theta+n\alpha)\phi(n+1)+v(\theta+n\alpha)\phi(n), \tag{33}$$

with $b(x), v(x) \in C^1(\mathbb{R})$, and periodic with period 1.

Let $M_{v,b,\alpha}$ be the direct sum of $H_{v,b,\alpha,\theta}$ over $\theta \in [0,1)$,

$$M_{v,b,\alpha} = \oplus_{\theta\in[0,1)} H_{v,b,\alpha,\theta}. \tag{34}$$

Continuity in α of $\sigma(M_{v,b,\alpha})$ in the Hausdorff metric was proved in [18]. Continuity of the measure of the spectrum is a more delicate issue, since, in particular, $|\sigma(M_\alpha)|$ can be (and is, for the almost Mathieu operator) discontinuous at rational α. Establishing continuity at irrational α requires quantitative estimates on the Hausdorff continuity of the spectrum. In the Schrödinger case, that is for $b = 1$, Avron, van Mouche, and Simon [20] obtained a very general result on Hölder-$\frac{1}{2}$ continuity (for arbitrary $v \in C^1$), improving Hölder-$\frac{1}{3}$ continuity obtained earlier by Choi, Elliott, and Yui [31]. It was argued in [20] that Hölder continuity of *any* order larger than 1/2 would imply the desired continuity property of the measure of the spectrum for *all* α. Lipshitz continuity of gaps was proved by Bellissard [23] for a large class of quasiperiodic operators, however without a uniform Lipshitz constant, thus not allowing to conclude continuity of the measure of the spectrum. In [57] (see also [63]) we showed a uniform almost Lipshitz continuity for Schrödinger operators with analytic potentials and Diophantine frequencies in the regime of positive Lyapunov exponents, which, in particular, allowed us to complete the proof of the Aubry-Andre conjecture for the non-critical case.

Namely, a Jacobi matrix (33) is called *singular* if for some θ_0, $b(\theta_0) = 0$. We assume that the number of zeros of b on its period is finite. In this case, uniform almost Lipshitz continuity (with a logarithmic correction) holds [58] which allows to conclude continuity of the measure of the spectrum for general singular Jacobi matrices:

Theorem 5.3. *For singular $H_{v,b,\alpha,\theta}$ as above, for any irrational α there exists a subsequence of canonical approximants $\frac{p_{n_j}}{q_{n_j}}$ such that*

$$|\sigma(M_{v,b,\alpha})| = \lim_{j\to\infty} \left|\sigma\left(M_{v,b,\frac{p_{n_j}}{q_{n_j}}}\right)\right|. \tag{35}$$

In the case of Schrödinger operators (i.e., for $b = 1$), the statement (35) was previously established in various degrees of generality in the regime of positive Lyapunov exponents [57, 66] and, in all regimes for analytic [64] or sufficiently smooth [98] v. Typically, proofs that work for $b = 1$ extend also to the case of non-vanishing b, that is *non-singular* Jacobi matrices, and there is no reason to believe the results of [64, 98] should be an exception. On the other hand, extending various Schrödinger results to the singular Jacobi case is technically non-trivial and adds a significant degree of complexity (e.g. [9, 44, 65]). Our proof however is based on showing that a singularity can be *exploited*, rather than circumvented, to establish enhanced continuity of spectra and therefore Theorem 5.3. Of course, Theorem 5.2 also follows immediately from the chiral gauge representation, the bound (36) below, and Theorem 5.3, providing a third proof of Problem 5 of [86].

Moreover, enhanced continuity combined with the chiral gauge representation allows to immediately prove Theorem 5.1 by an argument of [73]. Indeed, the original intuition behind Thouless' conjecture on the Hausdorff dimension 1/2 is based on another fascinating Thouless' conjecture [90, 91]: that for the critical almost Mathieu operator $H_{\alpha,\theta}$, in the limit $p_n/q_n \to \alpha$, we have $q_n|\sigma(M_{p_n/q_n})| \to c$ where $c = 32C_c/\pi$, C_c being the Catalan constant. Thouless argued that if $\sigma(M_\alpha)$ is "economically covered" by $\sigma(M_{p_n/q_n})$ and if all bands are of about the same size then the spectrum, being covered by q_n intervals of size $\frac{c}{q_n^2}$, has the box counting dimension 1/2. Clearly, the exact value of $c > 0$ is not important for this argument. An upper bound of the form

$$q_n|\sigma(M_{p_n/q_n})| < C, \qquad n = 1, 2, \ldots, \tag{36}$$

was proved by Last [73][8], which, combined with Hölder-$\frac{1}{2}$ continuity, led him in [73] to the bound $\leq \frac{1}{2}$ for the Hausdorff dimension for irrational α satisfying $\lim_{n\to\infty} |\alpha - p_n/q_n|q_n^4 = 0$. Such α form a zero measure set. The almost Lipschitz continuity and (36) allow us to obtain the result (Theorem 5.1) for *all* irrational α.

Since our proof of Theorem 5.1 only requires an estimate such as (36) and the existence of isospectral family of singular Jacobi matrices, it applies equally well to all other situations where the above two facts are present. For example, Becker et al [24] recently introduced a model of graphene as a quantum graph on the regular hexagonal lattice and studied it in the presence of a magnetic field with a constant flux Φ, with the spectrum denoted σ^Φ. Upon identification with the interval $[0, 1]$, the differential operator acting on each edge is then the maximal Schrödinger operator $\frac{d^2}{dx^2} + V(x)$ with domain H^2, where V is a Kato-Rellich potential symmetric with respect to 1/2. We then have

Theorem 5.4. *For any symmetric Kato-Rellich potential $V \in L^2$, the Hausdorff dimension $\dim_H(\sigma^\Phi) \leq 1/2$, for all irrational Φ.*

This result was proved in [24] for a topologically generic but measure zero set of α.

The basic idea behind the proof that singularity leads to enhanced continuity is that creating approximate eigenfunctions by cutting at near-zeros of the off-diagonal terms leads to smaller errors in the kinetic term. However, without apriori estimates on the behavior of solutions (and it is in fact natural for solutions to be large around the singularity) this in itself is insufficient to achieve an improvement over the Hölder exponent 1/2, so the argument ends up being not entirely straightforward.

6 Small denominators and arithmetic spectral transitions

In general, localization for quasiperiodic operators is a classical case of a small denominator problem, and has been traditionally approached in a per-

[8] with $C = 8e$.

turbative way: through KAM-type schemes for large couplings [36, 37, 88] (which, being KAM-type schemes, all required Diophantine conditions on frequencies). The opposite regime of very Liouville frequencies allowed proofs of delocalization by perturbation of periodic operators. Unlike the random case, where, in dimension one, localization holds for all couplings, a distinctive feature of quasiperiodic operators is the presence of metal-insulator transitions as couplings increase. Even when non-perturbative methods, for the almost Mathieu and then for general analytic potentials, were developed in the 90s [27, 51, 53], allowing to obtain localization for a.e. frequency throughout the regime of positive Lyapunov exponents, they still required Diophantine conditions, and exponentially approximated frequencies that are neither far from nor close enough to rationals remained a challenge, as for them there was nothing left to perturb about or to remove. It has gradually become clear that small denominators are not simply a nuisance, but lead to actual change in the spectral behavior.

The transitions in coupling between absolutely continuous and singular spectrum are governed by vanishing/non-vanishing of the Lyapunov exponent. It turns out that in the regime of positive Lyapunov exponents (also called supercritical in the analytic case, with the name inspired by the almost Mathieu operator) small denominators lead also to more delicate transitions: between localization (point spectrum with exponentially decaying eigenfunctions) and singular continuous spectrum. They are governed by the resonances: eigenvalues of box restrictions that are too close to each other in relation to the distance between the boxes, leading to small denominators in various expansions. All known proofs of localization, are based, in one way or another, on avoiding resonances and removing resonance-producing parameters, while all known proofs of singular continuous spectrum and even some of the absolutely continuous one are based on showing their abundance.

For quasiperiodic operators, one category of resonances are the ones determined entirely by the frequency. Indeed, for smooth potentials, large coefficients in the continued fraction expansion of the frequency lead to almost repetitions and thus resonances, regardless of the values of other parameters. Such resonances were first understood and exploited to show singular continuous spectrum for Liouville frequencies in [17], based on [40] . The strength of frequency resonances is measured by the arithmetic parameter

$$\beta(\alpha) = \limsup_{k \to \infty} -\frac{\ln ||k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{37}$$

where $||x||_{\mathbb{R}/\mathbb{Z}} = \inf_{\ell\in\mathbb{Z}} |x-\ell|$. Another class of resonances, appearing for all *even* potentials, was discovered in [67], where it was shown for the first time that the arithmetic properties of the phase also play a role and may lead to singular continuous spectrum even for the Diophantine frequencies. Indeed, for even potentials, phases with almost symmetries lead to resonances, regardless of the values of other parameters. The strength of phase resonances is measured by the arithmetic parameter

$$\delta(\alpha,\theta) = \limsup_{k\to\infty} -\frac{\ln ||2\theta + k\alpha||_{\mathbb{R}/\mathbb{Z}}}{|k|} \tag{38}$$

In both these cases, the strength of the resonances is in competition with the exponential growth controlled by the Lyapunov exponent. It was conjectured in 1994 [50] that for the almost Mathieu family- the prototypical quasiperiodic operator - the two above types of resonances are the only ones that appear, and the competition between the Lyapunov growth and resonance strength resolves, in both cases, in a sharp way.

Recall that α is called weakly Diophantine if $\beta(\alpha) = 0$, and θ is called α-Diophantine if $\delta(\alpha,\theta) = 0$. By a simple Borel-Cantelli argument, both weakly Diophantine and α-Diophantine numbers form sets of full Lebesque measure (for any α). Separating frequency and phase resonances, the frequency conjecture was that for α-Diophantine phases, there is a transition from singular continuous to pure point spectrum precisely at $\beta(\alpha) = L$, where L is the Lyapunov exponent. The phase conjecture was that for weakly Diophantine frequencies, there is a transition from singular continuous to pure point spectrum precisely at $\delta(\alpha,\theta) = L$.

Operator H is said to have Anderson localization if it has pure point spectrum with exponentially decaying eigenfunctions. We have

Theorem 6.1. *[Phase,[61]] For weakly Diophantine α,*

1. *$H_{\lambda,\alpha,\theta}$ has Anderson localization if $|\lambda| > e^{\delta(\alpha,\theta)}$,*
2. *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $1 < |\lambda| < e^{\delta(\alpha,\theta)}$.*
3. *$H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum if $|\lambda| < 1$.*

and

Theorem 6.2. *[Frequency, [60]]*
For α-Diophantine θ,

1. *$H_{\lambda,\alpha,\theta}$ has Anderson localization if $|\lambda| > e^{\beta(\alpha)}$,*

2. *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $1 < |\lambda| < e^{\beta(\alpha)}$.*

3. *$H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum if $|\lambda| < 1$.*

Remark

1. Part 2 of Theorem 6.1 holds for all irrational α, and part 2 of Theorem 6.2 ([14], see also a footnote in [7]) holds for all θ. Both also hold for general Lipshitz v replacing the $\cos$.

2. Part 3 of both theorems is known for all α, θ [6] and is included here for completeness.

3. Parts 1 and 2 of both Theorems put together verify the conjecture in [50], as stated there. The frequency half was first proved in [14] in a measure-theoretic sense (for a.e. θ).

For $\beta = \delta = 0$ (which is a.e. α, θ) the result follows from [53]. Proofs of the localization part of both theorems are based on the method developed in [53]. However, since the arithmetic transitions happen within the excluded measure zero set where the resonances are exponentially strong, new ideas were needed to handle those. A progress towards the localization side of the above conjecture in the frequency case was made in [7] (localization for $|\lambda| > e^{\frac{16}{9}\beta}$, as a step in solving the Ten Martini problem). The method developed in [7] that allowed to approach exponentially small denominators on the localization side was brought to its technical limits in [77], where the result for $|\lambda| > e^{\frac{3}{2}\beta}$ was obtained.

There have been no previous results on the transition in phase for $0 < \delta < \infty$. Singular continuous spectrum was first established for $1 < |\lambda| < e^{c\delta(\alpha,\theta)}$ (correspondingly, $1 < |\lambda| < e^{c\beta(\alpha)}$ for sufficiently small c [18, 67]. One can see that even with tight upper semicontinuity bounds the argument of [67] does not work for $c > 1/4$, New ideas to remove the factor of 4 and approach the actual threshold were required to prove Theorems 6.1, 6.2 in, correspondingly, [60, 61] . The singular continuous spectrum up to the threshold for frequency was established in [7, 14].

7 Exact asymptotics and universal hierarchical structure for frequency resonances

In this section we describe the universal self-similar exponential structure of eigenfunctions throughout the entire localization regime. We present explicit universal functions $f(k)$ and $g(k)$, depending only on the Lyapunov exponent and the position of k in the hierarchy defined by the denominators q_n of the continued fraction approximants of the flux α, that completely define the exponential behavior of, correspondingly, eigenfunctions and norms of the transfer matrices of the almost Mathieu operators, for all eigenvalues corresponding to a.e. phase, see Theorem 8.1. This result holds for *all* frequency and coupling pairs in the localization regime. Since the behavior is fully determined by the frequency and does not depend on the phase, it is the same, eventually, around any starting point, so is also seen unfolding at different scales when magnified around local eigenfunction maxima, thus describing the exponential universality in the hierarchical structure, see, for example, Theorems 7.2,7.4.

Since we are interested in exponential growth/decay, the behavior of f and g becomes most interesting in case of frequencies with exponential rate of approximation by the rationals.

These functions allow to describe *precise* asymptotics of *arbitrary* solutions of $H_{\lambda,\alpha,\theta}\varphi = E\varphi$ where E is an eigenvalue. The precise asymptotics of the norms of the transfer-matrices, provides the first example of this sort for non-uniformly hyperbolic dynamics. Since those norms sometimes differ significantly from the reciprocals of the eigenfunctions, this leads to further interesting and unusual consequencies, for example exponential tangencies between contracted and expanded directions at the resonant sites.

From this point of view, this analysis also provides the first study of the dynamics of Lyapunov-Perron non-regular points, in a natural setting. An artificial example of irregular dynamics can be found in [22], p.23, however it is not even a cocycle over an ergodic transformation, and we are not aware of other such, even artificial, ergodic examples where the dynamics has been studied. Loosely, for a cocycle A over a transformation f acting on a space X (Lyapunov-Perron) non-regular points $x \in X$ are the ones at which Oseledets multiplicative ergodic theorem does not hold coherently in both directions. They therefore form a measure zero set with respect to any invariant measure on X. Yet, it is precisely the non-regular points that are of interest in the study of Schrödinger cocycles in the non-uniformly hyperbolic (positive Lyapunov exponent) regime, since spectral measures, for every fixed phase, are always supported on energies where there exists a solution polynomially bounded in both directions, so the (hyperbolic) cocycle defined at such energies is always non-regular at precisely the relevant phases. Thus the non-regular points capture the entire action from the point of view of spectral theory, so become the most important ones to study. One can also discuss stronger non-regularity notions: absence of forward regularity and, even stronger, non-exactness of the Lyapunov exponent [22]. While it is not difficult to see that energies in the support of singular continuous spectral measure in the non-uniformly hyperbolic regime always provide examples of non-exactness, our analysis gave the first non-trivial example of non-exactness with non-zero upper limit (Corollary 7.12). Finally, as we understand, it also provided the first natural example of an even stronger manifestation of the lack of regularity, the exponential tangencies (Corollary 7.13). Tangencies between contracted and expanded directions are a characteristic feature of nonuniform hyperbolicity (and, in particular, always happen at the maxima of the eigenfunctions). They complicate proofs of positivity of the Lyapunov exponents and are viewed as a difficulty to avoid through e.g. the parameter exclusion [25, 97]. However, when the tangencies are only subexponentially deep they do not in themselves lead to non-exactness. Corollary 7.13 presents the first natural example of *exponentially* strong tangencies (with the rate determined by the arithmetics of α and the positions precisely along the sequence of resonances.)

For the almost Mathieu operator the k-step transfer matrix defined by (9),(10), becomes

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha)\cdots A(\theta) \quad (39)$$

and

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha) \quad (40)$$

for $k \geq 1$, where $A(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}$. As is clear from the definition, A_k also depends on θ and E but since those parameters will be usually fixed, we omit this from the notation.

Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ we define functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$ in the following way.

Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. For any $\frac{q_n}{2} \le k < \frac{q_{n+1}}{2}$, define $f(k), g(k)$ as follows:

Case 1 $q_{n+1}^{\frac{8}{9}} \ge \frac{q_n}{2}$ or $k \ge q_n$.

If $\ell q_n \le k < (\ell+1)q_n$ with $\ell \ge 1$, set

$$f(k) = e^{-|k-\ell q_n|\ln|\lambda|}\bar{r}_\ell^n + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\bar{r}_{\ell+1}^n, \tag{41}$$

and

$$g(k) = e^{-|k-\ell q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_\ell^n} + e^{-|k-(\ell+1)q_n|\ln|\lambda|}\frac{q_{n+1}}{\bar{r}_{\ell+1}^n}, \tag{42}$$

where for $\ell \ge 1$,

$$\bar{r}_\ell^n = e^{-(\ln|\lambda| - \frac{\ln q_{n+1}}{q_n} + \frac{\ln \ell}{q_n})\ell q_n}.$$

Set also $\bar{r}_0^n = 1$ for convenience.

If $\frac{q_n}{2} \le k < q_n$, set

$$f(k) = e^{-k\ln|\lambda|} + e^{-|k-q_n|\ln|\lambda|}\bar{r}_1^n, \tag{43}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{44}$$

Case 2 $q_{n+1}^{\frac{8}{9}} < \frac{q_n}{2}$ and $\frac{q_n}{2} \le k \le \min\{q_n, \frac{q_{n+1}}{2}\}$.

Set

$$f(k) = e^{-k\ln|\lambda|}, \tag{45}$$

and

$$g(k) = e^{k\ln|\lambda|}. \tag{46}$$

Notice that f, g only depend on α and λ but not on θ or E. $f(k)$ decays and $g(k)$ grows exponentially, globally, at varying rates that depend on the position of k in the hierarchy defined by the continued fraction expansion of α, see Fig.4 and Fig.5.

We say that ϕ is a generalized eigenfunction of H with generalized eigenvalue E, if

$$H\phi = E\phi, \text{ and } |\phi(k)| \le \hat{C}(1+|k|). \tag{47}$$

It turns out that in the entire regime $|\lambda| > e^{\beta}$, the exponential asymptotics of the generalized eigenfunctions and norms of transfer matrices at the generalized eigenvalues are completely determined by $f(k), g(k)$.

Theorem 7.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$. Suppose θ is Diophantine with respect to α, E is a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ and ϕ is the generalized eigenfunction. Let $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Then for any $\varepsilon > 0$, there exists K (depending on $\lambda, \alpha, \hat{C}, \varepsilon$) such that for any $|k| \geq K$, $U(k)$ and A_k satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leq ||U(k)|| \leq f(|k|)e^{\varepsilon|k|}, \tag{48}$$

and

$$g(|k|)e^{-\varepsilon|k|} \leq ||A_k|| \leq g(|k|)e^{\varepsilon|k|}. \tag{49}$$

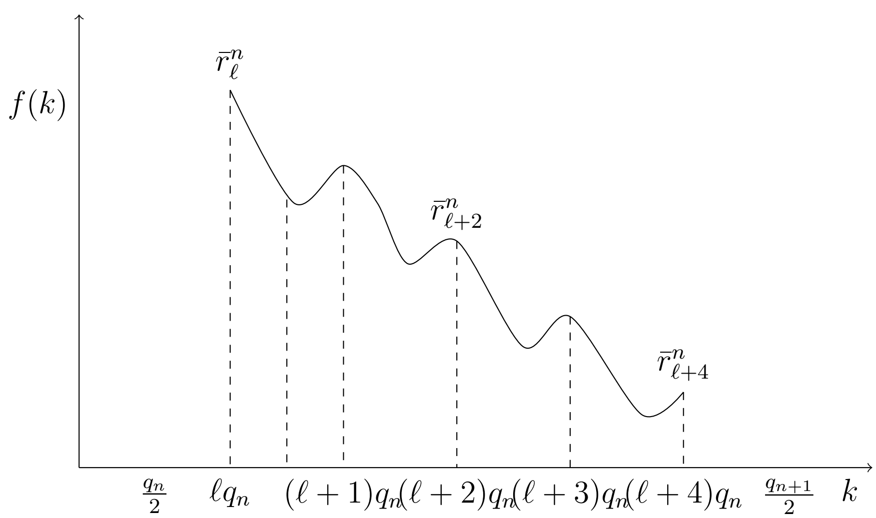

Figure 4

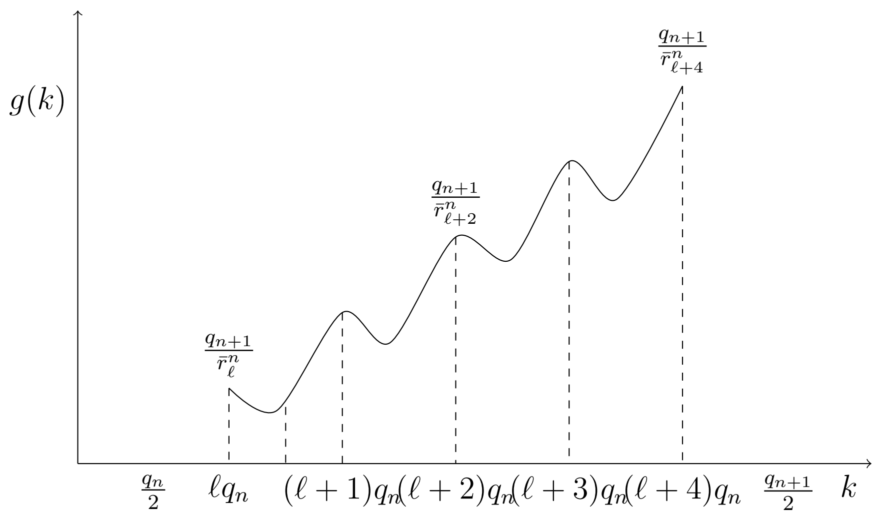

Figure 5

Certainly, there is nothing special about $k = 0$, so the behavior described in Theorem 8.1 happens around arbitrary point $k = k_0$. This implies the self-similar nature of the eigenfunctions): $U(k)$ behave as described at scale q_n but when looked at in windows of size $q_k, q_k \leq q_{n-1}$ will demonstrate the same universal behavior around appropriate local maxima/minima.

To make the above precise, let ϕ be an eigenfunction, and $U(k) = \begin{pmatrix} \phi(k) \\ \phi(k-1) \end{pmatrix}$. Let $I^j_{\varsigma_1,\varsigma_2} = [-\varsigma_1 q_j, \varsigma_2 q_j]$, for some $0 < \varsigma_1, \varsigma_2 \leq 1$. We will say k_0 is a local j-maximum of ϕ if $||U(k_0)|| \geq ||U(k)||$ for $k - k_0 \in I^j_{\varsigma_1,\varsigma_2}$. Occasionally, we will also use terminology (j, ς)-maximum for a local j-maximum on an interval $I^j_{\varsigma,\varsigma}$.

Fix $\kappa < \infty$, $\nu > 1$. We will say a local j-maximum k_0 is *nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{{q_{j-1}}^{\nu}},$$

for all $|k| \leq 2q_{j-1}$ and

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{50}$$

for all $2q_{j-1} < |k| \leq 2q_j$.

We will say a local j-maximum is *strongly nonresonant* if

$$||2\theta + (2k_0 + k)\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{51}$$

for all $0 < |k| \leq 2q_j$.

An immediate corollary of Theorem 8.1 is the universality of behavior at all (strongly) nonresonant local maxima.

Theorem 7.2. *Given $\varepsilon > 0$, there exists $j(\varepsilon) < \infty$ such that if k_0 is a local j-maximum for $j > j(\epsilon)$, then the following two statements hold:*

If k_0 is nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{52}$$

for all $2s \in I^j_{\varsigma_1,\varsigma_2}$, $|s| > \frac{q_{j-1}}{2}$.

If k_0 is strongly nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leq \frac{||U(k_0+s)||}{||U(k_0)||} \leq f(|s|)e^{\varepsilon|s|}, \tag{53}$$

for all $2s \in I^j_{\varsigma_1,\varsigma_2}$.

Remark 7.3.
1. For the neighborhood of a local j-maximum described in the Theorem 7.2 only the behavior of $f(s)$ for $q_{j-1}/2 < |s| \leq q_j/2$ is relevant. Thus f implicitly depends on j but through the scale-independent mechanism described in (41),(43) and (45).

2. Actually, one can formulate (52) in Theorem 7.2 with non-resonant condition (50) only required for $2q_{j-1} < |k| \leq q_j$ rather than for $2q_{j-1} < |k| \leq 2q_j$.

In case $\beta(\alpha) > 0$, Theorem 8.1 also guarantees an abundance (and a hierarchical structure) of local maxima of each eigenfunction. Let k_0 be a global maximum[9] .

[9] If there are several, what follows is true for each.

Universal hierarchical structure of an eigenfunction

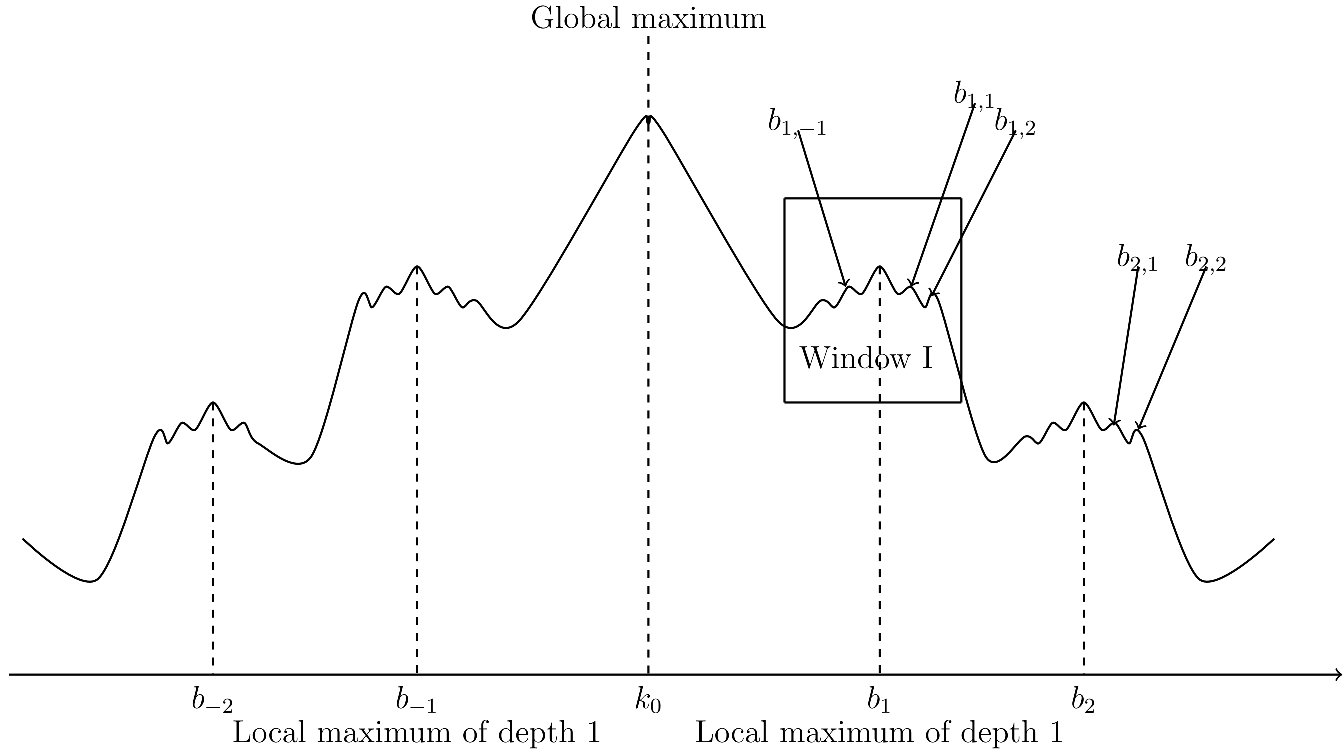

Figure 6

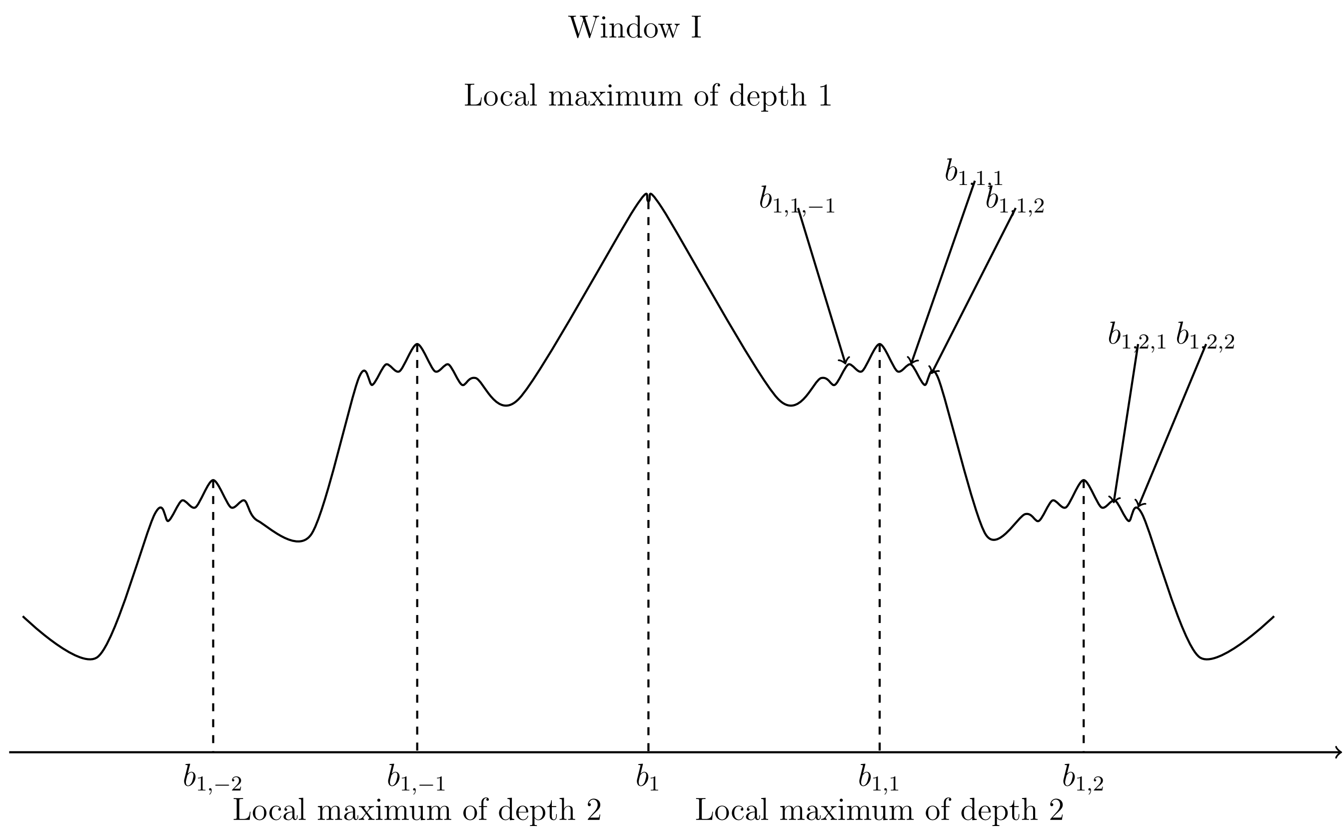

Figure 7

We first describe the hierarchical structure of local maxima informally. We will say that a scale n_{j_0} is exponential if $\ln q_{n_{j_0}+1} > cq_{n_{j_0}}$. Then there is a *constant* scale $\hat{n}_0$ thus a constant $C := q_{\hat{n}_0+1}$, such that for any exponential scale n_j and any eigenfunction there are local n_j-maxima within distance C of $k_0 + sq_{n_{j_0}}$ for each $0 < |s| < e^{cq_{n_{j_0}}}$. Moreover, these are all the local n_{j_0}-maxima in $[k_0 - e^{cq_{n_{j_0}}}, k_0 + e^{cq_{n_{j_0}}}]$. The exponential behavior of the eigenfunction in the local neighborhood (of size of order $q_{n_{j_0}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Note that only exponential behavior at the corresponding scale is determined by f and fluctuations of much smaller size are invisible. Now, let $n_{j_1} < n_{j_0}$ be another exponential scale. Denoting "depth 1" local maximum located near $k_0 + a_{n_{j_0}} q_{n_{j_0}}$ by $b_{a_{n_{j_0}}}$ we then have a similar picture around $b_{a_{n_{j_0}}}$: there are local n_{j_1}-maxima in the vicinity of $b_{a_{n_{j_0}}} + sq_{n_{j_1}}$ for each

$0 < |s| < e^{cq_{n_{j_1}}}$. Again, this describes all the local $q_{n_{j_1}}$-maxima within an exponentially large interval. And again, the exponential (for the n_{j_1} scale) behavior in the local neighborhood (of size of order $q_{n_{j_1}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Denoting those "depth 2" local maxima located near $b_{a_{n_{j_0}}} + a_{n_{j_1}} q_{n_{j_1}}$, by $b_{a_{n_{j_0}}, a_{n_{j_1}}}$ we then get the same picture taking the magnifying glass another level deeper, and so on. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ with each "depth $s+1$" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ being in the corresponding vicinity of the "depth s" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}}$, and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, yet the depth of the hierarchy that can be so achieved is at least $j/2 - C$, see Corollary 7.7. Fig. 6 schematically illustrates the structure of local maxima of depth one and two, and Fig. 7 illustrates that the neighborhood of a local maximum appropriately magnified looks like a picture of the global maximum.

We now describe the hierarchical structure precisely. Suppose

$$||2(\theta + k_0\alpha) + k\alpha||_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{54}$$

for any $k \in \mathbb{Z}\backslash\{0\}$. Fix $0 < \varsigma, \epsilon$, with $\varsigma + 2\epsilon < 1$. Let $n_j \to \infty$ be such that $\ln q_{n_j+1} \geq (\varsigma + 2\epsilon) \ln |\lambda| q_{n_j}$. Let $\mathfrak{c}_j = (\ln q_{n_j+1} - \ln |a_{n_j}|)/\ln |\lambda| q_{n_j} - \epsilon$. We have $\mathfrak{c}_j > \epsilon$ for $0 < a_{n_j} < e^{\varsigma \ln |\lambda| q_{n_j}}$. Then we have

Theorem 7.4. *There exists $\hat{n}_0(\alpha, \lambda, \kappa, \nu, \epsilon) < \infty$ such that for any $j_0 > j_1 > \cdots > j_k$, $n_{j_k} \geq \hat{n}_0 + k$, and $0 < a_{n_{j_i}} < e^{\varsigma \ln |\lambda| q_{n_{j_i}}}, i = 0, 1, \ldots, k$, for all $0 \leq s \leq k$ there exists a local n_{j_s}-maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ on the interval $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} + I^{n_{j_s}}_{\mathfrak{c}_{j_s}, 1}$ for all $0 \leq s \leq k$ such that the following holds:*

I $|b_{a_{n_{j_0}}} - (k_0 + a_{n_{j_0}} q_{n_{j_0}})| \leq q_{\hat{n}_0+1}$,

II *For any $1 \leq s \leq k$, $|b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} - (b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + a_{n_{j_s}} q_{n_{j_s}})| \leq q_{\hat{n}_0+s+1}$.*

III *if $2(x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}) \in I^{n_{j_k}}_{\mathfrak{c}_{j_k}, 1}$ and $|x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}| \geq q_{\hat{n}_0+k}$, then for each $s = 0, 1, \ldots, k$,*

$$f(x_s)e^{-\varepsilon|x_s|} \leq \frac{||U(x)||}{||U(b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}})||} \leq f(x_s)e^{\varepsilon|x_s|}, \tag{55}$$

where $x_s = |x - b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}|$ is large enough.

Moreover, every local n_{j_s}-maximum on the interval $b_{a_{n_j},a_{n_{j_1}},\ldots,a_{n_{j_{s-1}}}} + [-e^{\epsilon \ln \lambda q_{n_{j_s}}}, e^{\epsilon \ln \lambda q_{n_{j_s}}}]$ is of the form $b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}$ for some $a_{n_{j_s}}$.

Remark 7.5. By I of Theorem 7.4, the local maximum can be determined up to a constant $K_0 = q_{\hat{n}_0+1}$. Actually, if k_0 is only a local n_j+1-maximum, we can still make sure that I, II and III of Theorem 7.4 hold. This is the local version of Theorem 7.4

Remark 7.6. $q_{\hat{n}_0+1}$ is the scale at which *phase* resonances of $\theta + k_0\alpha$ still can appear. Notably, it determines the precision of pinpointing local n_{j_0}-maxima in a (exponentially large in $q_{n_{j_0}}$) neighborhood of k_0, for any j_0. When we go down the hierarchy, the precision decreases, but note that except for the very last scale it stays at least iterated logarithmically [10] small in the corresponding scale $q_{n_{j_s}}$

Thus for $x \in b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}} + [-\frac{c_{j_s}}{2}q_{n_{j_s}}, \frac{1}{2}q_{n_{j_s}}]$, the behavior of $\phi(x)$ is described by the same universal f in each $q_{n_{j_s}}$-window around the corresponding local maximum $b_{a_{n_{j_0}},a_{n_{j_1}},\ldots,a_{n_{j_s}}}, s = 0, 1, \ldots, k$. We call such a structure *hierarchical*, and we will say that a local j-maximum is k-hierarchical if the complete hierarchy goes down at least k levels. We then have an immediate corollary

Corollary 7.7. *There exists $C = C(\alpha, \lambda, \kappa, \nu, \epsilon)$ such that every local n_j-maximum in $[k_0 - e^{\varsigma \ln|\lambda| q_{n_j}}, k_0 + e^{\varsigma \ln|\lambda| q_{n_j}}]$ is at least $(j/2 - C)$-hierarchical.*

Remark 7.8. The estimate on the depth of the hierarchy in the corollary assumes the worst case scenario when all scales after $\hat{n}_0$ are Liouville. Otherwise the hierarchical structure will go even much deeper. Note that a local n_j-maximum that is not an n_{j+1}-maximum cannot be k-hierarchical for $k > j$.

Another interesting corollary of Theorem 8.1 is

[10] for most scales even much less

Theorem 7.9. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $|\lambda| > e^{\beta(\alpha)}$ and θ is Diophantine with respect to α. Then $H_{\lambda,\alpha,\theta}$ has Anderson localization, with eigenfunctions decaying at the rate* $\ln|\lambda| - \beta$.

This solves the arithmetic version of the second transition conjecture in that it establishes localization throughout the entire regime of (α, λ) where localization may hold for any θ (see the discussion in Section 6), for an arithmetically defined full measure set of θ.

Also, it could be added that, for all θ, $H_{\lambda,\alpha,\theta}$ has no localization (i.e., no exponentially decaying eigenfunctions) if $|\lambda| = e^{\beta}$

Let $\psi(k)$ denote any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ that is linearly independent with respect to $\phi(k)$. Let $\tilde{U}(k) = \begin{pmatrix} \psi(k) \\ \psi(k-1) \end{pmatrix}$. An immediate counterpart of (49) is the following

Corollary 7.10. *Under the conditions of Theorem 8.1 for large k vectors $\tilde{U}(k)$ satisfy*

$$g(|k|)e^{-\varepsilon|k|} \le ||\tilde{U}(k)|| \le g(|k|)e^{\varepsilon|k|}. \tag{56}$$

Thus every solution is exponentially expanding at the rate $g(k)$ except for one that is exponentially decaying at the rate $f(k)$.

It is well known that for E in the spectrum the dynamics of the transfer-matrix cocycle A_k is nonuniformly hyperbolic. Moreover, E being a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ already implies that the behavior of A_k is non-regular. Theorem 8.1 provides precise information on how the non-regular behavior unfolds in this case. We are not aware of other non-artificially constructed examples of non-uniformly hyperbolic systems where non-regular behavior can be described with similar precision.

The information provided by Theorem 8.1 leads to many interesting corollaries. Here we only want to list a few immediate sharp consequences.

Corollary 7.11. *Under the condition of Theorem 8.1, we have*

i)

$$\limsup_{k\to\infty} \frac{\ln||A_k||}{k} = \limsup_{k\to\infty} \frac{\ln||\tilde{U}(k)||}{k} = \ln|\lambda|,$$

ii)

$$\liminf_{k\to\infty} \frac{\ln||A_k||}{k} = \liminf_{k\to\infty} \frac{\ln||\tilde{U}(k)||}{k} = \ln|\lambda| - \beta.$$

iii) Outside an explicit sequence of lower density zero, [11]

$$\lim_{k\to\infty} \frac{\ln ||A_k||}{k} = \lim_{k\to\infty} \frac{\ln ||\tilde{U}(k)||}{k} = \ln |\lambda|.$$

Therefore the Lyapunov behavior for the norm fails to hold only along a sequence of density zero. It is interesting that the situation is different for the eigenfunctions. While, just like the overall growth of $\|A_k\|$ is $\ln |\lambda| - \beta$, the overall rate of decay of the eigenfunctions is also $\ln |\lambda| - \beta$, they however decay at the Lyapunov rate only outside a sequence of positive upper density. That is

Corollary 7.12. *Under the condition of Theorem 8.1, we have*

i)

$$\limsup_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda|,$$

ii)

$$\liminf_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda| - \beta.$$

iii) There is an explicit sequence of upper density $1 - \frac{1}{2}\frac{\beta}{\ln |\lambda|}$, [12], *along which*

$$\lim_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda|.$$

iv) There is an explicit sequence of upper density $\frac{1}{2}\frac{\beta}{\ln |\lambda|}$, [13]*along which*

$$\limsup_{k\to\infty} \frac{-\ln ||U(k)||}{k} < \ln |\lambda|.$$

The fact that g is not always the reciprocal of f leads also to another interesting phenomenon.

Let $0 \leq \delta_k \leq \frac{\pi}{2}$ be the angle between vectors $U(k)$ and $\tilde{U}(k)$.

[11] The sequence with convergence to the Lyapunov exponent contains $q_n, n = 1, \cdots$.

[12] The sequence contains $\lfloor \frac{q_n}{2} \rfloor, n = 1, \cdots$.

[13] This sequence can have lower density ranging from 0 to $\frac{1}{2}\frac{\beta}{\ln |\lambda|}$ depending on finer continued fraction properties of α.

Corollary 7.13. *We have*

$$\limsup_{k\to\infty} \frac{\ln \delta_k}{k} = 0, \tag{57}$$

and

$$\liminf_{k\to\infty} \frac{\ln \delta_k}{k} = -\beta. \tag{58}$$

Thus neighborhoods of resonances q_n are the places of exponential tangencies between contracted and expanded directions, with the rate approaching $-\beta$ along a subsequence.[14] This means, in particular, that A_k with $k \sim q_n$ is exponentially close to a matrix with the trace $e^{(\ln|\lambda|-\beta)k}$. Exponential tangencies also happen around points of the form jq_n but at lower strength.

8 Asymptotics of eigenfunctions and universal hierarchical structure for phase resonances

Our proof of localization is, again, based on determining the *exact asymptotics* of the generalized eigenfunctions in the regime $|\lambda| > e^{\delta(\alpha,\theta)}$. However, the asymptotics (and the methods required) are very different in the case of phase resonances.

For any ℓ, let x_0 (we can choose any one if x_0 is not unique) be such that

$$|\sin \pi(2\theta + x_0\alpha)| = \min_{|x|\le 2|\ell|} |\sin \pi(2\theta + x\alpha)|.$$

Let $\eta = 0$ if $2\theta + x_0\alpha \in \mathbb{Z}$, otherwise let $\eta \in (0,\infty)$ be given by the following equation,

$$|\sin \pi(2\theta + x_0\alpha)| = e^{-\eta|\ell|}. \tag{59}$$

Define $f : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: $x_0 \cdot \ell \le 0$. Set $f(\ell) = e^{-|\ell|\ln|\lambda|}$.

Case 2. $x_0 \cdot \ell > 0$. Set $f(\ell) = e^{-(|x_0|+|\ell-x_0|)\ln|\lambda|}e^{\eta|\ell|} + e^{-|\ell|\ln|\lambda|}$.

We say that ϕ is a generalized eigenfunction of H with generalized eigenvalue E, if

$$H\phi = E\phi, \text{ and } |\phi(k)| \le \hat{C}(1+|k|). \tag{60}$$

[14] In fact, the rate is close to $-\frac{\ln q_{n+1}}{q_n}$ for any large n.

For a fixed generalized eigenvalue E and corresponding generalized eigenfunction ϕ of $H_{\lambda,\alpha,\theta}$, let $U(\ell) = \begin{pmatrix} \phi(\ell) \\ \phi(\ell-1) \end{pmatrix}$. We have

Theorem 8.1. *Assume* $\ln|\lambda| > \delta(\alpha,\theta)$. *Then for any* $\varepsilon > 0$, *there exists* K *such that for any* $|\ell| \geq K$, $U(\ell)$ *satisfies*

$$f(\ell)e^{-\varepsilon|\ell|} \leq ||U(\ell)|| \leq f(\ell)e^{\varepsilon|\ell|}. \tag{61}$$

In particular, the eigenfunctions decay at the rate $\ln|\lambda| - \delta(\alpha,\theta)$.

Remark

- For $\delta = 0$ we have that for any $\varepsilon > 0$,

$$e^{-(\ln|\lambda|+\varepsilon)|\ell|} \leq f(\ell) \leq e^{-(\ln|\lambda|-\varepsilon)|\ell|}.$$

  This implies that the eigenfunctions decay precisely at the rate of Lyapunov exponent $\ln|\lambda|$.

- For $\delta > 0$, by the definition of δ and f, we have for any $\varepsilon > 0$,

$$f(\ell) \leq e^{-(\ln|\lambda|-\delta-\varepsilon)|\ell|}. \tag{62}$$

- By the definition of δ again, there exists a subsequence $\{\ell_i\}$ such that

$$|\sin\pi(2\theta + \ell_i\alpha)| \leq e^{-(\delta-\varepsilon)|\ell_i|}.$$

  By the DC on α, one has that

$$|\sin\pi(2\theta + \ell_i\alpha)| = \min_{|x|\leq 2|\ell_i|} |\sin\pi(2\theta + x\alpha)|.$$

  Then

$$f(\ell_i) \geq e^{-(\ln|\lambda|-\delta+\varepsilon)|\ell_i|}. \tag{63}$$

  This implies the eigenfunctions decay precisely at the rate $\ln|\lambda| - \delta(\alpha,\theta)$.

- If x_0 is not unique, by the DC on α, η is necessarily arbitrarily small. Then

$$e^{-(\ln|\lambda|+\varepsilon)|\ell|} \leq ||U(\ell)|| \leq e^{-(\ln|\lambda|-\varepsilon)|\ell|}.$$

The behavior described in Theorem 8.1 happens around arbitrary point. This, coupled with effective control of parameters at the local maxima, allows to uncover the self-similar nature of the eigenfunctions. Hierarchical behavior of solutions, despite significant numerical studies and even a discovery of Bethe Ansatz solutions [94] has remained an important open challenge even at the physics level. In the previous section we described universal hierarchical structure of the eigenfunctions for all frequencies α and phases with $\delta(\alpha,\theta)=0$. In studying the eigenfunctions of $H_{\lambda,\alpha,\theta}$ for $\delta(\alpha,\theta)>0$ Wencai Liu and I [61] obtained a different kind of universality throughout the pure point spectrum regime, which features a self-similar hierarchical structure upon proper *reflections*.

Assume phase θ satisfies $0<\delta(\alpha,\theta)<\ln\lambda$. Fix $0<\varsigma<\delta(\alpha,\theta)$.

Let k_0 be a global maximum of eigenfunction ϕ.[15] Let K_i be the positions of exponential resonances of the phase $\theta'=\theta+k_0\alpha$ defined by

$$||2\theta+(2k_0+K_i)\alpha||_{\mathbb{R}/\mathbb{Z}}\leq e^{-\varsigma|K_i|}, \tag{64}$$

This means that $|v(\theta'+\ell\alpha)-v(\theta'+(K_i-\ell)\alpha)|\leq Ce^{-\varsigma|K_i|}$, uniformly in ℓ, or, in other words, the potential $v_n=v(\theta+n\alpha)$ is $e^{-\varsigma|K_i|}$-almost symmetric with respect to $(k_0+K_i)/2$.

Since α is Diophantine, we have

$$|K_i|\geq ce^{c|K_{i-1}|}, \tag{65}$$

where c depends on ς and α through the Diophantine constants κ,τ. On the other hand, K_i is necessarily an infinite sequence.

Let ϕ be an eigenfunction, and $U(k)=\begin{pmatrix}\phi(k)\\ \phi(k-1)\end{pmatrix}$. We say k is a local K-maximum if $||U(k)||\geq||U(k+s)||$ for all $s-k\in[-K,K]$.

We first describe the hierarchical structure of local maxima informally. There exists a *constant* $\hat{K}$ such that there is a local cK_j-maximum b_j within distance $\hat{K}$ of each resonance K_j. The exponential behavior of the eigenfunction in the local cK_j-neighborhood of each such local maximum, normalized by the value at the local maximum is given by the *reflection* of f. Moreover, this describes the entire collection of local maxima of depth 1, that is K such that K is a cK-maximum. Then we have a similar picture in the

[15] Can take any one if there are several.

vicinity of b_j : there are local cK_i-maxima $b_{j,i}, i < j$, within distance $\hat{K}^2$ of each $K_j - K_i$. The exponential (on the K_i scale) behavior of the eigenfunction in the local cK_i-neighborhood of each such local maximum, normalized by the value at the local maximum is given by f. Then we get the next level maxima $b_{j,i,s}, s < i$ in the $\hat{K}^3$-neighborhood of $K_j - K_i + K_s$ and reflected behavior around each, and so on, with reflections alternating with steps. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{j_0,j_1,\ldots,j_s}$, with each "depth $s+1$" local maximum $b_{j_0,j_1,\ldots,j_s}$ being in the corresponding vicinity of the "depth s" local maximum $b_{j_0,j_1,\ldots,j_{s-1}} \approx k_0 + \sum_{i=0}^{s-1}(-1)^i K_{j_i}$ and with universal behavior at the corresponding scale around each. The quality of the approximation of the position of the next maximum gets lower with each level of depth, with $b_{j_0,j_1,\ldots,j_{s-1}}$ determined with $\hat{K}^s$ precision, thus it presents an accurate picture as long as $K_{j_s} \gg \hat{K}^s$.

We now describe the hierarchical structure precisely.

Theorem 8.2. *[61] Assume sequence K_i satisfies (64) for some $\varsigma > 0$. Then there exists $\hat{K}(\alpha, \lambda, \theta, \varsigma) < \infty$[16] such that for any $j_0 > j_1 > \cdots > j_k \geq 0$ with $K_{j_k} \geq \hat{K}^{k+1}$, for each $0 \leq s \leq k$ there exists a local $\frac{\varsigma}{2\ln\lambda}K_{j_s}$-maximum[17] $b_{j_0,j_1,\ldots,j_s}$ such that the following holds:*

I $|b_{j_0,j_1,\ldots,j_s} - k_0 - \sum_{i=0}^{s}(-1)^i K_{j_i}| \leq \hat{K}^{s+1}$.

II *For any $\varepsilon > 0$, if $C\hat{K}^{k+1} \leq |x - b_{j_0,j_1,\ldots,j_k}| \leq \frac{\varsigma}{4\ln\lambda}|K_{j_k}|$, where C is a large constant depending on $\alpha, \lambda, \theta, \varsigma$ and ε, then for each $s = 0, 1, \ldots, k$,*

$$f((-1)^{s+1}x_s)e^{-\varepsilon|x_s|} \leq \frac{||U(x)||}{||U(b_{j_0,j_1,\ldots,j_s})||} \leq f((-1)^{s+1}x_s)e^{\varepsilon|x_s|}, \qquad (66)$$

where $x_s = x - b_{j_0,j_1,\ldots,j_s}$.

Thus the behavior of $\phi(x)$ is described by the same universal f in each $\frac{\varsigma}{2\ln\lambda}K_{j_s}$ window around the corresponding local maximum $b_{j_0,j_1,\ldots,j_s}$ after alternating reflections. The positions of the local maxima in the hierarchy are determined up to errors that at all but possibly the last step are superlogarithmically small in K_{j_s}. We call such a structure *reflective hierarchy.*

[16] $\hat{K}$ depends on θ through $2\theta + k\alpha$, see (38).

[17] Actually, it can be a local $(\frac{\varsigma}{\ln\lambda} - \varepsilon)K_{j_s}$-maximum for any $\varepsilon > 0$.

reflective self-similarity of an eigenfunction

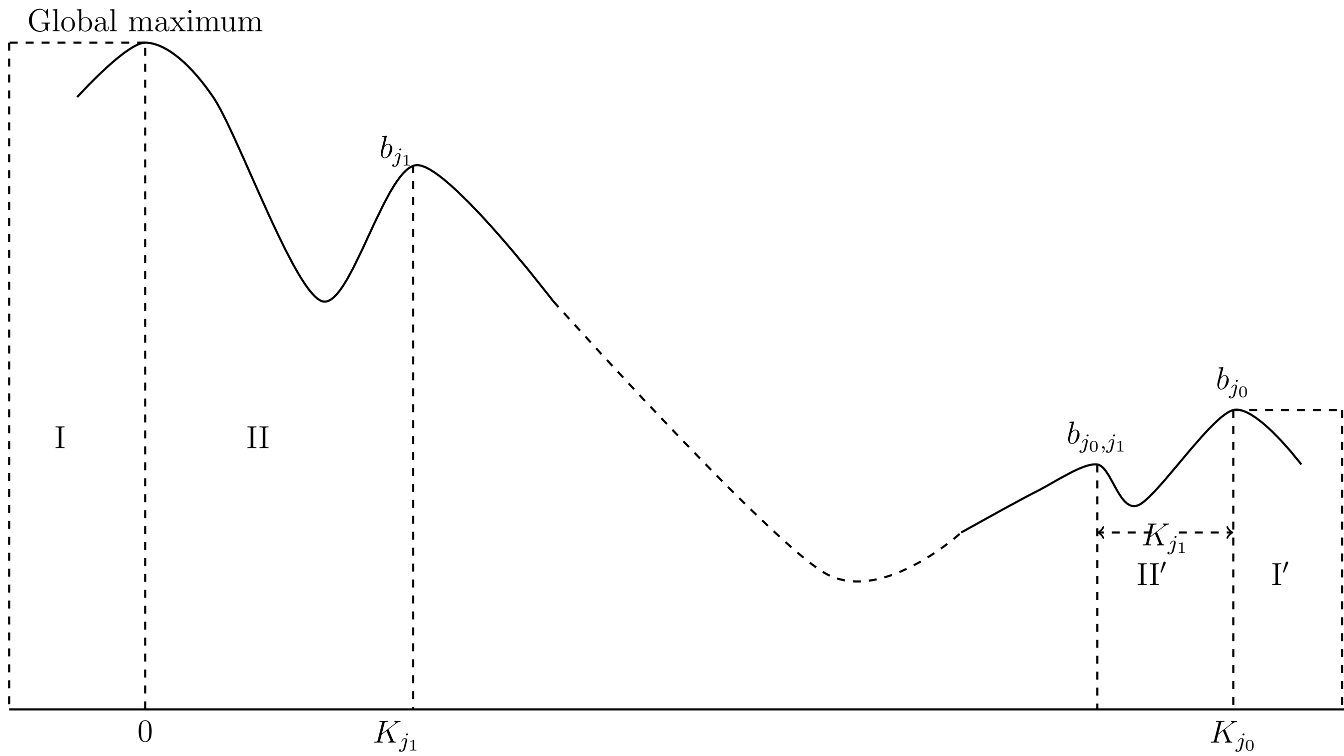

Figure 8: This depicts reflective self-similarity of an eigenfunction with global maximum at 0. The self-similarity: I′ is obtained from I by scaling the x-axis proportional to the ratio of the heights of the maxima in I and I′. II′ is obtained from II by scaling the x-axis proportional to the ratio of the heights of the maxima in II and II′. The behavior in the regions I′, II′ mirrors the behavior in regions I, II upon reflection and corresponding dilation.

Finally, as in the frequency resonance case, we discuss the asymptotics of the transfer matrices. Let, as before, $A_0 = I$ and for $k \geq 1$,

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha) \cdots A(\theta)$$

and

$$A_{-k}(\theta) = A_k^{-1}(\theta - k\alpha),$$

where $A(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}$. Thus A_k is the (k-step) transfer matrix. It also depends on α and E but since those parameters will be fixed, we omit them from the notation.

We define a new function $g : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: If $x_0 \cdot \ell \le 0$ or $|x_0| > |\ell|$, set

$$g(\ell) = e^{|\ell| \ln |\lambda|}.$$

Case 2: If $x_0 \cdot \ell \ge 0$ and $|x_0| \le |\ell| \le 2|x_0|$, set

$$g(\ell) = e^{(\ln\lambda - \eta)|\ell|} + e^{|2x_0 - \ell| \ln |\lambda|}.$$

Case 3: If $x_0 \cdot \ell \ge 0$ and $|\ell| > 2|x_0|$, set

$$g(\ell) = e^{(\ln\lambda - \eta)|\ell|}.$$

We have

Theorem 8.3. *Under the conditions of Theorem 8.1, we have*

$$g(\ell)e^{-\varepsilon|\ell|} \le ||A_\ell|| \le g(\ell)e^{\varepsilon|\ell|}. \tag{67}$$

Let $\psi(\ell)$ denote any solution to $H_{\lambda,\alpha,\theta}\psi = E\psi$ that is linearly independent with $\phi(\ell)$. Let $\tilde{U}(\ell) = \begin{pmatrix} \psi(\ell) \\ \psi(\ell-1) \end{pmatrix}$. An immediate counterpart of (67) is the following

Corollary 8.4. *Under the conditions of Theorem 8.1, vectors $\tilde{U}(\ell)$ satisfy*

$$g(\ell)e^{-\varepsilon|\ell|} \le ||\tilde{U}(\ell)|| \le g(\ell)e^{\varepsilon|\ell|}. \tag{68}$$

Our analysis also gives

Corollary 8.5. *Under the conditions of Theorem 8.1, we have,*

i)

$$\limsup_{k\to\infty} \frac{\ln ||A_k||}{k} = \limsup_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda|,$$

ii)

$$\liminf_{k\to\infty} \frac{\ln ||A_k||}{k} = \liminf_{k\to\infty} \frac{-\ln ||U(k)||}{k} = \ln |\lambda| - \delta.$$

iii) outside a sequence of lower density $1/2$*,*

$$\lim_{k\to\infty} \frac{-\ln ||U(k)||}{|k|} = \ln |\lambda|, \tag{69}$$

iv) outside a sequence of lower density 0*,*

$$\lim_{k\to\infty} \frac{\ln ||A_k||}{|k|} = \ln |\lambda|. \tag{70}$$

Note that (70) also holds throughout the pure point regime of [60]. As in the previous section, the fact that g is not always the reciprocal of f leads to exponential tangencies between contracted and expanded directions with the rate approaching $-\delta$ along a subsequence. Tangencies are an attribute of nonuniform hyperbolicity and are usually viewed as a difficulty to avoid through e.g. the parameter exclusion. Our analysis allows to study them in detail and uncovers the hierarchical structure of exponential tangencies positioned precisely at phase resonances. The methods developed to prove these theorems have made it possible to determine also the *exact* exponent of the exponential decay rate in expectation for the two-point function [59], the first result of this kind for any model.

9 Further extensions

While the almost Mathieu family is precisely the one of main interest in physics literature, it also presents the simplest case of analytic quasiperiodic operator, so a natural question is which features discovered for the almost Mathieu would hold for this more general class. Not all do, in particular, the ones that exploit the self-dual nature of the family $H_{\lambda,\alpha,\theta}$ often cannot be expected to hold in general. In case of Theorems 6.2 and 7.1, we conjecture that they should in fact hold for general analytic (or even more general) potentials, for a.e. phase and with $\ln|\lambda|$ replaced by the Lyapunov exponent $L(E)$, but with otherwise the same or very similar statements. The hierarchical structure theorems 7.2 and 7.4 are also expected to hold universally for most (albeit not all, as in the present paper) appropriate local maxima. Some of our qualitative corollaries may hold in even higher generality. Establishing this fully would require certain new ideas since so far even an arithmetic version of localization for the Diophantine case has not been established for the general analytic family, the current state-of-the-art result by Bourgain-Goldstein [27] being measure theoretic in α. However, some ideas of our method can already be transferred to general trigonometric polynomials [63]. Moreover, our method was used recently in [43] to show that the same f and g govern the asymptotics of eigenfunctions and universality around the local maxima throughout the a.e. localization regime in another popular object, the Maryland model, as well as in several other scenarios (work in progress).

So we expect the same arithmetic frequency transition for general analytic potentials, but as far as the arithmetic phase transitions, we expect the same results to hold for general *even* analytic potentials for a.e. frequency, see more detail below. We note that the singular continuous part up to the conjectured transition is already established, even in a far greater generality, in [14, 61].

The universality of the hierarchical structure described in Sections 7, 8 is twofold: not only it is the same universal function that governs the behavior around each exponential frequency or phase resonance (upon reflection and renormalization), it is the same structure for all the parameters involved: any (Diophantine) frequency α, (any α-Diophantine phase θ) with $\beta(\alpha) < L$, ($\delta(\alpha, \theta) < L$), and any eigenvalue E. The universal reflective-hierarchical structure requires evenness of the function defining the potential, and moreover, in general, resonances of other types may also be present. However, we conjectured in [61] that for general even analytic potentials for a.e. frequency only finitely many other exponentially strong resonances will appear, thus the structure described here will hold for the corresponding class, with the $\ln \lambda$ replaced by the Lyapunov exponent $L(E)$ throughout.

The key elements of the technique developed for the treatment of arithmetic resonances are robust and have made it possible to approach other scenarios, and in particular, study delicate properties of the singular continuous regime, obtaining upper bounds on fractal dimensions of the spectral measure and quantum dynamics for the almost Mathieu operator [62], as well as potentials defined by general trigonometric analytic functions [63].

Finally, we briefly comment that for Schrödinger operators with analytic periodic potentials, almost Lipshitz continuity of gaps holds for Diophantine α for all non-critical (in the sense of Avila's global theory [5]) energies [64]. For critical energies, we do not have anything better than Hölder-$\frac{1}{2}$ regularity that holds universally. For the prototypical critical potential, the critical almost Mathieu, almost Lipshitz continuity of spectra also holds, because of the hidden singularity. This leads to two potentially related questions for analytic quasiperiodic Schrödinger operators:

1. Does some form of uniform almost Lipshitz continuity always hold?
2. Is there always a singularity hidden behind the criticality?

A positive answer to the second question would lead to a statement that critical operators never have eigenvalues and that Hausdorff dimension of the critical part of the spectrum is always bounded by 1/2.

Acknowledgment

This work was supported by NSF DMS-1901462.

References

[1] S. Aubry, G. André: Analyticity breaking and Anderson localisation in incommensurate lattices. Ann. Israel Phys. Soc. **3**, 133–164 (1980).

[2] A. Avila. The absolutely continuous spectrum of the almost Mathieu operator. *arXiv preprint arXiv:0810.2965*, 2008.

[3] A. Avila. Almost reducibility and absolute continuity I. *arXiv preprint arXiv:1006.0704*, 2010.

[4] A. Avila. Preprint, 2012. In preparation.

[5] A. Avila. Global theory of one-frequency Schrödinger operators. *Acta Math.*, 215(1):1–54, 2015.

[6] A. Avila. On point spectrum at critical coupling. Preprint 2007.

[7] A. Avila and S. Jitomirskaya. The Ten Martini Problem. *Ann. of Math. (2)*, 170(1):303–342, 2009.

[8] A. Avila and S. Jitomirskaya. Almost localization and almost reducibility. *J. Eur. Math. Soc. (JEMS)*, 12(1):93–131, 2010.

[9] A. Avila, S. Jitomirskaya and C. Marx. Spectral theory of extended Harper's model and a question by Erdös and Szekeres. Invent. Math. **210.1**, 283–339 (2017)

[10] A. Avila, R. Krikorian. Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles. Ann. of Math. **164**, 911–940 (2006).

[11] A. Avila, R. Krikorian, Quasiperiodic SL(2, R) cocycles. Unpublished.

[12] A. Avila, Y. Last, M. Shamis, and Q. Zhou. On the abominable properties of the almost Mathieu operator with well approximated frequencies, in preparation.

[13] A. Avila, Y. Last, B. Simon, Bulk universality and clock spacing of zeros for ergodic Jacobi matrices with a.c. spectrum. Analysis & PDE **3**, 81-118 (2010).

[14] A. Avila, J. You, and Q. Zhou. Sharp phase transitions for the almost Mathieu operator. *Duke Math. J.*, 166(14):2697–2718 (2017).

[15] A. Avila, J. You, and Q. Zhou. In preparation.

[16] J. E. Avron, Topological quantum states and the 2016 Nobel prize, Bull. IAMP, 3-22, (July 2017)

[17] J. Avron, B. Simon: Singular continuous spectrum for a class of almost periodic Jacobi matrices. Bull. AMS **6**, 81-85 (1982).

[18] J.E. Avron, B. Simon: Almost periodic operators, II. The density of states. Duke Math J. **50**,369–391 (1983)

[19] J. Avron, D. Osadchy, R. Seiler, A Topological Look at the Quantum Hall Effect, Physics Today **56**, 8, 38 (2003)

[20] J. Avron, P. van Mouche, B. Simon: On the measure of the spectrum for the almost Mathieu operator. Commun. Math. Phys. **132**, 103–118 (1990).

[21] M. Y. Azbel. Energy spectrum of a conduction electron in a magnetic field. *Sov. Phys. JETP*, **19**, 634–645 (1964).

[22] L. Barreira and Y. Pesin. Nonuniform hyperbolicity: Dynamics of systems with nonzero Lyapunov exponents, volume 115. Cambridge University Press, 2007.

[23] J. Bellissard. Lipshitz continuity of gap boundaries for Hofstadter-like spectra. Commun. Math. Phys. **160**, 599–613 (1994).

[24] S. Becker, R. Han, S. Jitomirskaya, Cantor spectrum of graphene in magnetic fields. Inventiones math. **218**, 979–1041 (2019).

[25] M. Benedicks and L. Carleson. The dynamics of the Henon map. Annals of Mathematics, **133**, 73-169 (1991).

[26] J. Bourgain. *Green's function estimates for lattice Schrödinger operators and applications*, volume 158 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2005.

[27] J. Bourgain, M. Goldstein, On nonperturbative localization with quasiperiodic potential. Annals of Math., **152**, 835-879 (2000)

[28] J. Bourgain and S. Jitomirskaya, Absolutely continuous spectrum for 1D quasiperiodic operators. Invent. Math. **148** 453–463 (2002).

[29] J. Bourgain and S. Jitomirskaya. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Statist. Phys.*, **108**, 1203–1218 (2002) Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[30] V. Buslaev, A. Fedotov, The monodromization and Harper equation. Séminaire Équations aux dérivées partielles (Polytechnique) (1993-1994), exp. no 21, p. 1-21

[31] M.-D. Choi, G. A. Elliott, N. Yui: Gauss polynomials and the rotation algebra. Invent. Math. **99**, 225–246 (1990).

[32] W. Chojnacki, A generalized spectral duality theorem. Comm. Math. Phys., **143**, 527–544 (1992).

[33] P. Deift, B. Simon. Almost periodic Schrödinger operators III. The absolutely continuous spectrum in one dimension. Commun. Math. Phys. **90**, 389–411 (1983).

[34] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon. Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization. *Journal d'Analyse Mathématique*, **69**, 153–200 (1996).

[35] F. Delyon, Absence of localisation in the almost Mathieu equation. J. Phys. A **20** , L21–L23 (1987).

[36] Eliasson, L.H.: Reducibility and point spectrum for linear quasi-periodic skew products, Doc. Math, Extra volume ICM 1998 II, 779-787. operator. C. R. Math. Rep. Acad. Sci. Canada **4**, 225–259 (1982).

[37] Fröhlich, J., Spencer, T., Wittwer, P.: Localization for a class of one dimensional quasi-periodic Schrödinger operators, Commun. Math. Phys. **132**, 5-25 (1990).

[38] Geisel, T., Ketzmerick, R., Petshel, G.: New class of level statistics in quantum systems with unbounded diffusion. Phys. Rev. Lett. **66**, 1651–1654 (1991)

[39] M. Goldstein, W. Schlag: On resonances and the formation of gaps in the spectrum of quasi-periodic Schroedinger equations. Annals of Math **173** 337-475 (2011).

[40] A. Y. Gordon. The point spectrum of the one-dimensional Schrödinger operator. *Uspekhi Matematicheskikh Nauk*, **31**, 257–258 (1976).

[41] A.Y. Gordon, S. Jitomirskaya, Y. Last and B. Simon. Duality and singular continuous spectrum in the almost Mathieu equation, Acta Mathematica **178**, 169–183 (1997).

[42] R. Han, Absence of point spectrum for the self-dual extended Harper's model International Mathematics Research Notices, **9**, 2801–2809 (2018)

[43] R. Han, S. Jitomirskaya, F. Yang, Universal hierarchical structure of eigenfunctions for the Maryland model, In preparation

[44] R. Han, F. Yang, S. Zhang. Spectral dimension for
-almost periodic singular Jacobi operators and the extended Harper's model. J. d'Analyse, to appear.

[45] B. Helffer, Q. Liu, Y. Qu, Q. Zhou. Positive Hausdorff Dimensional Spectrum for the Critical Almost Mathieu Operator. Commun. Math. Phys., **368**, 369–382 (2019).

[46] B. Helffer, J. Sjöstrand. Semi-classical analysis for Harper's equation.III. Cantor structure of the spectrum. Mem. Soc. Math. France (N.S.) **39**, 1–139 (1989).

[47] M. R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractere local d'un théoreme d'Arnold et de Moser sur le tore de dimension 2. *Commentarii Mathematici Helvetici*, **58**, 453–502 (1983).

[48] D. R. Hofstadter. Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields. *Physical review B*, **14** 22-39 (1976).

[49] K. Ishii. Localization of eigenstates and transport phenomena in the one-dimensional disordered system. *Progress of Theoretical Physics Supplement*, **53**, 77–138 (1973).

[50] S. Jitomirskaya. Almost everything about the almost Mathieu operator. II. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, 373–382 (1994).

[51] S. Y. Jitomirskaya. Anderson localization for the almost Mathieu equation: a nonperturbative proof. *Comm. Math. Phys*, **165** 49–57 (1994).

[52] S. Jitomirskaya, Ergodic Schrödinger operators (on one foot). Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, 613–647, Proc. Sympos. Pure Math., 76, Part 2, Amer. Math. Soc., Providence, RI, 2007.

[53] S. Y. Jitomirskaya. Metal-insulator transition for the almost Mathieu operator. *Ann. of Math. (2)*, **150**, 1159–1175 (1999).

[54] S. Jitomirskaya, On point spectrum at critical coupling. Advances, to appear.

[55] S. Jitomirskaya, D.A. Koslover, M.S. Schulteis,

[56] S. Jitomirskaya, D.A. Koslover, M.S. Schulteis, Continuity of the Lyapunov Exponent for analytic quasiperiodic cocycles, ETDS, **29**, 1881–1905 (2009).

[57] S. Jitomirskaya and I. Krasovsky. Continuity of the measure of the spectrum for discrete quasiperiodic operators. Math. Res. Lett. **9**, 413–421 (2002).

[58] S. Jitomirskaya and I. Krasovsky. Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum. Preprint 2019.

[59] S. Jitomirskaya, H. Krüeger and W. Liu, Exact dynamical decay rate for the almost Mathieu operator. Math. Res. Lett., Volume 27, Number 3, 789–808, 2020.

[60] S. Jitomirskaya and W. Liu. Universal hierarchical structure of quasiperiodic eigenfunctions. *Ann. of Math. (2)*, **187**, 721–776 (2018).

[61] S. Jitomirskaya and W. Liu. Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase. *arXiv preprint arXiv:1802.00781*, 2018.

[62] S. Jitomirskaya, W. Liu, S. Tcheremchantsev, Upper bounds on the fractal spectral dimensions and singular continuous spectrum near the arithmetic transition. In preparation

[63] S. Jitomirskaya and W. Liu. In preparation

[64] S. Jitomirskaya and C. Marx. Analytic quasi-periodic schrödinger operators and rational frequency approximants. *Geometric and Functional Analysis*, **22**, 1407–1443 (2012).

[65] S. Jitomirskaya and C. Marx. Dynamics and spectral theory of quasi-periodic Schrödinger-type operators. Ergodic Theory and Dynamical Systems **37**, 2353-2393 (2017).

[66] S. Jitomirskaya and R. Mavi. Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials. *International Mathematics Research Notices*, **2017**, 96–120, (2016).

[67] S. Jitomirskaya and B. Simon. Operators with singular continuous spectrum: III. Almost periodic Schrödinger operators. *Communications in Mathematical Physics*, **165**, 201–205, (1994).

[68] S. Jitomirskaya and S. Zhang. Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schödinger operators. J. Eur. Math. Soc., to appear. *DOI 10.4171/JEMS/1139*

[69] R. Ketzmerick, K. Kruse, F. Steinbach, T. Geisel, Covering property of Hofstadter's butterfly, Phys. Rev. B. **58**. 10.1103 (1998)

[70] M. Kohmoto, Y. Hatsugai. Peierls stabilization of magnetic-flux states of two-dimensional lattice electrons. Phys. Rev. B **41**, 9527–9529 (1990).

[71] S. Kotani. Ljaponov indices determine absolutely continuous spectra of stationary one-dimensional Schrödinger operators. *Stochastic Analysis*, North-Holland Mathematical Library **32**, 225–248 (1984)

[72] H. Kunz and B. Souillard. Sur le spectre des opérateurs aux différences finies aléatoires. *Communications in Mathematical Physics*, **78**, 201–246, (1980).

[73] Y. Last. Zero measure spectrum for the almost Mathieu operator. *Communications in mathematical physics*, **164**, 421–432, (1994).

[74] Y. Last: A relation between a.c. spectrum of ergodic Jacobi matrices and the spectra of periodic approximants. Commun. Math. Phys. **151**, 183–192 (1993).

[75] Y. Last and M. Shamis. Zero Hausdorff dimension spectrum for the almost Mathieu operator. Commun. Math. Phys. **348**, 729–750 (2016).

[76] Y. Last, B. Simon. Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators. Invent. Math. **135**, 329–367 (1999).

[77] W. Liu and X. Yuan. Anderson localization for the almost Mathieu operator in the exponential regime. J. Spectr. Theory, **5**,89-112 (2015).

[78] V. A. Mandelshtam, S. Ya. Zhitomirskaya. 1D-quasiperiodic operators. Latent symmetries. Commun. Math. Phys. **139**, 589–604 (1991).

[79] L. A. Pastur. Spectral properties of disordered systems in the one-body approximation. *Communications in Mathematical Physics*, **75**, 179–196 (1980).

[80] R. Peierls. Zur Theorie des Diamagnetismus von Leitungselektronen. Zeitschrift für Physik A: Hadrons and Nuclei **80**, 763–791 (1933).

[81] J. Puig. Cantor spectrum for the almost Mathieu operator. Commun. Math. Phys. **244**, 297–309 (2004).

[82] M. A. Shubin, Discrete Magnetic Laplacian, Communications in Mathematical Physics, **164**, 259–275 (1994)

[83] B. Simon. Almost periodic Schrödinger operators: a review. *Advances in Applied Mathematics*, **3**, 463–490 (1982).

[84] B. Simon. Kotani theory for one dimensional stochastic Jacobi matrices. *Communications in mathematical physics*, **89**, 227–234, (1983).

[85] B. Simon. Fifteen problems in mathematical physics. Oberwolfach Anniversary Volume, 423–454 (1984).

[86] B. Simon. Schrödinger operators in the twenty-first century, Mathematical physics 2000, 283–288, Imp. Coll. Press, London, 2000.

[87] B. Simon. Fifty Years of the Spectral Theory of Schrödinger Operators. Linde Hall Inaugural Math Symposium, Caltech, Feb. 2019.

[88] Sinai, Ya.: Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential, J. Stat. Phys. **46**, 861-909 (1987).

[89] C. Tang, M. Kohmoto, Global scaling properties of the spectrum for a quasiperiodic Schrödinger equation, Phys. Rev. B **34**, 2041(R) (1986)

[90] D. J. Thouless. Bandwidths for a quasiperiodic tight-binding model. Phys. Rev. B **28**, 4272–4276 (1983)

[91] D. J. Thouless. Scaling for the discrete Mathieu equation. Commun. Math. Phys. **127**, 187–193 (1990).

[92] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential Phys. Rev. Lett. **49**, 405 –408 (1982).

[93] Y. Wang and J.You, Examples of discontinuity of Lyapunov exponent in smooth quasiperiodic cocycles. Duke Math. J. **162**, 2363-2412, (2013).

[94] P. B. Wiegmann and A. V. Zabrodin. Quantum group and magnetic translations Bethe ansatz for the Azbel-Hofstadter problem. Nucl. Phys. B **422**, 495–514 (1994).

[95] M. Wilkinson and E. J. Austin. J Phys. A: Math. Gen. **23** 2529–2554 (1990).

[96] J. You. Quantitative almost reducibility and applications, Proceedings of ICM (2018).

[97] L.-S. Young. Lyapunov exponents for some quasi-periodic cocycles. Ergodic Theory and Dynamical Systems, **17**, 483-504 (1997).

[98] X. Zhao. Continuity of the spectrum of quasi-periodic Schrödinger operators with finitely differentiable potentials. ETDS, **40**, 564-576 (2020).

[99] Zhitomirskaya, S.: Singular spectral properties of a one dimensional discrete Schrödinger operator with quasiperiodic potential. Advances of Sov. Math. v. 3, 215-254 (1991).

9. Random Jacobi Matrices

In this chapter and the next one, we discuss two rather new subjects of mathematical research: random and almost periodic operators. Those operators serve in solid state physics as models of disordered systems, such as alloys, glasses and amorphous materials. The disorder of the system is reflected by the dependence of the potential on some random parameters. Let us discuss an example. Suppose we are given an alloy, that is, a mixture of (say two) crystalline materials. Suppose furthermore that the atoms (or ions) of the two materials generate potentials of the type $\lambda_1 f(x - x_0)$ and $\lambda_2 f(x - x_0)$, respectively, where x_0 is the position of the atom. If atoms of the two kinds are spread randomly on the lattice $\mathbb{Z}^\nu$ with exactly one atom at each site, then the resulting potential should be given by

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^\nu} q_i(\omega) f(x - i) ,$$

where q_i are random variables assuming the values λ_1 and λ_2 with certain probabilities.

Schrödinger operators with stochastic (i.e. random or almost periodic) potentials show quite "unusual" spectral behavior. We will give examples, in this and the next chapter, of dense point spectrum, singular continuous spectrum and Cantor spectrum.

Despite intensive research by many mathematicians since the seventies, the theory of stochastic Schrödinger operators is far from being complete. In fact, most of the basic problems are unsolved in dimension $\nu > 1$.

We will not attempt to give a complete treatment of the subject, but rather introduce some of the basic problems, techniques and fascinating results in the field. We will not discuss stochastic Schrödinger operators, i.e. operators of the form $H_\omega = H_0 + V_\omega$ on $L^2(\mathbb{R}^\nu)$, but a discretized version of those operators acting on the sequence space $l^2(\mathbb{Z}^\nu)$, namely Jacobi matrices. The operator H_0 is replaced by a finite difference operator, and the potential becomes a function on $\mathbb{Z}^\nu$ rather than on $\mathbb{R}^\nu$. This model is known in solid state physics as the "tight binding approximation." We refer to Schrödinger operators on $L^2(\mathbb{R}^\nu)$ as the continuous case, and to the tight binding model as the discrete case.

The advantage of this procedure is twofold: Some technically difficult but unessential problems of the continuous case disappear, and our knowledge (especially for the almost periodic case) is larger for the discrete case.

Two recent reviews of random Schrödinger operators from distinct points of view from each other and from our discussion here are *Carmona* [61] and *Spencer* [347].

In this chapter, we assume some knowledge of basic concepts of probability theory. The required preliminaries can be found in any textbook on probability theory (for example, *Breiman* [53]).

9.1 Basic Definitions and Results

Let $u = \{u(n)\}_{n\in\mathbb{Z}^\nu}$ denote an element of $l^2(\mathbb{Z}^\nu)$; i.e. $\|u\| := [\sum_{n\in\mathbb{Z}}|u(n)|^2]^{1/2} < \infty$. Set $|n| = \max|n_j|$ and $|n|_+ := \sum_{j=1}^{\nu}|n_j|$ for $n\in\mathbb{Z}^\nu$. We define a discrete analog Δ_d of the Laplacian on $L^2(\mathbb{R}^d)$ by:

$$(\Delta_d u)(i) = \sum_{j;|j-i|_+=1}[u(j) - u(i)] = \left[\sum_{j;|j-i|_+=1} u(j)\right] - 2\nu u(i) \ . \tag{9.1}$$

Here Δ_d is a bounded operator on $l^2(\mathbb{Z}^\nu)$, a fact that makes life easier than in the continuous case. The spectrum of Δ_d is purely absolutely continuous, and $\sigma(\Delta_d) = \sigma_{ac}(\Delta_d) = [-4\nu, 0]$. This can be seen by Fourier transformation.

Note that

$$\langle u, -\Delta_d u\rangle = \sum_{\substack{i,j\\|i-j|_+=1}} |u(i) - u(j)|^2$$

(each pair occurring once in the sum), explaining why we can regard (9.1) as an analog of the Laplacian.

If V is a function on $\mathbb{Z}^\nu$ playing the role of a potential, a natural analog of the Schrödinger operator is the operator

$$\bar{H} = -\Delta_d + V \ . \tag{9.2}$$

However, it is common to consider $+\Delta_d$ instead of $-\Delta_d$, and furthermore, to subsume the diagonal terms of Δ_d into the potential. Since Δ_d is bounded, this procedure has no "essential" effect on the properties of $\bar{H}$. Indeed, since the operator $(-1)^N$ defined by $[(-1)^N u](n) = (-1)^{|n|_+}u(n)$ obeys $(-1)^N\bar{H}[(-1)^N]^{-1} = 4\nu + \Delta_d + V$, $\bar{H}$ and the operator H below are unitarily equivalent up to a constant. Thus, we consider the operators

$$(H_0 u)(n) = \sum_{j;|j-n|_+=1} u(j) \quad \text{and} \tag{9.3}$$

$$(Hu)(n) = (H_0 u)(n) + V(n)u(n) \ . \tag{9.4}$$

The potentials V we are interested in form a random field, i.e. for any $n\in\mathbb{Z}^\nu$, the potential $V(n)$ evaluated at n is a random variable (= measurable function) on a probability space (Ω, F, P). F is a σ-algebra on Ω, and P a probability measure on (Ω, F). We adopt the common use in probability theory to denote the integral with respect to P by $\mathbb{E}$ (for "expectation"), i.e. $\int f(\omega)\,dP(\omega) =: \mathbb{E}(f)$. Without loss of generality, we may (and will) assume that

$$\Omega = S^{\mathbb{Z}^\nu} = \bigtimes_{\mathbb{Z}^\nu} S \ , \tag{9.5}$$

where S is a (Borel-) subset of $\mathbb{R}$, and F is the σ-algebra generated by the cylinder sets, i.e. by sets of the form $\{\omega | \omega_{i_1} \in A_1, \ldots, \omega_{i_n} \in A_n\}$ for $i_1, \ldots, i_n \in \mathbb{Z}^\nu$ and $A_1, \ldots, A_n$ Borel set in $\mathbb{R}$. We define the *shift operators* T_i on Ω by

$$T_i \omega(j) = \omega(j - i) \ . \tag{9.6}$$

A probability measure P on Ω is called *stationary* if $P(T_i^{-1} A) = P(A)$ for any $A \in$ F. A stationary probability measure is called *ergodic*, if any shift invariant set A, i.e. a set A with $T_i^{-1} A = A$ for all $i \in \mathbb{Z}^\nu$, has probability, $P(A)$, zero or one.

If $V_\omega(n)$ is a real-valued random field on $\mathbb{Z}^\nu$, it can always be realized on the above probability space in such a way that $V_\omega(n) = \omega(n)$. V is called stationary (ergodic), if the corresponding probability measure P is stationary (ergodic).

An important example of an ergodic random field is a family of independent, identically distributed random variables. In this case, the measure P is just the product measure

$$\bigtimes_{i \in \mathbb{Z}^\nu} dP_0$$

of the common distribution P_0 of the random variables $V_\omega(i)$, i.e. $P_0(A) = P(V_\omega(i) \in A)$ for any $A \in B(\mathbb{R})$ and $i \in \mathbb{Z}^\nu$. We have, for example:

$$\begin{aligned} &\int f(\omega_{i_1}, \ldots, \omega_{i_n})\, dP(\omega) [= \mathbb{E}(f)] \\ &\quad = \int f(x_1, \ldots, x_n)\, dP_0(x_1)\, dP_0(x_2) \ldots dP_0(x_n) \ . \end{aligned}$$

The Hamiltonian H_ω with V_ω i.i.d. is referred to as the *Anderson model*.

Another important class of ergodic potentials are almost periodic potentials. We introduce and investigate those potentials in Chap. 10.

For a fixed ω, the operator H_ω is nothing but a discretized Schrödinger operator with a certain potential. Therefore, it may seem to the reader that the introduction of a probability space is useless since we could as well consider each V_ω as a deterministic potential. The point of random potentials is that we are no longer interested in properties of H_ω for a fixed ω, but only in properties for *typical* ω. More precisely, we are interested in theorems of the form: H_ω has a property, p, for all ω in a set $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$. This will be abbreviated by: H_ω has the property, p, P-almost surely (or P-a.s. or a.s.).

If not stated otherwise, V_ω is assumed to be a stationary ergodic random field satisfying $|V_\omega(n)| \leq C < \infty$ for all n and ω. However, the boundedness assumption can be omitted (or replaced by a moment condition) for many purposes.

We state and prove the following proposition for later use, as well as to demonstrate typical techniques concerning ergodicity. A random variable f is called *invariant* under T_i if $f(T_i \omega) = f(\omega)$ for all i.

Proposition 9.1. Suppose the family of measure preserving transformation, T_i, is ergodic. If a random variable, $f\colon \Omega \to \mathbb{R}$, is invariant under $\{T_i\}$, then f is constant P-a.s.

Remark. The proof extends easily to $f\colon \Omega \to \mathbb{R} \cup \{\infty\}$.

Proof. Define $\Omega_M = \{\omega | f(\omega) \leq M\}$. Since f is invariant, the set Ω_M is invariant under T_i and consequently has probability zero or 1. For $M \leq M'$ we have $\Omega_M \subset \Omega_{M'}$. Moreover,

$$\bigcup_{M \in \mathbb{R}} \Omega_M = \bigcup_{M \in \mathbb{Z}} \Omega_M = \Omega \quad \text{and}$$

$$\bigcap_{M \in \mathbb{R}} \Omega_M = \bigcap_{M \in \mathbb{Z}} \Omega_M$$

has probability zero. Thus,

$$M_0 = \inf_{P(\Omega_M)=1} M$$

is finite. Since

$$\Omega_{M_0} = \bigcap_{n \in \mathbb{N}} \Omega_{M_0+(1/n)} \quad \text{and}$$

$$\tilde{\Omega}_{M_0} := \{\omega | f(\omega) < M_0\} = \bigcup_{n \in \mathbb{N}} \Omega_{M_0-(1/n)}$$

we have $P(\Omega_{M_0}) = 1$, $P(\tilde{\Omega}_{M_0}) = 0$, and consequently

$$P(\{\omega | f(\omega) = M_0\}) = P(\Omega_{M_0} \backslash \tilde{\Omega}_{M_0}) = 1 \ . \quad \square$$

Let us define, for $u \in l^2(\mathbb{Z}^\nu)$

$$(U_i u)(n) = u(n - i) \ . \tag{9.7}$$

If V_ω is an ergodic potential with corresponding measure preserving transformations $\{T_i\}_{i \in \mathbb{Z}^\nu}$, then

$$H_{T_i\omega} = U_i H_\omega U_i^* \ , \tag{9.8}$$

a relation basic to some elementary properties of H_ω. Stochastic operators H_ω satisfying (9.8) are sometimes called *ergodic operators.*

The following theorem is a basic observation of *Pastur* [271] (with some technical supplements for the unbounded and the continuous case in [221] and [205]):

Theorem 9.2 (Pastur). Let V_ω be an ergodic potential. Then there exists a set $\Sigma \subset \mathbb{R}$ such that

$$\sigma(H_\omega) = \Sigma \quad P\text{-a.s.} \ .$$

Moreover,

$$\sigma_{\mathrm{dis}}(H_\omega) = \phi \quad P\text{-a.s.} \ .$$

To prove Theorem 9.2, we need a preparatory lemma. We say that a family $\{A_\omega\}_{\omega\in\Omega}$ of bounded operators on a Hilbert space H is *weakly measurable* if the mapping $\omega \mapsto \langle\varphi, A_\omega\psi\rangle$ is measurable for all $\varphi, \psi \in \mathrm{H}$.

Lemma 9.3. Suppose that $\{P_\omega\}_{\omega\in\Omega}$ is a weakly measurable family of orthogonal projections satisfying (9.8). Then

$\dim \mathrm{Ran}(P_\omega)$ is zero P-a.s. or

$\dim \mathrm{Ran}(P_\omega)$ is infinite P-a.s. .

Proof. The P_ω are positive operators, hence the trace $\mathrm{tr}\, P_\omega$ is uniquely defined (possibly $+\infty$). Fixing ω and choosing an orthonormal basis $a_1(\omega), a_2(\omega) \ldots$ of $\mathrm{Ran}(P_\omega)$ and an orthonormal basis $b_1(\omega), b_2(\omega) \ldots$ of $\mathrm{Ran}(P_\omega)^\perp$ we see that

$$\begin{aligned}\mathrm{tr}\, P_\omega &= \sum \langle a_i(\omega), P_\omega a_i(\omega)\rangle + \sum \langle b_i(\omega), P_\omega b_i(\omega)\rangle \\ &= \sum \langle a_i(\omega), a_i(\omega)\rangle = \dim \mathrm{Ran}(P_\omega).\end{aligned}$$

Now let $\{e_i, i \in \mathbb{Z}^\nu\}$ be the standard orthonormal basis in $l^2(\mathbb{Z}^\nu)$, i.e. $e_i(n) = \delta_{in}$. Then $\mathrm{tr}\, P_\omega = \sum \langle e_i, P_\omega e_i\rangle$ is a random variable ($=$ measurable). Moreover,

$$\begin{aligned}\mathrm{tr}\, P_{T_j\omega} &= \sum \langle P_{T_j\omega} e_i, P_{T_j\omega} e_i\rangle \\ &= \sum \langle P_\omega e_{i-j}, P_\omega e_{i-j}\rangle = \mathrm{tr}\, P_\omega \ .\end{aligned}$$

By Proposition 9.1, $\dim \mathrm{ran}\, P_\omega = \mathrm{tr}\, P_\omega$ is thus a.s. constant. Hence P-a.s.:

$$\begin{aligned}\mathrm{tr}\, P_\omega &= \mathbb{E}(\mathrm{tr}\, P_\omega) \geq \sum_{|i|\leq N} \mathbb{E}(\langle e_i, P_\omega e_i\rangle) \\ &= \sum_{|i|\leq N} \mathbb{E}(\langle e_0, P_{T_i\omega} e_0\rangle) = \sum_{|i|\leq N} \mathbb{E}(\langle e_0, P_\omega e_0\rangle) \ ,\end{aligned}$$

(where we use that T_i are measure preserving)

$$= (2N+1)^\nu \mathbb{E}(\langle e_0, P_\omega e_0\rangle)$$

Since N was arbitrary, $\mathrm{tr}\, P_\omega = 0$ or $\mathrm{tr}\, P_\omega = \infty$ according to $\mathbb{E}(\langle e_0, P_\omega e_0\rangle) = 0$ or not. ■

Proof of Theorem 9.2. Denote the spectral projections of H_ω by $E_\Delta(\omega)$. Equation (9.8) implies that

$$E_\Delta(T_i\omega) = U_i E_\Delta(\omega) U_i^* \ . \tag{9.9}$$

This follows from the fact that the right-hand side of (9.9) is the spectral resolution for the operator $U_i H_\omega U_i^*$.

We now prove that for a fixed Borel set Δ the function $\omega \mapsto E_\Delta(\omega)$ is weakly measurable. It is not difficult to see that products of bounded, weakly measurable functions are weakly measurable. So, in particular, H_ω^n is weakly measurable for any $n \in \mathbb{N}$. We can approximate $E_\Delta(\omega)$ in the strong topology by ω-independent polynomials in H_ω. Thus, $E_\Delta(\omega)$ is weakly measurable. Therefore, by Lemma 9.3, for fixed Δ dim Ran(E_Δ) is either zero a.s. or infinite a.s.

For any pair $\langle p,q \rangle$ of rational numbers, we set $\eta(p,q) := 0$ if $\dim \operatorname{Ran}(E_{(p,q)}(\omega)) = 0$ P-a.s. and $\eta(p,q) := \infty$ if $\dim \operatorname{Ran}(E_{(p,q)}(\omega)) = \infty$ P-a.s. According to Lemma 9.3, $\eta(p,q)$ is well defined. Define $\Omega_{p,q} := \{\omega | \dim \operatorname{Ran} E_{(p,q)}(\omega) = \eta(p,q)\}$ and

$$\Omega_0 = \bigcap_{p,q \in \mathbb{Q}} \Omega_{p,q} \; .$$

Since $\Omega_{p,q}$ has probability 1 and the intersection over p, $q \in \mathbb{Q}$ is countable, we have $P(\Omega_0) = 1$.

We claim that for $\omega_1, \omega_2 \in \Omega_0$ the spectra $\sigma(H_{\omega_1})$ and $\sigma(H_{\omega_2})$ coincide. Indeed, if $\lambda \notin \sigma(H_{\omega_1})$, then

$$\dim \operatorname{Ran} E_{(\lambda_1,\lambda_2)}(\omega_1) = 0$$

for all λ_1, λ_2 with $\lambda_1 < \lambda < \lambda_2$ sufficiently near to λ. Since $\omega_1, \omega_2 \in \Omega_0$ we have

$$\dim \operatorname{Ran} E_{(p,q)}(\omega_1) = \dim \mathbb{E}_{(p,q)}(\omega_2)$$

for p, $q \in \mathbb{Q}$, so

$$\dim \operatorname{Ran} E_{(\lambda_1,\lambda_2)}(\omega_2) = 0$$

for $\lambda_1, \lambda_2 \in \mathbb{Q}$ with $\lambda_1 < \lambda < \lambda_2$ sufficiently near to λ. This implies $\lambda \notin \sigma(H_{\omega_2})$. The claim follows by interchanging the roles of ω_1 and ω_2.

Now, suppose that $\lambda \in \sigma_{\text{dis}}(H_\omega)$ for an $\omega \in \Omega_0$. Then

$$0 < \dim \operatorname{Ran} E_{(\lambda_1,\lambda_2)}(\omega) < \infty$$

for some $\lambda_1 < \lambda < \lambda_2$, λ_1, $\lambda_2 \in \mathbb{Q}$. But this contradicts the choice of $\omega \in \Omega_0$. So $\sigma_{\text{dis}}(H_\omega) = \phi$ P-a.s. ∎

Remark. (1) The use of the countable set of pairs $\mathbb{Q} \times \mathbb{Q}$ in the above proof is essential, since an uncountable intersection of sets of full probability may have probability strictly less than 1.

(2) To prove the result for unbounded operators needs a bit more technique to prove the weak measurability of $E_\Delta(\omega)$ (see [205]).

The following theorem is due to *Kunz-Souillard* [221], and was extended to a more general context by *Kirsch* and *Martinelli* [205]:

Theorem 9.4 (Kunz-Souillard). Let V_ω be an ergodic potential. Then there exist sets $\sum_{\text{ac}}, \sum_{\text{sc}}, \sum_{\text{pp}} \subset \mathbb{R}$ such that

$$\sigma_{ac}(H_\omega) = \Sigma_{ac} \quad P\text{-a.s.}$$

$$\sigma_{sc}(H_\omega) = \Sigma_{sc} \quad P\text{-a.s.}$$

$$\sigma_{pp}(H_\omega) = \Sigma_{pp} \quad P\text{-a.s.}$$

Notational Warning. By $\sigma_{pp}(H)$ we denote the *closure* of the set $\varepsilon(H) := \{\lambda | \lambda \text{ is an eigenvalue of } H\}$. This notation disagrees with that of *Reed* and *Simon* I [292]. In fact, the above theorem would be wrong for $\varepsilon(H_\omega)$!

Remark. At first glance, given Theorem 9.2, Theorem 9.4 looks rather trivial. However, there is a pitfall: The necessary measurability of certain projections is nontrivial.

Proof. We define $E_\Delta^c(\omega) := E_\Delta(\omega)P_c(\omega)$ where $P_c(\omega)$ is the projector onto the continuous subspace w.r.t. H_ω. Analogously, we define $E_\Delta^{ac}(\omega) := E_\Delta(\omega)P_{ac}(\omega)$, etc. Then the proof of Theorem 9.2, with $E_\Delta(\omega)$ replaced by $E_\Delta^{ac}(\omega)$ etc., proves Theorem 9.4 except for one point: We have to prove the weak measurability of $E_\Delta^{ac}(\omega)$, $E_\Delta^{sc}(\omega)$ and $E_\Delta^{pp}(\omega)$. For this, it suffices to prove the weak measurability of $E_\Delta^c(\omega)$ and $E_\Delta^s(\omega)$.

It is not difficult to verify by the RAGE-theorem (Theorem 5.8) that

$$\langle \varphi, P_c(\omega)\psi \rangle = \lim_{J\to\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \langle \varphi, e^{itH_\omega} F(|j| > J) e^{-itH_\omega}, \psi \rangle \, dt \tag{9.10}$$

where $F(A)$ is multiplication with the characteristic function of A. Since

$$e^{itH_\omega} = \sum \frac{(itH_\omega)^n}{n!} ,$$

the right-hand side of (9.10) is measurable, hence $P_c(\omega)$ and therefore $E_\Delta^c(\omega) := E_\Delta(\omega)P_c(\omega)$ is weakly measurable.

We prove the weak measurability of $E_\Delta^s(\omega)$ using an argument of *Carmona* [61]. We need a lemma:

Lemma. Let $\mathscr{I}$ be the family of finite unions of open intervals, each of which has rational endpoints. Then, for any Borel set A,

$$\mu_{\mathrm{sing}}(A) = \lim_{n\to\infty} \sup_{I\in\mathscr{I}, |I|<n^{-1}} \mu(A \cap I) =: \nu(A) ,$$

where $|\cdot|$ = Lebesgue measure.

Proof of Lemma. Note first that the sup decreases as n increases, so the limit defining $\nu(A)$ exists.

Write $d\mu_{a.c.}(x) = f(x)dx$ and set $g(R) = \mu_{a.c.}(\{x | f(x) > R\})$ so $g(R) \searrow 0$ as

$R \to \infty$. Then $\mu_{\mathrm{a.c.}}(I) \le R|I| + g(R)$ so

$$\mu(A \cap I) \le \mu_s(A) + R|I| + g(R) \ .$$

Thus, for any R, $\nu(A) \le \mu_s(A) + g(R)$, and thus $\nu(A) \le \mu_s(A)$.

Conversely, find B with $|B| = 0$ and $\mu_s(\mathbb{R}\backslash B) = 0$. Find open sets C_m so $A \cap B \subset C_m$ and $|C_m| \searrow 0$. Given n and ε, find m so $|C_m| < n^{-1}$ and then $I \in \mathscr{I}$, so $\mu(C_m \backslash I) \le \varepsilon$. Then

$$\mu(A \cap I) \ge \mu(A \cap C_m) - \mu(C_m \backslash I) \ge \mu(A \cap B) - \varepsilon = \mu_s(A) - \varepsilon \ .$$

Thus, $\nu(A) \ge \mu_s(A) - \varepsilon$ so $\nu(A) \ge \mu_s(A)$. ■

Conclusion of the Proof of Theorem 9.4. For any φ,

$$(\varphi, E^s_\Delta(\omega)\varphi) = \lim_{n\to\infty} \sup_{I \in \mathrm{I}, |I| < n^{-1}} (\varphi, E_{\Delta \cap I}(\omega)\varphi) \ ,$$

since I is countable and $(\varphi, E_{\Delta\cap I}(\omega)\varphi)$ is measurable, we conclude that $(\varphi, E^s_\Delta(\omega)\varphi)$ is measurable. By polarization, $E^s_\Delta(\omega)$ is weakly measurable. ■

We close this section with the following observation due to *Pastur* [271] which is special to the one-dimensional case:

Theorem 9.5 (Pastur). If $\nu = 1$, then for any given λ, $P(\{\omega | \lambda$ is an eigenvalue of $H_\omega\}) = 0$.

Remarks (1). It does *not* follow from Theorem 9.5 that H_ω has no eigenvalues. An uncountable union of sets of probability zero may have positive probability (or may even be unmeasurable).

(2) While this observation of Pastur and the proof we give is one-dimensional, the result is true in any dimension. It follows from Theorem 9.9 below (see *Avron* and *Simon* [31]). This multidimensional result is more subtle, and is still not proven for the continuous case.

Proof. $\mathrm{tr}\, E_{\{\lambda\}} = 0$ a.s. or $\mathrm{tr}\, E_{\{\lambda\}} = \infty$ a.s. according to Lemma 9.3. But our one-dimensional finite difference equation has at most a two-dimensional space of solutions, hence $\mathrm{tr}\, E_{\{\lambda\}} = 0$ a.s. ■

Corollary. If the point spectrum $\Sigma_{\mathrm{pp}}[= \sigma_{\mathrm{pp}}(H_\omega)\text{a.s.}]$ is non-empty, then it is locally uncountable.

9.2 The Density of States

In this section, we briefly discuss an important quantity for disordered systems, the density of states $k(E)$. For recent surveys on this subject, see [202, 342]. The quantity $k(E)$ measures, in some sense, "how many states" correspond to energies below the level E.

Recall that our Hamiltonians H_ω model the motion of a single particle (electron) in a solid with infinitely many centers of forces (nuclei, ions) located at some fixed positions. This is the so-called one-body approximation. However, in a solid with infinitely many nuclei, we also have infinitely many electrons. We cannot handle directly a problem with infinitely many particles, but we should at least take into account the fermionic nature of the electrons via the Pauli exclusion principle. This principle states that two fermions (e.g. electrons or protons) cannot occupy the same quantum mechanical state (see also Chap. 3). This leads to the well-known fact that electrons in an atom do not all have the "ground-state energy", but fill up the energy levels starting from the ground state energy up to a certain level. Such a phenomenon also should occur in our disordered solid. However, we are faced with the problem of having to distribute *infinitely many* fermions on a *continuum* of *energy levels.* To get rid of this problem, we will restrict the problem first to a finite domain. In such a domain, we should have only finitely many electrons. To do this, let, as usual, $E_\Delta(\omega)$ denote a spectral projection measure associated with H_ω and denote, by χ_L, the characteristic function of the "cube" $C_L = \{i \in \mathbb{Z}^\nu | -L \le i_k \le L; k = 1, \dots, \nu\}$. The "number of electrons" in C_L should be a density times $\# C_L = (2L+1)^\nu$. We define a measure dk_L by

$$\int_A dk_L = \frac{1}{(2L+1)^\nu} \dim \operatorname{Ran}(\chi_L E_A(\omega)\chi_L) = \frac{1}{(2L+1)^\nu} \operatorname{tr}(E_A(\omega)\chi_L) \ , \qquad (9.11)$$

This measures "how many electrons per lattice site (i.e. per nucleus) can be put into energy levels in the set A without violating the Pauli principle if we restrict the whole problem to the cube C_L." We may hope that the measures dk_L converge (in some sense), if we send C_L to $\mathbb{Z}^d$ (i.e. $L \to \infty$).

The appropriate convergence of measures is the vague convergence, i.e. $d\mu_n \to d\mu$ if $\int f\, d\mu_n \to \int f\, d\mu$ for any continuous function f with compact support. We define the measure dk by

$$\int f(\lambda)\, dk(\lambda) := \mathbb{E}(f(H_\omega)(0,0)) \ ,$$

where $A(0,0)$ is a shorthand notation for $\langle \delta_0, A\delta_0 \rangle$.

Theorem 9.6. For any bounded measurable function f there is a set Ω_f of probability 1 such that

$$\int f(\lambda)\, dk_L(\lambda) \to \int f(\lambda)\, dk(\lambda) \quad \text{as } L \to \infty \ . \qquad (9.12)$$

Proof. Fix a bounded measurable function f. Define $\bar{f}(\omega) := f(H_\omega)(0,0)$. By stationarity,

$$f(H_\omega)(n,n) = \bar{f}(T_n\omega) \ .$$

We have

$$\int f(\lambda)\,dk_L(\lambda) = \frac{1}{(2L+1)^\nu}\operatorname{tr}(f(H_\omega)\chi_L)$$

$$= \frac{1}{(2L+1)^\nu}\sum_{|n|\le L} f(H_\omega)(n,n) = \frac{1}{(2L+1)^\nu}\sum_{|n|\le L} \bar{f}(T_n\omega) = (*) \ .$$

To the last expression we apply Birkhoff's ergodic theorem (see e.g. *Breiman* [53]) which states that

$$\frac{1}{(2L+1)^\nu}\sum_{|i|\le L} \varphi(T_n\omega) \text{ converges to } \mathbb{E}(\varphi)\ P\text{-a.s. if } \varphi\in L^1(P) \ .$$

Thus,

$$\lim(*) = \mathbb{E}(\bar{f}(\omega)) = \mathbb{E}(f(H_\omega)(0,0)) = \int f(\lambda)\,dk(\lambda) \ . \quad \blacksquare$$

Since the set Ω_f may depend on f, it is not clear that there is an $\omega\in\Omega$ such that (9.12) is true for all bounded measurable functions f. However, we have

Theorem 9.7. dk_L converges vaguely to dk for P-a.s.

Remark. Notice that the limit measure dk is non-random.

Proof. There exists a countable subset F, of C_0, the continuous functions with compact support, such that for any $f\in C_0$ there is a sequence $\{f_n\}$ in F with $f_n\to f$ uniformly, and $\bigcup_n \operatorname{supp} f_n$ is contained in a (f-dependent) compact set. Set

$$\Omega_0 := \bigcap_{g\in \mathrm{F}} \Omega_g \ .$$

We have $P(\Omega_0) = 1$. Moreover, one checks that (9.12) holds for any $\omega\in\Omega_0$ and any $f\in C_0$. $\blacksquare$

We define

$$k(E) := \int \chi_{(-\infty,E)}(\lambda)\,dk(\lambda)$$

and call this quantity the *integrated density of states*. (Note that it is sometimes this quantity that is called "density of states" in the literature.)

The following theorem states a connection between the spectrum and the density of states.

Theorem 9.8 (Avron and Simon [31]).

$$\operatorname{supp}(dk) = \Sigma[=\sigma(H_\omega)\text{a.s.}] \ .$$

Remark. From our intuition at the beginning of this section, the theorem certainly should hold.

Proof. If $\lambda_0\notin\Sigma$, there is a non-negative continuous function with $f(\lambda_0) = 1$ and $f = 0$ on Σ. Thus, $f(H_\omega)\equiv 0$ and so $\int f(\lambda)\,dk = (f(H_\omega)(0,0)) = 0$ so $\lambda_0\notin \operatorname{supp} dk$.

Conversely, if $\lambda_0 \notin \operatorname{supp}(dk)$, then there is a positive continuous function f with $f(\lambda_0) = 1$ and $\int f(\lambda)\,dk = 0$. Thus, for a.e. ω, $f(H_\omega)(0,0) = 0$, and so by $f(H_\omega)(n,n) = f(H_{T^n\omega})(0,0)$ we know that a.e. ω, $(\delta_n, f(H_\omega)\delta_n) = 0$ for all n. But since $f(H_\omega) \geq 0$, this implies that $f(H_\omega) = 0$. Since f is continuous and $f(\lambda_0) = 1$, this implies that $\lambda_0 \notin \Sigma$. □

It is easy to see that the measure dk is a continuous measure, i.e. the function $k(E)$ is continuous, in the one-dimensional case. It was proven by *Craig* and *Simon* [69] that in the multidimensional case, $k(E)$ is even log-Hölder continuous (see [69] for details). Those authors use a version of the Thouless formula (see Chap. 9.4) for a strip to establish this result. Recently *Delyon* and *Souillard* [86] found a very elementary proof for the continuity (not log-Hölder continuity) of k.

Theorem 9.9 (Craig-Simon, Delyon-Souillard). $k(E)$ is a continuous function.

Proof. We follow *Delyon* and *Souillard* [86]. Fix λ. Let f_n be a sequence of continuous functions with $f_n(\lambda) = 1$ and $f_n(x) \downarrow 0$ if $x \neq \lambda$. Then $f_n(H_\omega)(0,0) \downarrow E_{\{\lambda\}}(0,0)$ and $\int f_n(x)\,dk(x) \downarrow k(\lambda + 0) - k(\lambda - 0)$. Thus, by the definition of dk and Theorem 9.6, it is enough to prove that

$$\mathbb{E}(E_{\{\lambda\}}(0,0)) = \lim \frac{1}{(2L+1)^\nu} \operatorname{tr}(E_{\{\lambda\}}\chi_L) = 0 \ .$$

We remark that the set where the first equality holds may be λ-dependent, but that does not change the fact that for λ fixed it holds a.e., and we need only look at a typical point.

A solution ψ of $H_\omega\psi = E\psi$ is uniquely determined inside C_L by its values on

$$\tilde{C}_L = \bigcup_{j=1}^{\nu} \tilde{C}_{L,j} \ ,$$

where $\tilde{C}_{L,1} = \{i \in C_L | i_1 = -L \text{ or } -L+1\}$ and $\tilde{C}_{L,k} = \{i \in C_L | i_k = -L \text{ or } L\}$ if $k \geq 2$. For

$$\psi(\alpha) = [E - V(\alpha - \delta_1)]\psi(\alpha - \delta_1) - \psi(\alpha - 2\delta_1) - \sum_{j=2}^{\nu} [\psi(\alpha - \delta_1 + \delta_j) + \psi(\alpha - \delta_1 - \delta_j)]$$

allows us then to determine ψ inductively for $\alpha_1 = -L+2, \ldots, L$. Thus,

$$\dim \chi_L[\operatorname{Ran} E_{\{\lambda\}}] \leq \#(\tilde{C}_L) \leq 2\nu(2L+1)^{\nu-1} \ .$$

But

$$\operatorname{tr}(\chi_L E_{\{\lambda\}}) \leq \dim \chi_L(\operatorname{Ran} E_{\{\lambda\}}) \|\chi_L E_{\{\lambda\}}\| \leq \dim \chi_L(\operatorname{Ran} E_{\{\lambda\}})$$

so $(2L+1)^{-\nu} \operatorname{tr}(\chi_L E_{\{\lambda\}}) \to 0$. □

There are some more results on the regularity of the density of states. We mention Wegner's proof of the existence of a density for dk for the Anderson model when the common distribution has a density (see *Wegner* [366] or *Fröhlich* and *Spencer* [119] Appendix C). *Constantinescu Fröhlich* and *Spencer* [67] proved the analyticity of $k(E)$ if $|E|$ is large for i.i.d. Gaussian $V_\omega(n)$.

However, for $\nu > 1$, there seems to be no regularity result for $k(E)$ in the continuous case, so far. For $\nu = 1$, it is not difficult to show that $k(E)$ is continuous.

Frequently the density of states is defined in a slightly different way than above. Instead of restricting $E_\Lambda(\omega)$ to cubes, one restricts H_ω itself. Let us set $C_{N,M} := \{k \in \mathbb{Z}^\nu | N \le k_i \le M;\ i = 1, \ldots, \nu\}$ for $N, M \in \mathbb{Z}$. We then define an operator $H_\omega^{(N,M)}$ on $l^2(C_{N,M}) \simeq \mathbb{C}^{(M-N+1)^\nu}$ by its matrix elements:

$$H_\omega^{(N,M)}{}_{i,j} = \langle \delta_i, H_\omega^{(N,M)} \delta_j \rangle := \langle \delta_i, H_\omega \delta_j \rangle \tag{9.13}$$

for $i, j \in C_{N,M}$. Equation (9.13) can be looked upon as imposing "boundary conditions" $u(k) = 0$ for $k \notin C_{N,M}$, k nearest neighbors to $C_{N,M}$.

We set

$$\rho_{N,M}(A) := \#\{\lambda \in A | \lambda \text{ is an eigenvalue of } H_\omega^{(N,M)}\} \ . \tag{9.14}$$

It can be shown (see *Avron* and *Simon* [31]) that the measures $(\# C_{N,M})^{-1}\, d\rho_{N,M}$ converge vaguely to dk as $|M - N| \to \infty$.

The rigorous investigation of the density of states goes back to *Benderskii-Pastur* [44], who proved the existence of k as the limit of $d\rho_{-L,L}/(2L+1)^\nu$. The existence of dk was proven in increasing generality and by different methods by *Pastur* [269], *Nakao* [262], *Kirsch* and *Martinelli* [206] and others. The way of defining dk through dk_L is due to *Avron* and *Simon* [31]. The definition of dk via the rotation number in the one-dimensional continuous case was introduced by *Johnson* and *Moser* [186] (see *Delyon* and *Souillard* [85] for the discrete case).

There is large interest in the asymptotic behavior of $k(E)$ for large and small values in E. In the continuous case, $k(E)$ behaves like $\tau_\nu E^{\nu/2}/(2\pi)^\nu$ as $E \to \infty$ (τ_ν is the volume of the unit ball in $\mathbb{R}^\nu$); see *Pastur* [269], *Nakao* [262], *Kirsch* and *Martinelli* [206]. This is the same behavior as for the free operator H_0. However, as E goes to $E_0 := \inf \sigma(H_\omega)$, the behavior of $k(E)$ differs heavily from the free case. As a rule, $k(E)$ decays for $E \searrow E_0$ much faster than $k_0(E)$—the density of states for H_0. For $E_0 > -\infty$, it was predicted by *Lifshitz* [234] on the basis of physical arguments that $k(E)$ should behave like $C_1 \exp[-c_2(E - E_0)^{-\nu/2}]$ as $E \searrow E_0$, which is now called the Lifshitz behavior. For rigorous treatment of the Lifshitz behavior for the discrete case, see *Fukushima* [123], *Romerio* and *Wreszinski* [298] and *Simon* [341]; for the continuous case, see *Nakao* [262], *Pastur* [270], *Kirsch* and *Martinelli* [207] and *Kirsch* and *Simon* [208]. The behavior of $k(E)$ as $E \searrow E_0 = -\infty$ is treated in *Pastur* [269] (see also *Fukushima, Nagai* and *Nakao* [124], *Nakao* [262], *Kirsch* and *Martinelli* [206]).

9.3 The Lyaponov Exponent and the Ishii-Pastur-Kotani Theorem

For most of the rest of this chapter, we suppose that $\nu = 1$, for it is the one-dimensional case which is best understood. What makes the one-dimensional case accessible is that, for fixed E, a solution of $(H_\omega - E)u = 0$ is determined by its values at two succeeding points (initial value problem for a second-order difference equation).

Fix E. We consider the one-dimensional difference equation of second order

$$u(n+1) + u(n-1) + [V_\omega(n) - E]u(n) = 0 \ . \tag{9.15}$$

and introduce the vector-valued function

$$\underline{u}(n) = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} .$$

Define

$$A_n(E) := A_n(E, \omega) := \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} .$$

A function $u(n)$ is a solution of (9.15) if and only if

$$\underline{u}(n+1) = A_{n+1}(E)\underline{u}(n) \ .$$

Set

$$\Phi_n(E) := \Phi_n(\omega, E) := A_n(E)A_{n-1}(E)\ldots A_1(E) \ . \tag{9.16}$$

Then

$$\underline{u}(n) = \Phi_n(E)\underline{u}(0)$$

defines the solution of (9.15) "to the right" with initial condition

$$\underline{u}(0) = \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} .$$

Similarly $\underline{u}(-n) = \Phi_{-n}(E)\underline{u}(0)$ with $\Phi_{-n}(E) := A_{-n+1}(E)^{-1} \ldots A_0(E)^{-1}$ defines the solution to the left. Note that

$$A_n(E)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & E - V_\omega(n) \end{pmatrix} .$$

We now define

$$\overline{\gamma}^{\pm}(\omega, E) = \overline{\lim_{N\to\pm\infty}} \frac{1}{|N|} \ln \|\Phi_N(\omega, E)\| \tag{9.17a}$$

$$\underline{\gamma}^{\pm}(\omega, E) := \underline{\lim}_{N\to\pm\infty} \frac{1}{|N|} \ln \|\Phi_N(\omega, E)\| \ . \tag{9.17b}$$

Remark. For definiteness, let $\|A\|$ denote the operator norm of the matrix A. However, the limits (9.17a) and (9.17b) do not change, if we use another norm.

These quantities measure the growth of the matrix norm $\|\Phi_N(\omega, E)\|$. Since $\det A_i(E) = 1$ and hence $\det \Phi_N = 1$, it follows that $\|\Phi_N(\omega, E)\| \geq 1$, and consequently $0 \leq \underline{\gamma}^{\pm}(\omega, E) \leq \overline{\gamma}^{\pm}(\omega, E)$. Indeed, if $\det A = 1$, at least one eigenvalue of A satisfies $|\lambda| \geq 1$, thus, $\|A\| \geq 1$.

Theorem 9.10 (Furstenberg and Kesten [126]). For fixed E and P-almost all ω

$$\gamma^{\pm}(E) := \lim_{N\to\pm\infty} \frac{1}{|N|} \ln \|\Phi_N(\omega, E)\| \tag{9.18}$$

exists, is independent of ω and

$$\gamma^{+}(E) = \gamma^{-}(E) \ . \tag{9.19}$$

We call $\gamma(E) := \gamma^{\pm}(E)$ the *Lyaponov exponent* for H_ω. We will see in a moment that $\gamma(E)$ plays a central role in the investigation of one-dimensional stochastic Jacobi matrices.

To prove (9.18), we will exploit Kingman's subadditive ergodic theorem [200] which we state without proof. We remark that a multi-dimensional version of the subadditive ergodic theorem can be used to prove the existence of the (integrated) density of states by means of Dirichlet-Neumann bracketing (see [206, 325]).

Since now $v = 1$, we have $T_n = (T_1)^n$. We set $T := T_1$, so $T_n = T^n$. We call T ergodic if $\{T^n\}_{n\in Z}$ is ergodic. A sequence $\{F_N\}_{N\in\mathbb{N}}$ of random variables is called a *subadditive* process if

$$F_{N+M}(\omega) \leq F_N(\omega) + F_M(T^N\omega)$$

where T is a measure preserving transformation.

Theorem 9.11 (Subadditive Ergodic Theorem, Kingman [200]). If F_N is a subadditive process satisfying $\mathbb{E}(|F_N|) < \infty$ for each N, and $\Gamma(F) := \inf \mathbb{E}(F_N)/N > -\infty$, then $F_N(\omega)/N$ converges almost surely. If, furthermore, T is ergodic, then

$$\lim_{N\to\infty} \frac{1}{N} F_N(\omega) = \Gamma(F)$$

almost surely.

Proof of Theorem 9.10. Define $F_N(\omega) = \ln \|\Phi_N(\omega, E)\|$. Since

$$\begin{aligned} F_{N+M}(\omega) &= \ln \left\| \prod_{j=N+1}^{N+M} A_j(\omega, E) \cdot \prod_{i=1}^{N} A_i(\omega, E) \right\| \\ &= \ln \left\| \prod_{j=1}^{M} A_j(T_N\omega, E) \cdot \prod_{i=1}^{N} A_i(\omega, E) \right\| \\ &\le \ln(\|\Phi_M(T_N\omega, E)\| \, \|\Phi_N(\omega, E)\|) \\ &= F_M(T_N\omega) + F_N(\omega) \end{aligned}$$

the process F_N is subadditive.

Moreover,

$$\begin{aligned} \frac{1}{N} \mathbb{E}(|F_N|) &= \frac{1}{N} \mathbb{E}\left(\ln \left\| \prod_{j=1}^{N} A_j(\omega, E) \right\| \right) \\ &\le \frac{1}{N} \mathbb{E} \sum_{j=1}^{N} \ln \|A_j(\omega, E)\| \\ &= \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}(\ln \|A_j(\omega, E)\|) \\ &= \mathbb{E}(\ln \|A_0(\omega, E)\|) \ , \end{aligned}$$

where we used the stationarity of $V_\omega(n)$ in the last step. Moreover, $\mathbb{E}(\ln^+ |V_\omega(0)|) < \infty$ implies $\mathbb{E}(\ln \|A_0(\omega, E)\|) < \infty$. In addition, as noted above, $F_N \ge 0$ so $\inf[\mathbb{E}(F_N)/N] \ge 0 > -\infty$. Thus, Theorem 9.11 implies that

$$\lim_{N \to +\infty} \frac{1}{N} \ln \|\Phi_N(\omega, E)\| = \inf_{N>0} \frac{1}{N} \mathbb{E}(\ln \|\Phi_N(\omega, E)\|) \quad \text{a.s.}$$

and

$$\lim_{N \to -\infty} \frac{1}{|N|} \ln \|\Phi_N(\omega, E)\| = \inf_{N<0} \frac{1}{|N|} \mathbb{E}(\ln \|\Phi_N(\omega, E)\|) \quad \text{a.s.}$$

Now we prove (9.19). Since, for $N > 0$, $\Phi_{-N} = A_{-N}^{-1} \dots A_{-1}^{-1} A_0^{-1} = (A_0 A_{-1} \dots A_{-N})^{-1}$, we have by stationarity

$$\mathbb{E}(\ln \|\Phi_{-N+1}\|) = \mathbb{E}(\ln \|\Phi_N^{-1}\|) \ . \tag{9.20}$$

Moreover, for

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

we have

$$(J\Phi_N J^{-1})^t = \Phi_N^{-1} \ .$$

Thus, since $\|J\underline{u}\| = \|J^{-1}\underline{u}\| = \|\underline{u}\|$ and we have (9.20), it follows that $\gamma^+ = \gamma^-$. □

The following "multiplicative ergodic theorem" of *Osceledec* [267] connects the large N behavior of Φ_N with the behavior of solutions $\Phi_N \underline{u}$.

Theorem 9.12 (Multiplicative Ergodic Theorem, Osceledec). Suppose $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of real 2×2 matrices satisfying (i) $\lim_{n\to\infty} (1/n)\ln\|A_n\| = 0$ and (ii) $\det A_n = 1$. If $\gamma := \lim_{n\to\infty}(1/n)\ln\|A_n \cdot \ldots \cdot A_1\| > 0$, then there exists a one-dimensional subspace $V^- \subset \mathbb{R}^2$ such that

$$\lim \frac{1}{n} \ln \|A_n \cdot \ldots \cdot A_1 v\| = -\gamma \quad \text{for } v \in V_-, v \neq 0$$

and

$$\lim \frac{1}{n} \ln \|A_n \cdot \ldots \cdot A_1 v\| = \gamma \quad \text{for } v \notin V_- \ .$$

Osceledec proved a probabilistic version of this theorem (see Osceledec [267], *Raghunathan* [291]). Ruelle realized it was a deterministic theorem; for a proof, see *Ruelle* [301].

Osceledec's theorem tells us that under the hypothesis $\gamma(E) > 0$, there exists P-a.s. only "exponentially growing" and "exponentially decaying" solution (to the right) of the equation $H_\omega u = Eu$. The exponentially decaying solution occurs only for a particular initial condition $\lambda\underline{u}_+$; any other initial condition leads to an increasing solution. The same is true for solutions to the left with a particular initial condition $\lambda\underline{u}_-$. An (l^2-) eigenvector can only occur if the lines $\lambda\underline{u}_+$, $\lambda\underline{u}_-$ happen to coincide.

However, we have to be very careful with assertions as above, since these considerations are justified only for fixed E. If we allow E to vary through an uncountable set, it may happen that the exceptional ω for which $\gamma^\pm(\omega, E) \neq \gamma(E) > 0$ may add up to a set of measure 1!

For example, we *cannot* conclude that for P-a.a. ω any solution of $H_\omega u = Eu$ (with arbitrary $E \in \mathbb{R}$) is either exponentially increasing or decreasing! However, the Lyaponov exponent $\gamma(E)$ characterizes the absolutely continuous spectrum completely.

Suppose μ is a measure on $\mathbb{R}$, $\mu_{\text{a.c.}}$ its absolutely continuous part. We call a set A an *essential support* of $\mu_{\text{a.c.}}$ if: (1) There is a set, B, of Lebesgue measure zero such that $\mu(\mathbb{R}\backslash(A \cup B)) = 0$. (2) If $\mu(C) = 0$, then the Lebesgue measure $|A \cap C|$ is zero. The essential support is defined uniquely up to sets of Lebesgue measure zero. We define the *essential closure* $\bar{A}^{\text{ess}} := \{\lambda \mid |A \cap (\lambda - \varepsilon, \lambda + \varepsilon)| > 0 \text{ for all } \varepsilon > 0\}$.

Theorem 9.13 (Ishii-Pastur-Kotani). Suppose that V_ω is a bounded ergodic process. Then

$$\sigma_{\mathrm{a.c.}}(H_\omega) = \overline{\{E \mid \gamma(E) = 0\}}^{\mathrm{ess}} \ . \tag{9.21}$$

Moreover, the set $\{E \mid \gamma(E) = 0\}$ is the essential support of $E_\Delta^{\mathrm{a.c.}}(H_\omega)$.

Ishii [176] and *Pastur* [271] proved that $\sigma_{\mathrm{a.c.}}(H_\omega) \subset \overline{\{E \mid \gamma(E) = 0\}}^{\mathrm{ess}}$. Kotani [215] proved the converse for the continuous case. His method was adopted to the discrete case by *Simon* [336]. See *Minami* [247] for further information. The Kotani part of the proof requires the use of theorems on H^2-functions on the unit disc. We will only give the Ishii-Pastur part.

Proof. We only prove $\sigma_{\mathrm{a.c.}}(H_\omega) \subset \overline{\{E \mid \gamma(E) = 0\}}^{\mathrm{ess}}$. Suppose $\gamma(E) > 0$ for Lebesgue-almost all E in the open interval (a, b). Thus, $A_E = \{\omega \mid \gamma(\omega, E) = 0\}$ has P-measure zero for almost any $E \in (a, b)$.

Set $A := \{(\omega, E) \mid \gamma^+(\omega, E) \neq \gamma^-(\omega, E)$ or limit does not exist or $\gamma(\omega, E) = 0$; $E \in (a, b)\}$. Denote the Lebesgue measure by λ. Then

$$0 = \int_a^b P(A_E)\,dE = (\lambda \times P)(A) = (P \times \lambda)(A)$$

by Fubini's theorem. Therefore, for P-a.e. ω

$$\lambda(A_\omega) = \lambda(\{E \mid \gamma^+(\omega, E) \neq \gamma^-(\omega, E) \quad \text{or limit does not exist or}$$

$$\gamma(\omega, E) = 0;\ E \in (a, b)\}) = 0 \ ,$$

i.e. for P-a.e. ω, $\gamma(\omega, E) > 0$ for all E outside a set of Lebesgue measure zero.

We know (see Chap. 2) that
$S_\omega := \{E \mid H_\omega u = Eu$ has a polynomially bounded solution$\}$ satisfies

$$E_{\mathbb{R} \setminus S_\omega}(H_\omega) = 0 \ .$$

Moreover, since $\lambda(A_\omega) = 0$, it follows that

$$E_{A_\omega}^{\mathrm{a.c.}}(H_\omega) = 0 \ .$$

But for $E \notin A_\omega$ the only polynomially bounded solutions are exponentially decreasing because of Theorem 9.12; hence, they are l^2-eigenvectors. There are only countably many of them, hence $\lambda(S_\omega \cap ((a, b) \setminus A_\omega)) = 0$. Thus,

$$\begin{aligned} E_{(a,b)}^{\mathrm{a.c.}}(H_\omega) &= E_{(a,b) \cap S_\omega}^{\mathrm{a.c.}}(H_\omega) = E_{(a,b) \cap A_\omega}^{\mathrm{a.c.}}(H_\omega) + E_{(a,b) \cap S_\omega \setminus A_\omega}^{\mathrm{a.c.}}(H_\omega) \\ &= 0 \ . \quad \square \end{aligned}$$

Knowing whether $\gamma(E)$ is strictly positive or zero, we can answer the question for the measure theoretic nature of the spectrum of H_ω partially. However, in general, $\gamma(E)$ cannot distinguish between point spectrum and singularly continuous spectrum, as we shall see in Chap. 10.

Of course, to use Theorem 9.13 in concrete cases, we need a criterion to decide whether $\gamma(E) = 0$ or $\gamma(E) > 0$. The first such criterion was given by *Furstenberg* [125] for i.i.d. matrices $A_n(\omega, E)$. *Kotani* [215] proved that in a very general case, $\gamma(E) > 0$. (See *Simon* [336] for the discrete case.)

An ergodic potential, $V_\omega(n)$ is called *deterministic* if $V_\omega(0)$ is a measurable function of the random variables $\{V_\omega(n)\}_{n \leq -L}$ for all L. It is called *nondeterministic* if it is not deterministic.

Thus, an ergodic process $V_\omega(n)$ is deterministic if the knowledge of $V_\omega(n)$ arbitrary far to the left allows us to compute $V_\omega(0)$, and hence the whole process $V_\omega(n)$.

Theorem 9.14 (Kotani [215]). If $V_\omega(n)$ is nondeterministic, then $\gamma(E) > 0$ for Lebesgue-almost all $E \in \mathbb{R}$. Thus, $\sigma_{\text{a.c.}}(H_\omega) = \varnothing$.

For the proof, see *Kotani* [215] and *Simon* [336].

Example 1. If the $V_\omega(u)$ are i.i.d., then $\sigma_{\text{a.c.}}(H_\omega) = \varnothing$.

One might believe that Theorem 9.14 covers all interesting cases of random potentials. However, as was pointed out in *Kirsch* [201] and *Kirsch, Kotani* and *Simon* [203], there are interesting examples of stochastic potentials that are really random in an intuitive sense, but deterministic in the above precise sense. In [203], it is shown that $V_\omega(x) = \sum q_i(\omega) f(x - i)$—our introductory example—is "typically" deterministic even for i.i.d. $\{q_i\}$ if f has noncompact support. Here is a discrete example.

Example 2. Let φ be a bijection from $\mathbb{Z}$ to $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$. Set $f(n) = 3^{-\varphi(n)}$. Let $q_i(\omega)$ be i.i.d. random variables with $P(q_i(\omega) = 0) = p$; $P(q_i(\omega) = 1) = 1 - p$. Then for fixed $\lambda > 0$: $V_\omega(n) = \lambda \sum_m q_m(\omega) f(n - m)$ is an ergodic potential, which is random in an intuitive sense. However, q_m is essentially the decimal expansion of $\lambda^{-1} V_\omega(n)$ to the base 3, so the process V_ω is clearly deterministic. Especially for this example, the following theorem of *Kotani* [216] becomes useful.

To state Kotani's theorem, we regard our probability measures as measures on

$$\Omega = \mathop{\times}_{-\infty}^{\infty} [a, b] \quad \text{for some } a, b < \infty \ .$$

We can view Ω as a compact space under the topology of pointwise convergence. supp P can then be defined in the usual way.

Theorem 9.15 (Kotani). Suppose $V_\omega^{(1)}(n)$ and $V_\omega^{(2)}(n)$ are two bounded ergodic processes with corresponding probability measures P_1, P_2, corresponding spectra Σ_1, Σ_2 and absolutely continuous spectra $\Sigma_1^{\text{a.c.}}$ and $\Sigma_2^{\text{a.c.}}$. If supp $P_1 \subset$ supp P_2, then

(i) $\Sigma_1 \subset \Sigma_2$ and

(ii) $\Sigma_2^{\text{a.c.}} \subset \Sigma_1^{\text{a.c.}}$.

Part (i) follows essentially from *Kirsch* and *Martinelli* [204]. The more interesting part (ii) uses heavily *Kotani*'s earlier paper [215], usng again H^2-function theory.

Kotani proves this theorem in the continuous case. Using *Simon* [336], it can be carried over to the discrete case without difficulties.

Kirsch, *Kotani* and *Simon* [203] use Theorem 9.15 to prove the absence of absolutely continuous spectrum for a large class of random, but deterministic potentials.

Example 2 (continued). Taking $q_n \equiv 0$ and $q_n \equiv 1$, we see that $W_0 \equiv 0$ and $W_1 \equiv 3\lambda/2$ are periodic potentials in supp P. Hence, the point measure P_0 and P_1 on W_0 and W_1 respectively are ergodic measures with supp $P_i \subset$ supp P, $i = 0, 1$. By Theorem 9.15, we have $\sigma_{\text{a.c.}}(H_0 + W_0) = \sigma_{\text{a.c.}}(H_0) = [-2, 2] \supset \sigma_{\text{a.c.}}(H_\omega)$ a.s. and $\sigma_{\text{a.c.}}(H_0 + W_1) = [-2 + 3\lambda/2, 2 + 3\lambda/2] \supset \sigma_{\text{a.c.}}(H_\omega)$ a.s. Thus, $\sigma_{\text{a.c.}}(H_\omega) = \emptyset$ a.s. if $\lambda \geq 8/3$. □

Deift and *Simon* [79] investigate those energies with $\gamma(E) = 0$. Interesting examples for $\gamma(E) = 0$ on a set of positive Lebesgue measure occur in the context of almost periodic potentials (see Chap. 10). Among other results, Deift and Simon [79] show:

Theorem 9.16 (Deift-Simon). For a.e. pair $(\omega, E_0) \in \Omega \times \{E | \gamma(E) = 0\}$ there are linearly independent solutions $u_\pm$ of $H_\omega u = Eu$ such that

(i) $u_+ = \bar{u}_-$

(ii) $0 < \overline{\lim_{N\to\infty}} \left(\dfrac{1}{2N+1}\right) \sum_{n=-N}^{N} |u_\pm(n)|^2 < \infty$.

Moreover, $|u_\pm(n, \omega)| = |u_\pm(0, T^n\omega)|$.

For a proof, see [79].

9.4 Subharmonicity of the Lyaponov Exponent and the Thouless Formula

In this section, we establish an important connection of the Lyaponov exponent and the density of states: the Thouless formula. For the proof of this formula, as well as for other purposes, a certain regularity of the Lyaponov exponent—namely its subharmonicity—is useful.

Before proving this property of $\gamma(E)$, we recall some definitions and basic facts concerning subharmonic functions.

A function f on $\mathbb{C}$ with values in $\mathbb{R} \cup \{-\infty\}$ is called *submean* if

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \tag{9.22}$$

for $r > 0$ arbitrary. A function f is called *uppersemicontinuous* if, for any sequence $z_n \to z_0$, we have $\overline{\lim} f(z_n) \leq f(z_0)$. A function f is called *subharmonic* if it is both submean and uppersemicontinuous. It is an immediate consequence of the definitions that

$$f(z_0) \leq \underline{\lim}_{r\to 0} \frac{1}{\pi r^2} \int\limits_{|z-z_0|\leq r} f(z)\,dz \ , \tag{9.23a}$$

if f is submean, and that

$$f(z_0) = \lim_{r\to 0} \frac{1}{\pi r^2} \int\limits_{|z-z_0|\leq r} f(z)\,dz \ , \tag{9.23b}$$

if f is subharmonic.

Proposition 9.17. (i) If f_n are submean functions with

$$\sup_{|z|<R} |f_n(z)| < \infty \text{ for any } R, \quad \text{and}$$

$$f_0(z) = \overline{\lim_{n\to\infty}} f_n(z) \ ,$$

then f_0 is submean.

(ii) If $\{f_n\}$ is a decreasing sequence of subharmonic functions, then

$$f_0(z) = \inf_n f_n(z)$$

is subharmonic.

Proof. (i) For any n and any $N \leq n$

$$\begin{aligned} f_n(z_0) &\leq \frac{1}{2\pi} \int_0^{2\pi} f_n(z_0 + \mathrm{r}e^{i\theta})\,d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \sup_{j\geq N} f_n(z_0 + \mathrm{r}e^{i\theta})\,d\theta \ . \end{aligned}$$

Thus,

$$\begin{aligned} f_0(z_0) = \inf_N \sup_{j\geq N} f_j(z_0) &\leq \frac{1}{2\pi} \inf_N \int_0^{2\pi} \sup_{j\geq N} f_j(z_0 + \mathrm{r}e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f_0(z_0 + \mathrm{r}e^{i\theta})\,d\theta \end{aligned}$$

by the monotone convergence theorem.

(ii) follows from (i) since the inf of uppersemicontinuous functions is uppersemicontinuous. □

After these preliminaries, we come to the basic result of this paragraph, which is due to *Craig* and *Simon* [70]. Craig and Simon were motivated by *Herman* [162] who extensively used subharmonicity of γ in various auxiliary parameters.

Theorem 9.18 (Craig and Simon). (i) $\bar{\gamma}^{\pm}(\omega, E)$ is submean.
(ii) $\gamma(E)$ is subharmonic.

Proof. (i) The matrix-valued function $\Phi_N(\omega, E)$ is obviously analytic in E for any N. Thus, $\ln\|\Phi_N(\omega, E)\|$ is subharmonic [see e.g. *Katznelson* [199], Chapter III, Equation (3.2)]. Thus, by Proposition 9.17(i),

$$\bar{\gamma}^{\pm}(\omega, E) = \overline{\lim_{N\to\pm\infty}} \frac{1}{N} \ln\|\Phi_N(\omega, E)\|$$

is submean.
(ii) By the subadditive ergodic theorem (Theorem 9.11)

$$\gamma(E) = \inf \frac{1}{N} \mathbb{E}(\ln\|\Phi_N(\omega, E)\|) \ .$$

By Fubini's theorem and Fatou's lemma, $\mathbb{E}(\ln\|\Phi_N(\omega, E)\|)$ is subharmonic. By 9.17(ii), $\gamma(E)$ is subharmonic. □

We come to a first application of Theorem 9.18:

Theorem 9.19 (Craig and Simon [70]). For P-almost all ω and all E

$$\bar{\gamma}^{\pm}(\omega, E) \le \gamma(E) \ .$$

Proof. From the very definition of $\bar{\gamma}^{\pm}(\omega, E)$ and $\gamma(E)$, it is obvious that for fixed E: $\bar{\gamma}^{\pm}(\omega, E) = \gamma(E)$ P-a.s. By use of Fubini's theorem, we conclude from this that $\bar{\gamma}(\omega, E) = \gamma(E)$ for P-almost all ω and Lebesgue-almost all E. Thus,

$$\int_{|E-E_0|\le r} \bar{\gamma}^{\pm}(E, \omega)\, d^2E = \int_{|E-E_0|\le r} \gamma(E)\, d^2E \quad P\text{-a.s.}$$

(d^2E indicates that we integrate over a complex domain). Thus, using Theorem 9.18 by (9.23), we know (P-a.s.)

$$\begin{aligned}\bar{\gamma}^{\pm}(E_0, \omega) &\le \lim_{r\to 0} \frac{1}{\pi r^2} \int_{|E-E_0|\le r} \bar{\gamma}^{\pm}(E, \omega)\, d^2E \\ &= \lim_{r\to 0} \frac{1}{\pi r^2} \int_{|E-E_0|\le r} \gamma(E)\, d^2E = \gamma(E_0) \ . \quad \square\end{aligned}$$

Finally, we discuss an important connection between the Lyaponov exponent γ and the density of states k: The Thouless formula. It is named after Thouless, who gave a not completely rigorous proof of it [355]. The Thouless formula was discovered independently by *Herbert* and *Jones* [153]. Thouless' proof was made rigorous by *Avron* and *Simon* [31]. We follow *Craig* and *Simon* [70], who simplified the proof of [31] by using the subharmonicity of γ. In the continuous case, *Johnson* and *Moser* [186] have an alternative proof.

Theorem 9.20 (Thouless Formula).

$$\gamma(E) = \int \ln|E - E'|\, dk(E') \ . \tag{9.24}$$

Proof. We first prove (9.24) for $E \in \mathbb{C} \setminus \mathbb{R}$. For those E, the function $f(E') := \ln|E - E'|$ is continuous on $\mathrm{supp}(dk) \subset \mathbb{R}$. By the definition of $\Phi_L(E)$ [see (9.16)], it is easy to see that $\Phi_L(E)$ is of the form

$$\Phi_L(\omega, E) = \begin{pmatrix} P_L(\omega, E) & Q_{L-1}(\omega, E) \\ P_{L-1}(\omega, E) & Q_{L-2}(\omega, E) \end{pmatrix} ,$$

where P_l and Q_l are polynomials in E of degree l with leading coefficient 1. Moreover,

$$P_l(\omega, E) = 0 \ ,$$

if and only if $H_\omega u = Eu$ has a solution u satisfying $u(0) = 0$, $u(l + 1) = 0$ and $Q_l(\omega, E) = 0$, if and only if there exists a solution with $u(1) = 0$ and $u(l + 2) = 0$. Thus,

$$P_l(E) = \prod_{j=1}^{l} (E - E_j^{(l)}), \quad Q_l(E) = \prod_{j=1}^{l} (E - \tilde{E}_j^{(l)}) \ ,$$

where $E_j^{(l)}$ (resp. $\tilde{E}_j^{(l)}$) are the eigenvalues of H_ω restricted to $\{1, \dots, l\}$ (resp. $\{2, \dots, l + 1\}$) with boundary condition $u(0) = u(l + 1) = 0$ [resp. $u(1) = u(l + 2) = 0$]. Thus, we conclude that [see (9.14)]

$$\ln|P_L(E)| = \int \ln|E - E'|\, d\rho_{1,L}(E') \quad \text{and}$$
$$\ln|Q_L(E)| = \int \ln|E - E'|\, d\rho_{2,L+1}(E') \ .$$

By (9.14), we conclude that

$$\frac{1}{L} \ln|P_L(E)| \to \int \ln|E - E'|\, dk(E')$$

and the same for $Q_L(E)$ if $E \in \mathbb{C} \setminus \mathbb{R}$. Thus, (9.24) follows for those values of E.

Now let E be arbitrary. We need

Lemma. The function $f(E) = \int \ln|E - E'|\, dk(E')$ (with $f(E) = -\infty$ if the integral diverges to $-\infty$) is subharmonic.

Before we prove this lemma, we continue the proof of Theorem 9.20: Since we know (9.24) for $E \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\frac{1}{\pi r^2} \int_{|E-\tilde{E}| \leq r} \gamma(\tilde{E}) d^2 \tilde{E} = \frac{1}{\pi r^2} \int_{|E-\tilde{E}| \leq r} f(\tilde{E}) d^2 \tilde{E}$$

[$f(E)$ as in the lemma].

Taking the limit $r \to 0$ on both sides of the above equation, we obtain

$$\gamma(E) = f(E) = \int \ln |E - E'| \, dk(E') \ ,$$

since both γ and f are subharmonic. □

Proof of the Lemma. The function $\varphi(E) = \ln |E - E'|$ is subharmonic [see *Katznelson* [199] III, Equation (3.2)]. Thus, f is submean by Fubini. Define for $M > 0$

$$f_M(E) = \int \max\{\ln |E - E'|, -M\} \, dk(E') \ .$$

Here f_M is obviously continuous. By the monotone convergence theorem,

$$f(E) = \inf_{M>0} f_M(E) \ .$$

Thus, f is uppersemicontinuous. □

Craig and *Simon* [70] use the Thouless formula to prove that $k(E)$ is log-Hölder continuous in the one-dimensional case. In [69], these authors prove a version of the Thouless formula for strips in arbitrary dimension. From this result, they obtain the log-Hölder continuity of k in the multidimensional case.

9.5 Point Spectrum for the Anderson Model

In this section, we show that the Anderson model has pure point spectrum.

We first prove a criterion for point spectrum of H_ω that allows us to reduce the problem to uniform estimates for $H_\omega^{(N)} := H_\omega^{(0,N)}$ [see (9.13)].

We set

$$a(n,m) := \mathbb{E}\left(\sup_t |e^{-itH_\omega}(n,m)| \right) \ .$$

[If A is a bounded operator on $l^2(\mathbb{Z}^\nu)$, we denote by $A(n,m) = \langle \delta_n, A\delta_m \rangle$ with $\delta_n(i) = 0$ for $i \neq n$, $\delta_n(n) = 1$ the matrix elements of A.]

We say that *physical localization* holds if $\sum_{n \in \mathbb{Z}} |a(n,m)| < \infty$ for $m = 0$ and $m = 1$.

Theorem 9.21 (Kunz-Souillard [221]). Physical localization implies mathematical localization (i.e. $\sigma_c(H_\omega) = \phi$).

Proof. By the RAGE theorem (Sect. 5.4), in particular, formula (9.10) in Sect. 9.1, we conclude:

$$\|P^c(H_\omega)\varphi\|^2 = \lim_{J\to\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{|j|\geq J} |\langle e^{-itH_\omega}\varphi, \delta_j\rangle|^2 \, dt \ .$$

Thus, if $\sum_{n\in Z} |a(n,0)|^2 < \infty$, we have (since $|\exp(-itH_\omega)(m,n)| \leq 1$)

$$\mathbb{E}(\|P^c(H_\omega)\delta_0\|^2) \leq \lim_{J\to\infty} \sum_{|j|\geq J} |a(j,0)| = 0 \ .$$

Hence δ_0 is P-a.s. orthogonal to the continuous subspace. A similar argument shows that $P^c(H_\omega)\delta_1 = 0$ almost surely. It is easy to see that any δ_j can be written as

$$\delta_j = \sum_{n=0}^{N} \alpha_n(\omega) H_\omega^n \delta_0 + \sum_{n=0}^{N} \beta_n(\omega) H_\omega^n \delta_1$$

(e.g. $\delta_2 = H_\omega\delta_1 - V_\omega\delta_1 - \delta_0$). Hence, $P^c(H_\omega)\delta_j = 0$ a.s. for any j. Thus, $\sigma_c(H_\omega) = \phi$ a.s. □

Remark. In terms of a direct physical interpretation, it would be better to define a with a square inside $\mathbb{E}$; the statement and proof of Theorem 9.21 still go through. Since it is easier to estimate a as we define it, we have used that definition.

The next result shows that $a(n,m)$ even determines the decay of the eigenfunctions:

Theorem 9.22. If, for $m = 0$ and $m = 1$,

$$|a(n,m)| \leq C e^{-D|n|} \ , \tag{9.25}$$

then P-a.s. any eigenfunction φ_ω of H_ω satisfies

$$|\varphi_\omega(n)| \leq C_{\omega,\varepsilon} e^{-(D-\varepsilon)|n|}$$

for any $\varepsilon > 0$.

Remark. The constant $C_{\omega,\varepsilon}$ may depend on the eigenvalue. We say that the eigenfunction φ_ω is *exponentially localized.*

Proof. Set

$$\beta(\omega,n,m) := \sup_t |e^{-itH_\omega}(n,m)| \ .$$

We first prove that (9.25) implies

$$\beta(\omega,n,m) \leq \tilde{C}_{\omega,\varepsilon} e^{-(D-\varepsilon)|n|} \tag{9.26}$$

for $m = 0$ and $m = 1$ P-a.s.

Equation (9.26) holds if we show that

$$P(\beta(\omega,n,m) > e^{-(D-\varepsilon)|n|} \text{ for infinitely many } n) = 0 \ .$$

This, in turn, follows by the Borel-Cantelli Lemma (see any book on probability theory, e.g. *Breiman* [53]) if we show that

$$\sum_n P(\beta(\omega,n,m) > e^{-(D-\varepsilon)|n|}) < \infty \tag{9.27}$$

for $m = 0$ and $m = 1$.

Since, by Tschebycheff's inequality

$$\begin{aligned} P(\beta(\omega,n,m) > e^{-(D-\varepsilon)|n|}) &\leq e^{+(D-\varepsilon)|n|}\mathbb{E}(\beta(\omega,n,m)) \\ &= e^{-\varepsilon|n|}[e^{D|n|}a(n,m)] \leq Ce^{-\varepsilon|n|} \ , \end{aligned}$$

(9.27) follows. Thus, we have proven (9.26).

Now we use the formula

$$P_{\{E\}}(H) = s - \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{isE}e^{-isH}\,ds \ . \tag{9.28}$$

This follows from continuity of the functional calculus and the fact that functions

$$f_T(x) = \frac{1}{T}\int_0^T e^{isE}e^{-isx}\,ds$$

obey $|f_T(x)| \leq 1$ and $f_T(x) \to 0$ (resp. 1) as $T \to \infty$ for $x \neq E$ (resp. $x = E$).

Suppose now that E is an eigenvalue of H_ω. Since $\nu = 1$, any eigenvalue is simple. Denote by $\varphi_{\omega,E}$ the normalized eigenfunction corresponding to E. Then (9.28) implies

$$\begin{aligned} |\varphi_{\omega,E}(0)||\varphi_{\omega,E}(n)| &= |\langle\delta_0,\varphi_{\omega,E}\rangle\langle\varphi_{\omega,E},\delta_n\rangle| \\ &= |\langle\delta_0,P_{\{E\}}(H_\omega)\delta_n\rangle| \leq \beta(\omega,n,0) \leq \tilde{C}_{\omega,\varepsilon}e^{-(D-\varepsilon)|n|} \end{aligned}$$

by (9.26). This proves the theorem if $\varphi_{\omega,E}(0) \neq 0$. If $\varphi_{\omega,E}(0) = 0$, we have $\varphi_{\omega,E}(1) \neq 0$, and obtain the above estimate for $m = 1$. $\square$

Now we consider H_ω restricted to $l^2(-L,\dots,L)$. As usual, we denote the corresponding operator by $H_\omega^{(L)}$ [see (9.13)]. We define

$$a_L(n,m) := E\left(\sup_t |\exp[-itH_\omega^{(L)}](n,m)|\right) \ .$$

Proposition 9.23: $a(n,m) \leq \overline{\lim}_{L\to\infty}\, a_L(n,m)$.

Remark. By Proposition 9.23 and Theorems 9.21 and 9.22, we can conclude that

H_ω has pure point spectrum with exponentially localized eigenfunctions if we have an estimate

$$a_L(n,m) \le Ce^{-D|n|} \quad \text{for } m = 0 \text{ and } m = 1$$

uniformly in L.

Proof. $H_\omega^{(L)}$ converges strongly to H_ω (with the understanding that $H_\omega^{(L)}(n,m) = 0$ for $|n| > L$ or $|m| > L$). Hence,

$$\exp[-itH_\omega^{(L)}](n,m) \to \exp(-itH_\omega)(n,m)$$

(cf. *Reed* and *Simon* I, VIII.20 [292]), and thus by Fatou's lemma

$$\mathbb{E}\left(\sup_t |\exp(-itH_\omega)(n,m)|\right) \le \varliminf \mathbb{E}\left(\sup_t |\exp[-itH_\omega^{(L)}](n,m)|\right) . \qquad \square$$

We denote by $\{E_\omega^{L,k}\}$ the eigenvalues of $H_\omega^{(L)}$ in increasing order. $\varphi_\omega^{L,k}$ denotes "the" normalized eigenfunction corresponding to $E_\omega^{L,k}$.

Finally, we define for any (Borel) set $A \subset \mathbb{R}$:

$$\rho_L(n,m,A) := \mathbb{E}\left(\sum_k |\varphi_\omega^{L,k}(n)|\,|\varphi_\omega^{L,k}(m)|\,\chi_A(E_\omega^{L,k})\right) . \tag{9.29}$$

Note that the sum over k goes only over $2L + 1$ terms since H_ω^L is a $(2L + 1) \times (2L + 1)$-matrix.

It is immediately clear that

$$a_L(n,m) \le \rho_L(n,m,\mathbb{R}) , \quad \text{since}$$

$$\exp(-itH_\omega^L)(n,m) = \sum_k \exp(-itE_\omega^{L,k})\overline{\varphi_\omega^{L,k}(n)}\varphi_\omega^{L,k}(m) .$$

Note that $\rho_L(n,m,\mathbb{R}) = \rho_L(n,m,[-M,M])$ for M large enough since the operators H_ω are uniformly bounded.

Theorem 9.24 (Kunz-Souillard). Suppose the $V_\omega(n)$ are independent random variables with a common distribution $r(x)dx$. If $r \in L^\infty$ and r has compact support, then

$$H_\omega u(n) = u(n + 1) + u(n - 1) + V_\omega(n)u(n)$$

has pure point spectrum (P-a.s.). The eigenfunctions of H_ω are P-a.s. exponentially localized.

Remark. The above theorem was conjectured by theoretical physicists since the early sixties. A continuous analog, where the potential $V_\omega(x)$ is a rather complicated diffusion process, was proven by *Goldsheid*, *Molchanov* and *Pastur* [138], and *Molchanov* [248] (see also *Carmona* [59]). The above theorem is due to *Kunz* and *Souillard* [221]. *Delyon Kunz* and *Souillard* [81] simplified this proof and extended the theorem to other types of disorder. We mainly follow their proof. Their proof has some elements in common with an earlier approach of *Wegner* [365].

Proof. We will give a uniform (in L) estimate on $\rho_L(n,m) := \rho_L(n,m,\mathbb{R})$ of the form

$$\rho_L(n,m) \le Ce^{-D|n|} \quad m = 0, 1 \ . \tag{9.30}$$

This implies the theorem by Proposition 9.23 and Theorems 9.21 and 9.22.

The proof of (9.30) is broken into three steps. First, we rewrite $\rho_L(n,m)$ as a multiple product of integral operators T_0 and T_1. This will be done by changing variables from $V(-L), \ldots, V(L)$ to $\lambda, x_{-L}, \ldots, x_{-1}, x_1, \ldots, x_L$ where λ is the eigenvalue and the x_i are simple expressions in terms of the eigenfunctions. Note that the expectation $\mathbb{E}$ in (9.29) is nothing but an integral in the variables $V(k)$: $|k| \le L$.

In the second step (Proposition 9.25), we explore some properties of the operators T_0, T_1. This investigation allows us to estimate ρ_L in the last step.

We start with

$$\begin{aligned}\rho_L(n,m) &= \rho_L(n,m,\mathbb{R})\\ &= \int \sum_{k=0}^{2L+1} |\varphi_{\underline{V}}^{L,k}(n)|\,|\varphi_{\underline{V}}^{L,k}(m)| \prod_{n=-L}^{L} r(V(n))\, d^{2L+1}\underline{V} \ ,\end{aligned}$$

where $\underline{V} = (V(-L), \ldots, V(L))$ and the $\varphi_{\underline{V}}^{L,k}$ denote the eigenfunctions for the potential $\underline{V}$. For definiteness, we now assume $m = 0$, $n > 0$. The other cases are similar. After interchanging sum and integrals, we change variables from $\{V(n)\}_{n=-L}^{L}$ to $\{x_{-L}, \ldots, x_{-1}, \lambda, x_1, \ldots, x_L\}$ where

$$\lambda := E_{\underline{V}}^{L,k} \ , \tag{9.31a}$$

$$x_m := \frac{\varphi_{\underline{V}}^{L,k}(m-1)}{\varphi_{\underline{V}}^{L,k}(m)} \quad \text{for } m > 0 \text{ , and} \tag{9.31b}$$

$$x_m := \frac{\varphi_{\underline{V}}^{L,k}(m+1)}{\varphi_{\underline{V}}^{L,k}(m)} \quad \text{for } m < 0 \ . \tag{9.31c}$$

The Schrödinger equation $u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$ then yields

$$V(m) = \begin{cases} \lambda - x_{m+1}^{-1} - x_m, & m > 0 \\ \lambda - x_1^{-1} - x_{-1}^{-1}, & m = 0 \\ \lambda - x_{m-1}^{-1} - x_m, & m < 0 \end{cases} \tag{9.32}$$

with the understanding that $x_{L+1}^{-1} = x_{-L-1}^{-1} = 0$. Equation (9.32) allows us to compute the Jacobian J with respect to the change of variables $\underline{V} \mapsto (\underline{x}, \lambda)$. It is straightforward to see that

$$\begin{aligned}\det J &= 1 + x_1^{-2}\{1 + x_2^{-2}[1 + \ldots x_{L-1}^{-2}(1 + x_L^{-2})\ldots]\} \\ &\quad + x_{-1}^{-2}\{1 + x_{-2}^{-2}[1 + \ldots x_{-L+1}^{-2}(1 + x_{-L}^{-2})\ldots]\} \\ &= \varphi_{\underline{V}}^{L,k}(0)^{-2} \ ,\end{aligned}$$

where we used that $\varphi_{\underline{V}}^{L,k}$ is normalized. Moreover,

$$|\varphi_{\underline{V}}^{L,k}(0)|^{-1}\,|\varphi_{\underline{V}}^{L,k}(n)| = |x_1^{-1}\cdot x_2^{-1}\cdot\ldots\cdot x_n^{-1}| \ .$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^{2L+1}} |\varphi_{\underline{V}}^{L,k}(0)|\,|\varphi_{\underline{V}}^{L,k}(n)| \prod_{n=-L}^{L} r(V(n))\,d^{2L+1}\underline{V} \\ &= \int_{\Sigma_0} d\lambda \int_{\mathbb{R}^{2L}} |x_1^{-1}x_2^{-1}\cdot\ldots\cdot x_n^{-1}| \left[\prod_{m>0} r(\lambda - x_{m+1}^{-1} - x_m)\right]\cdot r(\lambda - x_1^{-1} - x_{-1}^{-1}) \\ &\quad \otimes \left[\prod_{m<0} r(\lambda - x_{m-1}^{-1} - x_m)\right] dx_{-L}\ldots dx_{-1}\,dx_1\ldots dx_L \ ,\end{aligned}$$

where $\Sigma_0 = [-2 - \|V\|_\infty, 2 + \|V\|_\infty]$ and $\|V\|_\infty = \sup\{|\lambda|; \lambda\in\operatorname{supp} r\}$. The possible eigenvalues that occur always lie in this range, so we can restrict the λ integration to this region.

Now we fix λ for awhile and consider

$$\begin{aligned}\rho_L(0,n;\lambda) &= \int \prod_{i=1}^{n} |x_i^{-1}| \left[\prod_{m>0} r(\lambda - x_{m+1}^{-1} - x_m)\right] \\ &\quad \times \left[\prod_{m<0} r(\lambda - x_{m-1}^{-1} - x_m)\right] r(\lambda - x_1^{-1} - x_{-1}^{-1})\,d^{2L}\underline{x} \ .\end{aligned} \tag{9.33}$$

We introduce the integral operators T_0, T_1 by

$$T_0 f(x) := \int r(\lambda - x - y^{-1}) f(y)\,dy \ , \tag{9.34a}$$

$$T_1 f(x) := \int r(\lambda - x - y^{-1})|y|^{-1} f(y)\,dy \ . \tag{9.34b}$$

Set $\varphi(x) := r(\lambda - x)$ and $U_0 f(x) := |x|^{-1} f(x^{-1})$. Note that U_0 is a unitary operator on $L^2(\mathbb{R})$. From these definitions and (9.33), we see that

$$\begin{aligned}\rho_L(0,n,\lambda) &= \int_{\mathbb{R}} T_0^L \varphi(x_1^{-1})|x_1|^{-1}\, T_1^{n-1}\, T_0^{L-n}\varphi(x_1)\,dx_1 \\ &= \langle T_1^{n-1} T_0^{L-n}\varphi, U T_0^L \varphi\rangle \ .\end{aligned} \tag{9.35}$$

Observe that both T_0 and T_1 depend on the parameter λ. We now investigate $T_0 = T_0(\lambda)$ and $T_1 = T_1(\lambda)$.

Proposition 9.25:

(a) $\|T_0 f\|_1 \le \|f\|_1$
(b) $\|T_0 f\|_2 = \|T_0(\lambda) f\|_2 \le C\|f\|_1$ uniformly in λ
(c) $\|T_1 f\|_2 \le \|f\|_2$
(d) T_1^2 is a compact operator
(e) $\|T_1^2 f\|_2 \le q\|f\|_2$ with a $q < 1$ uniformly in λ .

Before demonstrating Proposition 9.25, we show how this proposition yields the exponential estimate of $\rho(0, n, \lambda)$.

Proof of Theorem 9.24 (continued). We note that $\|\varphi\|_1 = 1$. From (9.35) and the unitarity of U, we see

$$\begin{aligned}\rho_L(0, n, \lambda) &\le \|T_0^L \varphi\|_2 \|T_1^{n-1} T_0^{L-n} \varphi\|_2 \\ &\le \|T_0\|_{L^1, L^2} \cdot \|T_0^{L-1}\|_{L^1, L^1} \cdot \|T_1^{n-1}\|_{L^2, L^2} \|T_0\|_{L^1, L^2} \cdot \|T_0^{L-n+1}\|_{L^1, L^1} \\ &\le C^2 \|T_1^{n-1}\|_{L^2, L^2} \le C^2 q^{(n-2)/2} = \tilde{c} \exp\left(-\frac{1}{2} n |\ln q|\right) .\end{aligned}$$

Hence, $\rho_L(0, n, \lambda)$ decays exponentially in n. Moreover, Proposition 9.25 gives uniformity in λ on compact sets. This finishes the proof of Theorem 9.24, since Σ_0 is bounded. □

Proof (*Proposition 9.25*):

(a) $\|T_0 f\|_1 \le \iint r(\lambda - x - y^{-1}) |f(y)| \, dy \, dx = \int r(x) \, dx \int |f(y)| \, dy = \|f\|_1$
(b) $\|T_0 f\|_2^2 \le \iint r(\lambda - x - y^{-1}) |f(y)| \, dy \cdot \int r(\lambda - x - z^{-1}) \cdot |f(z)| \, dz \, dx$
$\le \|r\|_\infty \cdot \|r\|_1 \cdot \|f\|_1^2$
(c) Defining $Kf(x) := \int r(\lambda - x + y) f(y) \, dy = r_\lambda * f(x)$ where $r_\lambda(x) = r(\lambda - x)$ and $\tilde{U} f(x) := |x|^{-1} f(-x^{-1})$, we can write T_1 as $T_1 = K\tilde{U}$, so

$$\|T_1 f\|_2 = \|K\tilde{U} f\|_2 = \|r * \tilde{U} f\|_2 \le \|r\|_1 \cdot \|\tilde{U} f\|_2 = \|r\|_1 \cdot \|f\|_2 .$$

(d) Let F denote the Fourier transformation. We set $\hat{K} := \mathrm{F} K \mathrm{F}^{-1}$ and $\hat{U} := \mathrm{F} \tilde{U} \mathrm{F}^{-1}$. Since $\tilde{U}$—and hence $\hat{U}$—is unitary, we have to show that $\hat{K}\hat{U}\hat{K}$ is compact. Since K is a convolution operator, we have $\hat{K}\hat{f}(p) = \hat{r}_\lambda(p) \hat{f}(p)$ where $\hat{r}_\lambda(p) = \int \exp(-\mathrm{i}xp) r_\lambda(x) \, dx$. Formally we have

$$\hat{U}\hat{f}(k) = \int a(k, p) \hat{f}(p) \, dp \quad \text{with}$$

$$a(k, p) = \frac{1}{2\pi} \int \exp[-\mathrm{i}(kx + px^{-1})] \frac{dx}{|x|} .$$

However, the integral is not absolutely convergent, so $a(k, p)$ requires a careful interpretation: Let g_1 be a C_0^∞-function which is 1 near 0, $g_2 := 1 - g_1$. Set

$U_i\varphi(x) := g_i(x)\tilde{U}\varphi(x)$. Then $\tilde{U}\varphi = U_1\varphi + U_2\varphi$. Define furthermore

$$a_i(k,p) = \frac{1}{2\pi}\int \exp[-i(kx + px^{-1})]\frac{g_i(x)}{|x|}dx \ .$$

For fixed p, $a_2(k,p)$ is the Fourier transform of $f_p := (2\pi)^{-1}\exp(-ipx^{-1})|x|^{-1}g_2(x)$, which is an L^2-function whose L^2-norm is independent of p. Hence,

$$\sup_p \int |a_2(k,p)|^2\,dk < \infty \ . \tag{9.36a}$$

Moreover, for fixed k

$$\int_{|x|>(1/n)} \exp[-i(kx + px^{-1})]\frac{g_1(x)}{|x|}dx$$

$$= \int_{|x|<n} \exp[-i(kx^{-1} + px)]\frac{g_1(x^{-1})}{|x|}dx \ ,$$

which is convergent in the L^2-sense to the Fourier transform of the L^2-function

$$\exp(-ikx^{-1})\frac{g_1(x^{-1})}{|x|} \ .$$

Hence,

$$\sup_k \int |a_1(k,p)|^2\,dp < \infty \ . \tag{9.36b}$$

Define A_i by $(A_i\varphi)(k) = \int a_i(k,p)\varphi(p)\,dp$. Now we show $\hat{U}_i = A_i$. We will handle freely integrals—such as $\int \exp(-ikx)f(x)\,dx$ for $f \in L^2$—that exist only in an L^2-sense. The reader can easily verify those manipulations. We have

$$(U_1\varphi)\hat{}\,(k) = \int \exp(-ikx)\frac{g_1(x)}{|x|}\varphi(-x^{-1})\,dx$$

$$= \int \exp(-ikx^{-1})\frac{g_1(x^{-1})}{|x|}\varphi(-x)\,dx$$

$$= \frac{1}{2\pi}\int \exp(-ikx^{-1})\frac{g_1(x^{-1})}{|x|}\int \exp(-ixp)\hat{\varphi}(p)\,dp\,dx$$

$$= \frac{1}{2\pi}\iint \exp[-i(kx^{-1} + xp)]\frac{g_1(x^{-1})}{|x|}dx\,\hat{\varphi}(p)\,dp$$

$$= \frac{1}{2\pi} \iint \exp[-\mathrm{i}(kx + px^{-1})] \frac{g_1(x)}{|x|} dx\, \hat{\varphi}(p)\, dp$$

$$= \int a_1(k,p)\hat{\varphi}(p)\, dp \ .$$

Thus, $\hat{U}_1 = A_1$. The proof of $\hat{U}_2 = A_2$ is similar (and even simplier). Therefore, $(K\tilde{U}K)\hat{}$ has an integral kernel $b(k,p) = \hat{r}_\lambda(k)a_1(k,p)\hat{r}_\lambda(p) + \hat{r}_\lambda(k)a_2(k,p)\hat{r}_\lambda(p)$ since $r \in L^1 \cap L^\infty$ and hence $r \in L^2$, we have $\hat{r} \in L^2 \cap L^\infty$, so

$$\|b(k,p)\|_{L^2(dk) \times L^2(dp)}$$

$$\leq \|\hat{r}\|_2 \sup_k \|a_1(k,p)\|_{L^2(dp)} \|\hat{r}\|_\infty$$

$$+ \|\hat{r}\|_\infty \sup_p \|a_2(k,p)\|_{L^2(dk)} \|\hat{r}\|_2 \ .$$

Thus, $(K\tilde{U}K)\hat{}$ is Hilbert Schmidt and consequently T_1^2 is compact.

(e) Since $|T_1^2|$ is positive and compact, $\|T_1^2\|$ is an eigenvalue of $|T_1^2|$. Since $|\hat{r}_\lambda(k)| < 1$ for $k \neq 0$, we have $\|T_1 f\| < \|f\|$ for any $f \in L^2$. Therefore, 1 is not an eigenvalue for $|T_1^2|$. So $\|T_1^2\| < 1$. Moreover, since $\hat{r}_\lambda(k) = \exp(-\mathrm{i}\lambda k)\hat{r}_0(k)$, the norm $\|T_1^2\|$ is independent of λ. □

By a refinement of the methods of the above proof, one can prove the following results:

Theorem 9.26 (Delyon, Kunz and Souillard [81]). Suppose that $V_\omega(n)$ satisfies the assumptions of Theorem 9.24. Let $V_0(n)$ be a bounded function on $\mathbb{Z}$. Then

$$H_\omega := H_0 + V_0 + V_\omega$$

has P-a.s. dense point spectrum with exponentially decaying eigenfunctions, and possibly in addition, isolated eigenvalues.

Theorem 9.27 (Simon [333]). Suppose $V_\omega(n)$ satisfies the assumptions of Theorem 9.24. Let a_n be a sequence with $|a_n| \geq C|n|^{-1/2+\delta}$ and set $W_\omega(n) := a_n V_\omega(n)$. Then

$$H_\omega := H_0 + W_\omega(n)$$

has only dense point spectrum.

Remark. (1) Observe that the potentials $V_0 + V_\omega$ and W_ω are not stationary. So, the corresponding H_ω will have a random spectrum in general. The proofs of these theorems can be found in [81] and [333] respectively.

(2) More recently, *Delyon, Simon* and *Souillard* [84] and *Delyon* [80] have studied the operators of Theorem 9.27, but with different a_n. If $|a_n| \leq C|n|^{-1/2-\delta}$, then H_ω has no point spectrum [84], and if $a_n \sim \lambda n^{-1/2}$ with λ small, the operator has some singular continuous spectrum [80]!

In the above proof, the assumption that the distribution P_0 of $V(0)$ has a density $r(\lambda)$ was necessary. For example, if $V(0) = 0$ with probability p, and $V(0) = 1$ with probability $1 - p$, the above proof does not apply but a recent paper of R. Carmona, A. Klein and F. Martinelli shows there is also only point spectrum in this case.

We now survey briefly some further results on random potentials.

Brossard [56] proves pure point spectrum (in the continuous case) for certain potentials of the form $V_\omega(x) = V_0(x) + W_\omega(x)$, $x \in \mathbb{R}^1$; where V_0 is a periodic potential and W_ω is a certain random one.

Carmona [60] considers random, but not stationary, potentials (continuous case). For example, suppose $V_\omega^{(1)}(x)$, $x \in \mathbb{R}^1$ is a random potential such that $-(d^2/dx^2) + V_\omega^{(1)}$ has pure point spectrum, and suppose $V^{(2)}(x)$ is periodic. Consider

$$V_\omega(x) = \begin{cases} V_\omega^{(1)}(x) & \text{for } x \le 0 \\ V^{(2)}(x) & \text{for } x > 0 \ . \end{cases}$$

Carmona [60] proves that

$$\sigma_{\text{a.c.}}\left(-\frac{d^2}{dx^2} + V_\omega\right) = \sigma_{\text{a.c.}}\left(-\frac{d^2}{dx^2} + V^{(2)}\right) = \sigma\left(-\frac{d^2}{dx^2} + V^{(2)}\right) \ ,$$

$$\sigma_{\text{s.c.}}\left(-\frac{d^2}{dx^2} + V_\omega\right) = \phi \quad \text{and}$$

$$\sigma_{\text{p.p.}}\left(-\frac{d^2}{dx^2} + V_\omega\right) = \overline{\sigma_{\text{p.p.}}\left(-\frac{d^2}{dx^2} + V_\omega^{(1)}\right)\Big\backslash \sigma\left(-\frac{d^2}{dx^2} + V^{(2)}\right)} \ .$$

There has been large interest in operators with constant electric field and stochastic potential. Suppose q_n are i.i.d. random variables, f a C^2-function with support in $(-\frac{1}{2}, \frac{1}{2})$ and $f \le 0$, ($f \not\equiv 0$).

$$H_\omega = -\frac{d^2}{dx^2} + Fx + \sum_{n \in Z} q_n(\omega) f(x - n) \ .$$

We have seen, using Mourre-estimates, that the spectrum of H_ω is absolutely continuous if $F \ne 0$ (see Chap. 4 and [45]). *Bentosela* et al. prove that for $F = 0$, the operator H_ω has a.s. pure point spectrum (with exponentially decaying eigenfunctions) provided the distribution P_0 of q_0 has continuous density with compact support.

If f is the δ-function, then H_ω has a.s. pure point spectrum even for $F \ne 0$, but $|F|$ small. However, in this case the eigenfunctions are only polynomially localized (but they are exponentially localized for $F = 0$). For $|F|$ large, the spectrum of H_ω is continuous (*Delyon, Simon* and *Souillard* [84]).

For the case $\nu > 1$, much less is known than for $\nu = 1$. The physicists' belief is that the nature of the spectrum depends on the magnitude of disorder. For small disorder, one expects that the spectrum is pure point at the boundaries of the spectrum, while it should be continuous (absolutely continuous?) away from the boundary, at least if $\nu \geq 3$. Those values where the nature of the spectrum changes are called *mobility edges*. If the disorder is increased, the continuous spectrum is supposed to shrink in favor of the pure point one. Finally, at a certain degree of disorder, the spectrum should become a pure point one.

Recently, *Fröhlich* and *Spencer* [119] proved that for the multidimensional Anderson model (with absolutely continuous distribution p_0), the kernel $G(E + i\varepsilon; 0, n)$ of the resolvent $[H_\omega - (E + i\varepsilon)]^{-1}$ decays P-a.s. exponentially in n uniformly as $\varepsilon \to 0$, provided that either $E \to \pm\infty$ (this corresponds to the boundary of the spectrum) or the disorder is large enough.

Martinelli and *Scoppola* [239] observed that the estimates of *Fröhlich* and *Spencer* [119] actually suffice to prove the absence of absolutely continuous spectrum for $|E|$ large or for large disorder. Corresponding results for a continuous model are contained in *Martinelli* and *Holden* [167]. *Fröhlich, Martinelli, Scoppola* and *Spencer* [118] have proven that in the same regime, H_ω has only pure point spectrum, and *Goldsheid* [137] has announced a similar result.

New insight on localized has come from work of *Kotani* [217, 218], *Delyon, Levy* and *Souillard* [82, 83], *Simon* and *Wolf* [345] and *Simon* [343] which has its roots in the work of *Carmona* [60]. The key remark is that the spectral measure averaged over variations of the potential in a bounded region is absolutely continuous with respect to Lebesgue measure, so the sets of measure zero where the Osceledec theorem fails are with probability 1 irrelevant. In any event, the reader should be aware that the state of our understanding of localization was changing rapidly as this book was being completed.

Kunz and Souillard [222] have studied the case of random potentials on the Bethe lattice.

For additional information, see [347].

10. Almost Periodic Jacobi Matrices

This chapter deals with almost periodic Hamiltonians. Those operators have much in common with random Hamiltonians; consequently, Chaps. 9 and 10 are intimately connected. Almost periodic Jacobi matrices, as well as their continuous counterparts, have been the subject of intensive research in the last years. They show surprising phenomena such as singular continuous spectrum, pure point spectrum and absolutely continuous spectrum that is nowhere dense!

Despite much effort, almost periodic Hamiltonians are not well understood. Virtually all the really interesting results concern a small class of examples.

10.1 Almost Periodic Sequences and Some General Results

We consider the space l^∞ of bounded (real-valued) sequences $\{c(n)\}_{n\in\mathbb{Z}^\nu}$. For $c\in l^\infty$, we define c_m to be the sequence $\{c(n-m)\}_{n\in\mathbb{Z}^\nu}$. A sequence c is called almost periodic if the set $\Omega_0=\{c_m|m\in\mathbb{Z}^\nu\}$ has a compact closure in l^∞. The closure of Ω_0 is called the *hull* of c.

A convenient way to construct examples goes as follows: Take a continuous periodic function F: $\mathbb{R}\to\mathbb{R}$ with period 1. We can think of F as a function on the torus $\mathbb{T}=\{\exp(2\pi\mathrm{i}x)|x\in[0,1)\}$, i.e. $F(x)=\tilde{F}(\exp(2\pi\mathrm{i}x))$. Now choose a real number α and define $F^{(\alpha)}(n):=F(\alpha n)$. $F^{(\alpha)}$ as a function on $\mathbb{Z}$ will not be periodic if $\alpha\notin\mathbb{Q}$. It is, however, an almost periodic sequence. To see this, define $F^{(\alpha),\theta}(n):=F(\alpha n+\theta)$. For α fixed, $S:=\{F^{(\alpha),\theta}\}_{\theta\in[0,2\pi]}$ is a continuous image of the circle, and is thus compact. The translates of $F^{(\alpha)}$ lie in S, so their closure is compact. In fact, if α is irrational, S is precisely the hull of $F^{(\alpha)}$.

Similarly, if F is a continuous periodic function on $\mathbb{R}^d$, then for $\alpha\in\mathbb{R}^d$, $F^{(\alpha)}(n):=F(n\alpha)$ defines an almost periodic sequence on $\mathbb{Z}$.

Let us define $\mathbb{T}^d$—the d-dimensional torus—to be the set $\{(\exp(2\pi\mathrm{i}x_1),\ldots,\exp(2\pi\mathrm{i}x_d))|(x_1,\ldots,x_d)\in[0,1]^d\}$, i.e. $\mathbb{T}^d$ is $[0,1]^d$ with opposite surface identified. We say that $(c_1,\ldots,c_n)$ are *independent over the rationals* $\mathbb{Q}$, if, for $\gamma_i\in\mathbb{Q}$: $\sum\gamma_i c_i=0$ implies that $\gamma_i=0$ for all i. If $(1,\alpha_1,\alpha_2,\ldots,\alpha_d)$ are independent over the rationals, then the set $\{[\exp(2\pi\mathrm{i}\alpha_1 n),\exp(2\pi\mathrm{i}\alpha_2 n),\ldots,\exp(2\pi\mathrm{i}\alpha_d n)]|n\in\mathbb{Z}\}$ is dense in $\mathbb{T}^d$. From this it is not difficult to see that the hull $\Omega_{F^{(\alpha)}}$ of $F^{(\alpha)}$ (F continuous periodic) is given by $\{F(\alpha n+\theta)|\theta\in[0,2\pi]^d\}\simeq\mathbb{T}^d$, if $(1,\alpha,\ldots,\alpha_d)$ are independent over the rationals.

Now let c be an arbitrary, almost periodic sequence on $\mathbb{Z}^\nu$. On $\Omega_0=\{c_m|m\in\mathbb{Z}^\nu\}$ we define an operation $\circ$ by: $c_m\circ c_{m'}:=c_{m+m'}$. By density of Ω_0 in the hull Ω, this operation can be extended to Ω in a unique way. The operation $\circ$

makes Ω a compact topological group. It is well known that any compact topological group Ω carries a unique Baire measure μ, which satisfies

$$\int f(gg')\,d\mu(g') = \int f(g')\,d\mu(g')$$

and $\mu(\Omega) = 1$. This invariant measure is called the *Haar measure* (see [260] or [292] for details). We may (and will) look upon Ω, P as a probability space. We define, for $f \in \Omega$: $T_n f = f_n$. The invariance property of the Haar measure μ tells us that

$$\mu(T_n A) = \mu(A) \ .$$

Thus, the T_n are measure-preserving transformations. It is not difficult to see that any set A with $T_n A = A$ for all A has Haar measure 0 or 1. Indeed, for such an A, $\tilde{\mu}(B) = \int_B \chi_A \, d\mu$ would define another Haar measure on Ω. But up to a constant, the Haar measure is unique. Hence $\{T_n\}$ are ergodic. We may therefore apply Theorems 9.2 and 9.4 to almost periodic Jacobi matrices, i.e. to operators H of the form $H = H_0 + V$ where H_0 is the discretized Laplacian and V is an almost periodic sequence.

Proposition 10.1. Suppose V is an almost periodic sequence.

(i) For all W in the hull Ω of V, the spectrum $\sigma(H_0 + W)$ is the same. The discrete spectrum is empty.

(ii) There is a subset $\tilde{\Omega}$ of Ω of full Haar measure, such that for all $W \in \tilde{\Omega}$ the pure point spectrum (singular continuous, absolutely continuous spectrum) is the same.

That (i) is true for all W rather than merely for a set of measure 1 comes from an easy approximation argument. This argument is not applicable to (ii) since the absolutely continuous spectrum, etc. may change discontinuously under a perturbation.

The above consideration emphasizes some similarity between stochastic and almost periodic Jacobi matrices. However, to get deeper results, more specific methods are required.

Most of the rest of this chapter deals with examples of the type $F^{(\alpha)}(n) = F(\alpha n)$ for a periodic function F (with period 1). The spectral properties of $H = H_0 + \lambda F^{(\alpha)}$ depend on the coupling constant λ, and on "Diophantine" properties of α (an observation of *Sarnak* [306]). More precisely, (suppose $\nu = 1$) if α is rational, $F^{(\alpha)}$ is periodic and we have only absolutely continuous spectrum. If α is irrational but "extremely well approximated" by rational numbers, then $H = H_0 + \lambda F^{(\alpha)}$ has a tendency to singular continuous spectrum for large λ (see Sect. 10.2), while for "typical" irrational α and λ large, the operator should have pure point spectrum (Sect. 10.3). This picture has not been generally proven, but rather for specific examples. We will see such examples below. It is even less well understood what happens in the range between "extremely well approximated" by rationals and "typical" α, and for small λ. Moreover, the spectrum has a tendency to be a Cantor set, that is, a closed set without isolated points, but with empty interior. We will discuss this phenomenon in Sect. 10.4.

We will be able to discuss only a few aspects of the theory in this chapter. Our main goal is to show the flavor and the richness of the field. For further reading, we recommend the survey [335] which has some more material. The reader will, however, realize that some results discussed here were found after the writing of [335], which shows the rapid development of the subject.

10.2 The Almost Mathieu Equation and the Occurrence of Singular Continuous Spectrum

In what follows, we will examine the following one-dimensional example of an almost periodic potential:

$$V_\theta(n) = \sum_{k=1}^{K} a_k \cos[2\pi k(\alpha n + \theta)] \tag{10.1}$$

with $\theta \in [0, 1] \simeq$ the hull of V_0.

For the case $k = 1$, the corresponding (discretized) Schrödinger equation is called the "almost Mathieu equation." It is actually the almost Mathieu equation that we will investigate in detail.

Our first theorem in this section is due to *Herman* [162], and provides an estimate of the Lyaponov exponent γ of (10.1) from below.

Theorem 10.2 (Herman). If $\alpha \notin \mathbb{Q}$, then the Lyaponov exponent γ corresponding to (10.1) satisfies

$$\gamma(E) \geq \ln\left(\frac{|a_K|}{2}\right) .$$

Remarks (1) By Theorem 9.13, we conclude that $H_\theta := H_0 + V_\theta$ for almost all θ has no absolutely continuous spectrum if α is irrational and $|a_K| > 2$.

(2) Prior to Herman, another proof of the case $K = 1$ was given by *Andre* and *Aubry* [15] (with points of rigor clarified by *Avron* and *Simon* [30]).

Proof. For notational convenience, we suppose $K = 1$, i.e.

$$\begin{aligned} V_\theta(n) &= a\cos 2\pi(\alpha n + \theta) \\ &= \frac{a}{2}(e^{2\pi i\alpha n}e^{2\pi i\theta} + e^{-2\pi i\alpha n}e^{-2\pi i\theta}) \\ &= \frac{a}{2}(e^{2\pi i\alpha n}z + e^{-2\pi i\alpha n}z^{-1}) , \end{aligned}$$

where we set $z := \exp(2\pi i\theta)$. For a fixed value E the transfer matrices (see (9.16) $\Phi_L(z)$ are given by

$$\Phi_L(z) = A_L(z)A_{L-1}(z)\cdot\ldots\cdot A_2(z)A_1(z) \quad \text{with}$$

$$A_n(z) = \begin{bmatrix} E - \frac{a}{2}(e^{2\pi i\alpha n}z + e^{-2\pi i\alpha n}z^{-1}) & -1 \\ 1 & 0 \end{bmatrix} .$$

We define

$$F_L(z) := z^L\Phi_L(z) = \prod_{n=1}^{L} zA_n(z) \ .$$

The matrix-valued function $F_L(z)$ is obviously analytic in the whole complex plane, and furthermore satisfies

$$\|F_L(z)\| = \|\Phi_L(z)\|$$

for all z of the form $\exp(i\theta)$. Since $F_L(z)$ is analytic, the function $\ln\|F(z)\|$ is subharmonic (see e.g. *Katznelson* [199] III.3.2), thus

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\|F_L(e^{i\theta})\|\, d\theta \geq \ln\|F_L(0)\| = L\ln\left(\frac{|a|}{2}\right) . \tag{10.2}$$

Because of $\alpha \notin \mathbb{Q}$, the flow $\tau_n(\theta) = (\theta + \alpha n) \bmod 1$ is ergodic (see Sect. 10.1); hence the subadditive ergodic theorem [200] tells us that for almost all θ

$$\gamma = \lim_{L\to\infty} \frac{1}{L}\ln\|\Phi_L(e^{i\theta})\| = \lim_{L\to\infty}\frac{1}{L}\int_0^{2\pi} \ln\|\Phi_L(e^{i\theta})\|\frac{d\theta}{2\pi}$$

$$= \lim_{L\to\infty}\frac{1}{L}\int_0^{2\pi} \ln\|F_L(e^{i\theta})\|\frac{d\theta}{2\pi} .$$

Therefore, we obtain the bound

$$\gamma \geq \ln\left(\frac{|a|}{2}\right)$$

because of (10.2). □

The next theorem will enable us to exclude also point spectrum for H_θ for special values of α. For those values, H_θ has neither point spectrum nor absolutely continuous spectrum; consequently, the spectrum of H_θ is purely singular continuous!

The theorem we use to exclude eigenvalues is due to *Gordon* [139]. It holds—with obvious modifications—in the continuous case [i.e. on $L^2(\mathbb{R})$] as well (for this case, see *Simon* [335]).

Theorem 10.3 (Gordon). Let $V(n)$ and $V_m(n)$ for $m \in \mathbb{N}$ be bounded sequences on $\mathbb{Z}$ (i.e. $n \in \mathbb{Z}$). Furthermore, let

(i) V_m be periodic, with period $T_m \to \infty$.
(ii) $\sup_{n,m} |V_m(n)| < \infty$.
(iii) $\sup_{|n| \leq 2T_m} |V_m(n) - V(n)| \leq m^{-T_m}$.

Then any solution $u \neq 0$ of

$$Hu = (H_0 + V)u = Eu$$

satisfies

$$\overline{\lim_{|n| \to \infty}} \frac{u(n+1)^2 + u(n)^2}{u(1)^2 + u(0)^2} \geq \frac{1}{4} .$$

Remark. The assumptions of the theorem roughly require that the potential V is extremely well approximated by periodic potentials. The conclusion, in particular, implies that $H = H_0 + V$ does not have (l^2-) eigenfunction, i.e. the point spectrum of H is empty.

Before we give a proof of Gordon's theorem, we apply the theorem to the almost Mathieu equation. We first need the definition:

Definition. A number $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ is called a *Liouville number* if, for any $k \in \mathbb{N}$, there exist $p_k, q_k \in \mathbb{N}$ such that

$$\left| \alpha - \frac{p_k}{q_k} \right| \leq k^{-q_k}.$$

Thus, a Liouville number is an irrational number that is extremely well approximated by rational ones. The set of Liouville numbers is small from an analyst's point of view: It has Lebesgue measure zero. However, from a topologist's point of view, it is rather big: It is a dense G_δ-set. (Recall that F is a G_δ-set if it is a countable intersection of open sets.)

Theorem 10.4 (Avron and Simon [30]). It α is a Liouville number, $|\lambda| > 2$ and

$$V_\theta(n) := \lambda \cos[2\pi(\alpha n + \theta)] ,$$

then $H_\theta = H_0 + V_\theta$ has purely singular continuous spectrum for almost all θ.

Proof. By Theorem 10.2, we know that H_θ does not have absolutely continuous spectrum for a.e. θ. Assume that α is well approximated by p_k/q_k in the sense of the above Definition. By choosing a subsequence $p_{k'}/q_{k'}$ of p_k/q_k, we may assume

$$\left| \alpha - \frac{p_{k'}}{q_{k'}} \right| < q_{k'}^{-1} k^{-q_{k'}} .$$

We set

$$V_k(n) := \lambda \cos\left[2\pi\left(\frac{p_{k'}}{q_{k'}} n + \theta\right)\right] .$$

Then $T_k = q_{k'}$ is a period for V_k.

We estimate:

$$\begin{aligned}\sup_{|n| \le 2q_{k'}} |V_k(n) - V(n)| &= \sup_{|n| \le 2q_k} \left|\cos\left(2\pi \frac{p_{k'}}{q_{k'}} n\right) - \cos(2\pi\alpha n)\right| \\ &\le \sup_{|n| \le 2q_{k'}} 2\pi|n| \left|\frac{p_{k'}}{q_{k'}} - \alpha\right| \\ &\le 4\pi k^{-T_k} .\end{aligned}$$

Thus, V satisfies the assumptions of Gordon's theorem. □

Now we turn to the proof of Gordon's theorem. We start with an elementary lemma:

Lemma. Let A be an invertible 2×2 matrix, and x a vector of norm 1. Then

$$\max(\|Ax\|, \|A^2 x\|, \|A^{-1} x\|, \|A^{-2} x\|) \ge \tfrac{1}{2} .$$

Proof. The matrix A obeys its characteristic equation

$$a_1 A^2 + a_2 A + a_3 = 0 . \tag{10.3}$$

We may assume that $a_i = 1$ for some $i \in \{1, 2, 3\}$ and $|a_j| \le 1$ for all $j \ne i$.

Let us suppose $a_2 = 1$ and that $|a_1|, |a_3| \le 1$, the other cases are similar. Then (10.3) gives

$$x = -a_1 Ax - a_3 A^{-1} x$$

Since x has norm one and $|a_1|, |a_3| \le 1$, it follows that $\|Ax\| \ge \frac{1}{2}$ or $\|A^{-1}x\| \ge \frac{1}{2}$. □

Proof of Theorem 10.3. Let u be the solution of $(H_0 + V)u = Eu$ with a particular initial condition. Let u_m be the solution of $(H_0 + V_m)u = Eu$ with the same initial condition. Define

$$\phi(n) := \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}, \quad \phi_m(n) := \begin{pmatrix} u_m(n+1) \\ u_m(n) \end{pmatrix}$$

and

$$A_n = \begin{pmatrix} E - V(n) & -1 \\ +1 & 0 \end{pmatrix}, \quad A_n^{(m)} = \begin{pmatrix} E - V_m(n) & -1 \\ +1 & 0 \end{pmatrix} .$$

Then

$$\sup_{|n|\leq 2T_m} \|\phi_m(n) - \phi(n)\|$$

$$\leq \sup_{|n|\leq 2T_m} \|A_n A_{n-1} \cdots A_1 - A_n^{(m)} A_{n-1}^{(m)} \cdots A_1^{(m)}\| \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|$$

$$\leq \sup_{|n|\leq 2T_m} |n| e^{C|n|} m^{-T_m} = 2T_m e^{2CT_m} m^{-T_m} .$$

Thus,

$$\max_{a=\pm 1, \pm 2} \|\phi(aT_m) - \phi_m(aT_m)\| \to 0 \quad \text{as } m \to \infty .$$

By the above lemma, we have

$$\max_{a=\pm 1, \pm 2} \|\phi_m(aT_m)\| \geq \tfrac{1}{2}\|\phi_m(0)\| = \tfrac{1}{2}(|u(0)|^2 + |u(1)|^2)^{1/2}$$

Thus

$$\overline{\lim} \frac{|u(n)|^2 + |u(n+1)|^2}{|u(0)|^2 + |u(1)|^2} \geq \overline{\lim} \frac{\max_{a=\pm 1, \pm 2} \|\phi(aT_m)\|^2}{\|\phi(0)\|^2} \geq \frac{1}{4} . \quad \square$$

10.3 Pure Point Spectrum and the Maryland Model

We now turn to an almost periodic (discretized) Schrödinger operator that, to a certain extent, admits an explicit solution. We call this operator the Maryland model, after the place of its creation by Grempel, Prange and Fishman at the University of Maryland. The potential in this model is given by

$$V(n) = V_{\alpha,\theta,\lambda}(n) := \lambda \tan[\pi(\alpha \cdot n) + \theta] \tag{10.4}$$

for $n \in \mathbb{Z}^\nu$. Here $\alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{Z}^\nu$, $\alpha \cdot n$ denotes the scalar product, and $\theta \in [0, 2\pi]$. To have $V(n)$ finite for all $n \in \mathbb{Z}^\nu$, we require $\theta \neq \pi(\alpha \cdot n) + \pi/2 \bmod \pi$. Then, $V(n)$ will be unbounded (unless all components of α are rational). Therefore V is not an almost periodic function in the sense of Sect. 10.1. We will think of V as a "singular almost periodic function." Since H_0 is a bounded operator, there is no difficulty to define $H = H_0 + V$ properly.

Recently the potential (10.4) was studied extensively by *Fishman*, *Grempel* and *Prange* [111, 112], *Grempel*, *Fishman* and *Prange* [142], *Prange*, *Grempel* and *Fishman* [288], *Figotin* and *Pastur* [110, 272] and *Simon* [338, 340]. We note that Figotin and Pastur even obtain an explicit formula for the Green's function.

There is an explicit expression for the density of states $k_\lambda(E)$ of H. It is not difficult to compute $k_0(E)$, i.e. the density of states for H_0. In momentum space, H_0 is nothing but multiplication by $\phi(k) = 2\sum_{i=1}^{\nu} \cos k_i$; hence, its spectral resolution $P_{(-\infty,E)}$ is multiplication by $\chi_{(-\infty,E)}(\phi(k))$. Using this and the definition of the density of states, one learns

$$k_0(E) = \frac{1}{(2\pi)^\nu} |\{k \in [0, 2\pi]^\nu | \phi(k) < E\}| \ , \tag{10.5}$$

where $|\cdot|$ denotes the Lebesgue measure.

Let us now give the explicit expression for k_λ:

Theorem 10.5. Suppose that $\{1, \alpha_1, \ldots, \alpha_\nu\}$ are independent over the rationals. Then

$$k_\lambda(E) = \frac{1}{\pi} \int \frac{\lambda}{(E - E')^2 + \lambda^2} k_0(E')\, dE' \ . \tag{10.6}$$

Corollary. Suppose $\nu = 1$. Then the Lyaponov exponent $\gamma_\lambda(E)$ of $H_0 + \lambda \tan(\pi\alpha n + \theta)$, $\alpha \notin \mathbb{Q}$ is given by

$$\gamma_\lambda(E) = \frac{1}{\pi} \int \frac{\lambda}{(E - E')^2 + \lambda^2} \gamma_0(E')\, dE' \ , \tag{10.6'}$$

where $\gamma_0(E)$ is the Lyaponov exponent of H_0.

Remarks. (1) As long as $(1, \alpha_1, \ldots, \alpha_\nu)$ are independent over the rationals, $k_\lambda(E)$ [and for $\nu = 1$: $\gamma_\lambda(E)$] is independent of α, θ.

(2) $p_\lambda(x) := 1/\pi(\lambda/(x^2 + \lambda^2))$ is the density of a probability measure known as the Cauchy-distribution or the Lorentz-distribution among probabilists and theoretical physicists respectively. Equation (10.6) tells us that $k_\lambda(E)$ is just the convolution $p_\lambda * k_0(E)$. From this, we see that k_λ is a strictly monotone function in E from $(-\infty, +\infty)$ onto $(0, 1)$. Thus, $\sigma(H) = \operatorname{supp} k_\lambda(dE) = (-\infty, \infty)(\lambda \neq 0)$.

(3) For $\nu = 1$, $\gamma_\lambda(E)$ is strictly positive since $\gamma_0(E')$ is positive outside the spectrum of H_0. Thus, H has no absolutely continuous spectrum (for $\nu = 1$).

Proof of the Corollary. The Thouless formula (see Chap. 9) tells us that $\gamma_\lambda(E)$ is the convolution of $f(E) = \ln(E)$ with $dk_\lambda(E)/dE$. Thus,

$$\gamma_\lambda = f * \frac{dk_\lambda}{dE} = f * p_\lambda * \frac{dk_0}{dE} = p_\lambda * \gamma_0 . \quad \square$$

To prove the theorem, we will make use of the following lemma:

Lemma. Fix arbitrary reals $\alpha_1, \ldots, \alpha_k$ and positive numbers $\psi_1, \ldots, \psi_k$ with

$\sum_{j=1}^{k} \psi_j = 1$.

Let $\varphi(\theta) := \sum_{j=1}^{k} \psi_j \tan(\alpha_j + \theta)$. Then $\frac{1}{2\pi} \int_0^{2\pi} e^{it\varphi(\theta)} d\theta = e^{-|t|}$.

The proof of the lemma is left to the reader as an exercise in complex integration (for a proof see *Simon* [338]).

Proof of Theorem 10.5. We prove that the Fourier transform $\hat{k}_\lambda(t)$ of k_λ is given by $\exp(-\lambda|t|)\hat{k}_0(t)$ which implies the theorem.

The operators $H_0 + V$ restricted to a finite box are just finite matrices. Thus, we have (for the restricted matrices)

$$\exp[it(H_0 + V) = \exp(itV) + i \int_0^t \exp[is(H_0 + V)] H_0 \exp[i(t - s)V] \, ds \; .$$

Iterating this formula, we obtain a series

$$\begin{aligned} &\exp[it(H_0 + V) \\ &\quad = \exp(itV) + i \int_0^t \exp(is_1 V) H_0 \exp[i(t - s_1)V] \, ds \\ &\qquad + i^2 \int_0^t \int_0^{s_1} \exp(is_2 V) H_0 \exp[i(s_2 - s_1)V] H_0 \exp[i(t - s_1)V] \, ds_2 \, ds_1 , \\ &\qquad + \cdots \end{aligned}$$

which is easily seen to be convergent.

Taking expectation of the matrix element $\exp[it(H_0 + V)](n, m)$, we see that $\exp[it(H_0 + V)]$ is a series of integrals of the type evaluated in the lemma. Therefore,

$$\mathbb{E}(e^{it(H_0+V)}(n,m)) = e^{-\lambda|t|} e^{itH_0}(n,m) \; . \quad \square$$

Remark. The argument shows that $k_\lambda(e)$ is the density of states for a large variety of models; for example, in the Anderson model with a potential distribution p_λ. This model is known as the Lloyd model, after work of *Lloyd* [235], who computed k_λ in this model. *Grempel, Fishman* and *Prange* [142] obtained Theorem 10.5 for their model by rather different means. Our proof follows *Simon* [340], who investigated the question of why the two models had the same $k(e)$.

We suppose from now on that $(1, \alpha_1, \ldots, \alpha_\nu)$ are independent over $\mathbb{Q}$. While the density of states $k_\lambda(E)$ of $H_0 + \lambda V_{\alpha,\theta}$ does not depend on α, the spectral properties do, at least in dimension $\nu = 1$. We saw already that no absolutely continuous spectrum occurs. If α is a Liouville number, one can apply Gordon's theorem to prove that no (l^2-) eigenfunctions of $H_{\alpha,\theta,\lambda} = H_0 + \lambda V_{\alpha,\theta}$ occur, thus showing that the spectrum is singular continuous in this case. We show now (even in higher dimension) that pure point spectrum occurs for certain other choices of α.

To prove that $H_{\alpha,\theta,\lambda}$ has pure point spectrum for certain values of α, we will transform the eigenvalue equation in a number of steps. Finally, we will arrive at an equation that will make the dependence of the solution on α rather explicit, or more precisely, a sequence ψ_n that determines the solution of our eigenvalue equation will be given by

$$\psi_n = (e^{i\pi(\alpha\cdot n)} - 1)^{-1}\zeta_n ,$$

where ζ_n is a known sequence exponentially decaying in $|n|$. To prevent ψ_n from blowing up, the denominator must approach zero more slowly than ζ_n. This is a typical small divisor problem. Indeed, methods to overcome those problems (KAM-methods) dominate many proofs concerning almost periodic operators. In our case, it is natural to demand the following condition on α.

Definition. We say that α has *typical Diophantine properties* if there exist constants $C, k > 0$ such that

$$\left|\sum_{i=1}^{\nu} m_i\alpha_i - n\right| \geq C\left(\sum_{i=1}^{\nu} m_i^2\right)^{-k/2} \tag{10.7}$$

holds for all $n, m_1, \ldots, m_\nu \in \mathbb{Z}$.

As the name suggests, $\{\alpha|\alpha$ has typical Diophantine properties$\}$ has a complement of Lebesgue measure zero in $\mathbb{R}^\nu$. We will show

Theorem 10.6. If α has typical Diophantine properties, then $H_{\alpha,\theta,\lambda}$ has pure point spectrum for all $\lambda > 0$ and all θ. Moreover, the eigenvalues are precisely the solutions of

$$k_\lambda(E) = \left(\alpha\cdot m + \frac{1}{2} - \frac{\theta}{\pi}\right)_f ,$$

where $(x)_f$ means the fractional part of x, and m runs through $\mathbb{Z}^\nu$. All eigenfunctions decay exponentially.

Theorem 2.9 and 2.10 in Chap. 2 tell us to seek polynomially bounded solutions u of

$$\lambda^{-1}(E - H_0)u(n) = \tan[\pi(\alpha\cdot n) + \theta]u(n) . \tag{10.8}$$

Let us introduce the shorthand notations $A := \lambda^{-1}(E - H_0)$ and $B := \tan[\pi(\alpha\cdot n) + \theta]$. Then formally, (10.8) implies, for $c = (1 + iB)u$

$$\frac{(1-\mathrm{i}A)}{(1+\mathrm{i}A)}c=\frac{(1-\mathrm{i}B)}{(1+\mathrm{i}B)}c\ . \tag{10.9}$$

The advantage of (10.9) lies in the following simple expression of its right-hand side:

$$\frac{(1-\mathrm{i}B)}{(1+\mathrm{i}B)}=\exp(-2\pi\mathrm{i}\alpha n-2\mathrm{i}\theta)\ .$$

Before we continue, we convince ourselves that the above formal calculation can be justified. Denote by $\mathscr{P}$ the space of all polynomially bounded sequences, i.e. $\mathscr{P}=\{\{u(n)\}_{n\in\mathbb{Z}^\nu}\,|\,|u(n)|\le A(1+|n|)^k$ for some $A,k\}$.

Proposition 10.7. If $Au=Bu$ has a solution $u\in\mathscr{P}$, then $c=(1+\mathrm{i}B)u\in\mathscr{P}$ and

$$\frac{1-\mathrm{i}A}{1+\mathrm{i}A}c=\frac{1-\mathrm{i}B}{1+\mathrm{i}B}c\ .$$

Conversely, if

$$\frac{1-\mathrm{i}A}{1+\mathrm{i}A}c=\frac{1-\mathrm{i}B}{1+\mathrm{i}B}c\ .$$

has a solution $c\in\mathscr{P}$, then c is of the form $c=(1+\mathrm{i}B)u$ for a $u\in\mathscr{P}$, and u solves $Au=Bu$.

Remarks. (1) The above equations are a priori to be read pointwise as relations between numbers $u(n)$ rather than as equations in a certain space of sequences. The operator $(1+\mathrm{i}A)^{-1}$ is well defined on $l^2(\mathbb{Z}^\nu)$. It has a kernel $K(n-m)$ there with K decaying faster than any polynomial, as can be seen by Fourier transform. So $(1+\mathrm{i}A)^{-1}$ can be defined on $\mathscr{P}$ as well as via its kernel K.

(2) If we consider B as a self-adjoint operator on $l^2(\mathbb{Z}^\nu)$ with domain $D(B)$, and A as an everywhere defined bounded operator, then above $u\in D(B)$ if $c\in l^2(\mathbb{Z}^\nu)$.

Proof. Observe first that both $(1+\mathrm{i}A)$ and $(1+\mathrm{i}A)^{-1}$ map $\mathscr{P}$ into $\mathscr{P}$. Thus, $u\in\mathscr{P}$ and $Au=Bu$ implies $c:=(1+B)u=(1+A)u\in\mathscr{P}$ and

$$\frac{1-\mathrm{i}A}{1+\mathrm{i}A}c=\frac{1-\mathrm{i}B}{1+\mathrm{i}B}c\ .$$

Suppose now that

$$\frac{1-\mathrm{i}A}{1+\mathrm{i}A}c=\frac{1-\mathrm{i}B}{1+\mathrm{i}B}c\text{ for a }c\in\mathscr{P}\ .$$

Here $u=(1+\mathrm{i}B)^{-1}c$ makes perfectly good sense as a (pointwise) equation between sequences (although we do not know $u \in \mathscr{P}$ a priori). Therefore, we obtain

$$\frac{1 - iA}{1 + iA}(1 + iB)u = (1 - iB)u \ ,$$

thus $Au = Bu$. This, in turn, implies that

$$u = (1 + iB)^{-1}c = (1 + iA)^{-1}c \in \mathscr{P}. \quad \square$$

Now that we know that (10.8) is equivalent to (10.9), we apply a Fourier transform to (10.9). Let us define

$$\hat{f}(k) := \sum_{n \in \mathbb{Z}} f(n) e^{-ink} \ ,$$

where $\hat{f}$ is well defined if $\sum |f(n)| < \infty$. Moreover, for a continuous function φ on $\mathbb{T}^\nu = [0, 2\pi]^\nu$, we define

$$\check{\varphi}(n) := \frac{1}{2\pi} \int_{\mathbb{T}^\nu} \varphi(k) e^{ink} d^\nu k \ .$$

If the sequence f is merely in $\mathscr{P}$, we define $\hat{f}$ to be the distribution

$$\langle \hat{f}, \varphi \rangle := \sum f(n) \check{\varphi}(n)$$

for $\varphi \in C^\infty(\mathbb{T}^\nu)$. Here $\mathbb{T}^\nu$ is the ν-dimensional torus. Applying the Fourier transform to (10.9), we obtain in the distribution sense

$$q(k)\hat{c}(k) = e^{-2i\theta}\hat{c}(k + 2\pi\alpha) \quad \text{with} \tag{10.10a}$$

$$q(k) := -\frac{2\sum_{i=1}^{\nu} \cos k_i - E - i\lambda}{2\sum_{i=1}^{\nu} \cos k_i - E + i\lambda} \ . \tag{10.10b}$$

Here $q(k)$ is an analytic function of $z_i = \exp(ik_i)$ near $|z_i| = 1$, $|q(k)| = 1$ and q does not take the value -1. Thus, $q(k) = \exp[-i\zeta(k)]$ for a function $\zeta(k)$ analytic in $z_i = \exp(ik_i)$ near $|z_i| = 1$ satisfying $-\pi < \zeta(k) < \pi$.

Summarizing, we have shown that if the equation $(H_0 + V)u = Eu$ has a polynomially bounded solution, then

$$e^{-i\zeta(k)}\hat{c}(k) = e^{-2i\theta}\hat{c}(k + 2\pi\alpha) \tag{10.11}$$

has a distributional solution $\hat{c}$.

We will now concentrate on continuous solutions of (10.11) for a while. Since $2\alpha n (\text{mod}\ 2\pi)$ is dense in $\mathbb{T}^n$, we read off from (10.11) that $|\hat{c}|$ is constant. We may suppose that $|\hat{c}(k)| = 1$. Thus, $\hat{c}(k)$ has the form $\hat{c}(k) = \exp[-im \cdot k - i\psi(k)]$ for a continuous periodic function $\psi(k)$. So (10.11) implies

$$\psi(k + 2\pi\alpha) - \psi(k) = \zeta(k) - 2\theta - 2\pi(m \cdot \alpha) + 2\pi m_0 \tag{10.12}$$

for suitably chosen $m_0 \in \mathbb{Z}$.

Applying the Fourier transform to equation (10.12), we get

$$(e^{-2\pi i \alpha \cdot n} - 1)\check{\psi}_n = \check{\zeta}_n \quad \text{for } n \neq 0 \tag{10.13a}$$

$$\check{\zeta}_0 = 2\theta + 2\pi(m \cdot \alpha) - 2\pi m_0 \ . \tag{10.13b}$$

To solve (10.13b), we observe

Proposition 10.8. $\check{\zeta}_0 = 2\pi k_\lambda(E) - \pi$, $k_\lambda(E)$ being the integrated density of states.

Proof:

$$\frac{\partial}{\partial E}\check{\zeta}_0 = \frac{1}{(2\pi)^\nu}\int_{\mathbf{T}^\nu}\frac{\partial}{\partial E}\zeta(k)d^\nu k = \frac{1}{(2\pi)^\nu}\int_{\mathbf{T}^\nu}\frac{2\lambda}{(2\sum\cos k_i - E)^2 + \lambda^2}d^\nu k$$

[the formula for $\partial\zeta/\partial E$ can be obtained by differentiating (10.10b)]. On the other hand, from (10.6) we know

$$\frac{\partial k_\lambda}{\partial E} = \frac{1}{\pi}\int\frac{\lambda}{(y - E)^2 + \lambda^2}\frac{\partial k_0}{\partial E}(y)dy \ .$$

From (10.5) we can read off that $(\partial k_0/\partial E)(y)$ is $1/(2\pi)^\nu$ times the surface measure of the surface $\{k|2\sum\cos k_i = y\}$. Thus,

$$\begin{aligned}\frac{\partial k_\lambda}{\partial E} &= \frac{1}{\pi}\left(\frac{1}{(2\pi)^\nu}\int\frac{\lambda}{(2\sum\cos k_i - E) + \lambda^2}dk\right)\\ &= \frac{1}{2\pi}\frac{\partial\zeta}{\partial E} \ .\end{aligned}$$

Therefore the assertion follows from $\zeta \to -\pi$ as $E \to -\infty$ while $k_\lambda \to 0$. □

Proposition 10.8 tells us that (10.13b) is equivalent to

$$k_\lambda(E) = \left(\alpha \cdot m + \frac{\theta}{\pi} + \frac{1}{2}\right)_f$$

[$(x)_f$ is the fractional part of x].

Equation (10.13a) is solved, of course, by

$$\check{\psi}_n = (e^{-2\pi i \alpha n} - 1)^{-1}\check{\zeta}_n \ . \tag{10.14}$$

Since ζ is an analytic function, $\check{\zeta}_n$ decays exponentially. Moreover, since α has typical Diophantine properties, we have

$$|e^{-2\pi i\alpha n} - 1| = 2|\sin \pi\alpha \cdot n| \geq \tilde{C}\left(\sum_{i=1}^{\nu} m_i^2\right)^{-k/2} .$$

This follows from the estimate

$$|\sin x| \geq \tilde{\tilde{C}}|x|$$

for $|x| \leq \pi/2$. Therefore, $\check{\psi}_n$ decays exponentially. This, in turn, implies that, for any solution $\check{\psi}_n$ of (10.13a, b), the function $\psi(k) = \sum \check{\psi}_n \exp(-ink)$ is analytic and solves (10.12). Thus, we have shown

Proposition 10.9. The equation

$$e^{-i\zeta(k)}\hat{c}(k) = e^{-2i\theta}\hat{c}(k + 2\pi\alpha) \tag{10.15}$$

has a continuous solution $\hat{c}$ if and only if

$$k_\lambda(E) = \left(\alpha \cdot m + \frac{\theta}{\pi} + \frac{1}{2}\right)_f .$$

Any continuous solution $\hat{c}$ of (10.15) is analytic and of the form $\hat{c}(k) = \exp[-ik \cdot m - i\psi(k)]$, and the Fourier coefficients $\check{\psi}_n$ of ψ decay exponentially.

Now we show that the above solutions are the only ones of interest:

Proposition 10.10. Suppose the sequence c is polynomially bounded, and $\hat{c}$ fulfills (10.15); then $\hat{c}$ is analytic, and c decays exponentially.

Proof. We choose θ_0 such that

$$k_\lambda(E) = \left(\frac{1}{2} + \frac{\theta_0}{\pi}\right)_f .$$

From our considerations above, it follows that there is an analytic function $\hat{d}(k)$ such that $|\hat{d}(k)| = 1$ and

$$\hat{d}(k + 2\pi\alpha) = \exp[-i\zeta(k) + 2i\theta]_0\hat{d}(k) .$$

Suppose now $\hat{c}$ is a distributional solution of (10.15). Then $l = \hat{c}/\hat{d}$ is also a distribution and satisfies

$$l(k + 2\pi\alpha) = e^{2i(\theta-\theta_0)}l(k) \ ; \quad \text{hence,}$$

$$e^{-2\pi i\alpha n}\check{l}_n = e^{2i(\theta-\theta_0)}\check{l}_n \ ;$$

thus, $\check{l}_n = 0$ for all but one n. Therefore, $l(k) = \exp(-in_0k)$ for some n_0, i.e.

$$\hat{c}(k) = e^{in_0 k}\hat{d}(k) \ ;$$

thus, $\hat{c}$ is analytic and $c(n) = d(n + n_0)$ decays exponentially. □

We now complete the proof of Theorem 10.6:

Proof (*Theorem 10.6*). Suppose u is a polynomially bounded solution of

$$(H_0 + \lambda V_{\alpha,\theta})u = Eu \ .$$

Thus $c = (1 + iB)u$ is a polynomially bounded solution of

$$\frac{1 - iA}{1 + iA}c = \frac{1 - iB}{1 + iB}c \ . \tag{10.16}$$

with $A = \lambda^{-1}(E - H_0)$ and $B = \tan[2\pi(\alpha \cdot n) + \theta]$.

We have shown that any polynomially bounded solution of (10.16) is exponentially decaying. Thus,

$$u = (1 + iB)^{-1}c = (1 - iA)^{-1}c$$

is exponentially decaying.

From Theorem 2.10 in Chap. 2 we know that the spectral measures are supported by $S = \{E | Hu = Eu$ has a polynomially bounded solution$\}$. Since any polynomially bounded solution of $Hu = Eu$ is exponentially decaying, S is a countable set; thus, H has pure point spectrum. □

Besides various cleverly chosen transformations of the problem, the very heart of the proof of Theorem 10.6 is the solution of (10.13a), i.e. to control the behavior of

$$\check{\psi}_n = (e^{-2\pi i\alpha\cdot n} - 1)^{-1}\check{\zeta}_n \ .$$

This is a typical problem of small divisors. Above we ensured that $\check{\psi}_n$ decays exponentially by requiring α to have typical Diophantine properties. Virtually all proofs for pure point spectrum of almost periodic Hamiltonians rely upon handling such small divisor problems. We can only mention some of those works: *Sarnak* [306], *Craig* [68], *Bellissard*, *Lima* and *Scoppola* [40], *Pöschel* [286]. Those authors construct examples of almost periodic Hamiltonians with dense point spectrum. They use Kolmogoroff-Arnold-Moser (KAM)-type methods to overcome the small divisor problem. Among their examples are, for any $\lambda \in [0, 1]$, almost periodic V's so that $H_0 + V$ has only dense point spectrum and $\sigma(H_0 + V)$ has Hausdorff dimension λ.

The first use of KAM-methods in the present context was made by *Dinaburg* and *Sinai* [87]. They proved that absolutely continuous spectrum occurs for certain almost periodic Schrödinger operators, and moreover, that certain solutions of their Schrödinger equation have Floquet-type structure. Their work was extended considerably by *Russmann* [304] and *Moser* and *Pöschel* [253].

Bellissard, Lima and *Testard* [41] applied KAM-ideas to the almost Mathieu equation (see Sect. 10.2). They proved, for typical Diophantine α and small coupling constant there is some absolutely continuous spectrum. Moreover, for typical Diophantine α and large coupling they found point spectrum of positive Lebesgue measure for almost all values of θ. In neither case could they exclude additional spectrum of other types.

10.4 Cantor Sets and Recurrent Absolutely Continuous Spectrum

General wisdom used to say that Schrödinger operators should have absolutely continuous spectrum plus some discrete point spectrum, while singular continuous spectrum is a pathology that should not occur in examples with V bounded. This general picture was proven to be wrong by *Pearson* [275, 276], who constructed a potential V such that $H = H_0 + V$ has singular continuous spectrum. His potential V consists of bumps further and further apart with the height of the bumps possibly decreasing. Furthermore, we have seen the occurrence of singular continuous spectrum in the innocent-looking almost Mathieu equation (Sect. 10.2).

Another correction to the "general picture" is that point spectrum may be dense in some region of the spectrum rather than being a discrete set. We have seen this phenomenon in Chap. 9 as well as in Sect. 10.3. Thus, so far we have four types of spectra: "thick" point spectrum and singular continuous spectrum, which are the types one would put in the waste basket if they did not occur in natural examples, and "thin" point spectrum and absolutely continuous spectrum, the two types that are expected according to the above picture.

It was *Avron* and *Simon* [28] who proposed a further splitting of the absolutely continuous spectrum into two parts: The transient a.c. spectrum, which is the "expected" one, and the recurrent a.c. spectrum, which is the "surprising" one usually coming along with Cantor sets.

To motivate their analysis, we construct some examples, at the same time fixing notations.

A subset C of the real line is called a *Cantor set* if it is closed, has no isolated points (i.e. is a perfect set), and furthermore, is nowhere dense (i.e. $\bar{C} = C$ has an empty interior). "The" Cantor set is an example for this: Remove from $[0, 1]$ the middle third. From what remains, remove the middle third in any piece, and so on. What finally remains is a perfect set with empty interior. This set is well known to have Lebesgue measure zero.

The construction of "removing the middle third" can be generalized easily. Choose a sequence n_j of real numbers, $n_j > 1$. From $S_0 = [0, 1]$ remove the open interval of size $1/n_1$ about the point $\frac{1}{2}$. The new set is called

$$S_1 : S_1 = [0, 1] \setminus \left(\frac{1}{2} - \frac{1}{2n_1}, \frac{1}{2} + \frac{1}{2n_1}\right) = \left[0, \frac{1}{2}\left(1 - \frac{1}{n_1}\right)\right] \cup \left[\frac{3}{2}\left(1 - \frac{1}{n_1}\right), 1\right] .$$

Having constructed S_j, a disjoint union of 2^j closed intervals of size α_j, remove from each of these intervals the open interval of size $\alpha_j n_{j+1}^{-1}$ about the center of the interval. The union of the remaining 2^{j+1} intervals is called S_{j+1}. We define

$$S = S(\{n_i\}) = \bigcap_{j=0}^{\infty} S_j .$$

It is not difficult to see that S is a Cantor set (in the above defined sense; see *Avron* and *Simon* [28]). Moreover, since

$$\alpha_j = 2^{-j} \prod_{k=1}^{j} [1 - (1/n_k)] ,$$

the Lebesgue measure of the set S_j is given by

$$|S_j| = \prod_{k=1}^{j} [1 - (1/n_k)] .$$

Hence

$$|S| = \prod_{k=1}^{\infty} [1 - (1/n_k)] .$$

The infinite product is zero if and only if $\sum_{k=1}^{\infty}(1/n_k) = \infty$. Thus, the above procedure allows us to construct Cantor sets of arbitrary Lebesgue measure (< 1). The "middle third" Cantor set, our starting example, has $n_k = 3$ for all k, and thus zero Lebesgue measure. It can be used to construct a singular continuous measure carried by it (see e.g. *Reed* and *Simon* I [292]).

Suppose now S is a Cantor set with $0 < |S| < \infty$. Let χ_S be its characteristic function [$\chi_S(x) = 1$ if $x \in S$, and zero otherwise]. Then $\mu_S := \chi_S(x)\,dx$ defines an absolutely continuous measure (with respect to Lebesgue measure). So μ_S is an absolutely continuous measure with nowhere dense support!

The idea of Avron and Simon was to single out measures like μ_S by looking at their Fourier transform.

It is well known that the Fourier transform $F_\mu(t) = \int \exp(itx)\,d\mu(x)$ goes to zero as $|t| \to \infty$ if μ is an absolutely continuous measure. $F_\mu(t)$ goes to zero at least in the averaged sense that $1/2T \int_{-T}^{T} F_\mu(t)\,dt \to 0$ as $T \to \infty$ if μ is a continuous (a.c. or s.c.) measure. We will now distinguish two types of a.c. measures by the fall-off of their Fourier transform. We call two measures, μ and ν, *equivalent* if they are mutually absolutely continuous, that is to say, there exists functions $f \in L^1(\mu)$ and $g \in L^1(\nu)$ such that $d\nu = f\,d\mu$ and $\mu = g\,d\nu$.

Proposition 10.11. (1) Suppose μ is an absolutely continuous measure supported by a Cantor set S, then $F_\mu(t)$ is not in L^1.

(2) Consider the measure $\nu = \chi_A\,dx$ where ∂A has Lebesgue measure zero. Then there exists a measure $\tilde{\nu}$ equivalent to ν such that $F_{\tilde{\nu}}(t) = O(t^{-N})$ for all $N \in \mathbb{N}$.

Proof. (1) $\mu = f(x)\,dx$ for a function f supported by S. Since S is a Cantor set, f cannot be continuous. But if $F_\mu(t) = \int \exp(itx) f(x)\,dx$ were in L^1, then f would be continuous.

(2) There exists a function $f \in S(\mathbb{R})$, the Schwartz functions, with $\operatorname{supp} f = A$ and $f > 0$ on the interior A^{int} of A, such that $\tilde{v} := f(x)\,dx$ is equivalent to v. Then $F_{\tilde{v}}(t) = \int f(x)\exp(itx)\,dx = O(t^{-N})$, which can be seen by integration by parts. □

The above considerations motivate the following definition:

Definition. Let H be a self-adjoint operator on a separable Hilbert space H. The quantity $\varphi \in$ H is called a *transient vector* for H if

$$\langle \varphi, e^{-tH}\varphi \rangle = O(t^{-N}) \quad \text{for all } N \in \mathbb{N} .$$

The closure of the set of transient vectors is called H_{tac} (*transient absolutely continuous subspace*). Thus, φ is a transient vector if the spectral μ_φ measure associated with φ has rapidly decaying Fourier transform. Proposition 10.11 would equally well suggest to define φ as a transient vector if the Fourier transform of its spectral measure is L^1. Fortunately, this leads to the same set H_{tac}.

Proposition 10.12:

(i) H_{tac} is a subspace of H,
(ii) $H_{tac} \subset H_{ac}$
(iii) $H_{tac} = \overline{\{\varphi \mid F_{\mu_\varphi} \in L^1\}}$.

For a proof, see *Avron* and *Simon* [28].

Definition. We define $H_{rac} = H_{tac}^{\perp} \cap H_{ac}$. H_{rac} is called the *recurrent absolutely continuous subspace*. Both H_{tac} and H_{rac} are invariant subspaces under H. We can therefore define $\sigma_{tac}(H) = \sigma(H|_{H_{tac}})$ and $\sigma_{rac}(H) = \sigma(H|_{H_{rac}})$.

As the reader might expect, the occurrence of σ_{rac} and Cantor sets are intimately related:

Proposition 10.13. Suppose that H has nowhere dense spectrum. Then $\sigma_{tac}(H) = \phi$.

This is actually a corollary to Proposition 10.11. It is, of course, easy to construct operators with $\sigma_{rac} \neq \phi$. Take H $= L^2(\mathbb{R})$ and consider the operator $T_A = x\chi_A(x)$ where $A \in B(\mathbb{R})$. We have $\sigma(T_A) = \bar{A}$. If A is an interval $[a, b]$, $(a < b)$, then the spectrum is purely transient absolutely continuous. There are, however, vectors φ with bad behavior of $\int \exp(itx)\,d\mu_\varphi(x)$. For example, take $\varphi = \chi_S$, S a Cantor set of positive Lebesgue measure in $[a, b]$. This shows clearly that not all $\varphi \in H_{tac}$ show fast decay of $\int \exp(itx)\,d\mu_\varphi(x)$, but rather a dense subset of φ's does. If A is a Cantor set of positive Lebesgue measure, then the spectrum is recurrent absolutely continuous.

One might think that σ_{rac} is always a nowhere dense set. This is wrong!

Avron and *Simon* [28] constructed a set A such that $\sigma_{\mathrm{rac}}(T_A) = (-\infty, +\infty)$. This means, in particular, that $(-\infty, \infty)$ is the support of an absolutely continuous measure $d\mu = f(x)\,dx$, $\operatorname{supp} f = \mathbb{R}$, but dx is not absolutely continuous with respect to $d\mu$.

So far, we worked in a quite abstract setting, and one might think that Cantor sets and recurrent absolutely continuous spectrum do not occur for Schrödinger operators. However, there is some evidence that Cantor sets as spectra of one-dimensional almost periodic operators are very common, although recurrent absolutely continuous spectrum might be less generic.

Chulaevsky [64], *Moser* [252] and *Avron* and *Simon* [29] have constructed examples of limit periodic potentials whose spectra are Cantor sets. A sequence $\{c_n\}_{n\in\mathbb{Z}}$ is called *limit periodic* if it is a uniform limit (i.e. a limit in l^∞) of periodic sequences. For example,

$$V(n) = \sum_{j=-\infty}^{+\infty} a_j \cos\left(\frac{2\pi n}{2^j}\right), \tag{10.17}$$

for $\sum |a_j| < \infty$ is such a sequence.

We denote by L the space of all limit periodic sequences, and by L_0 the space of all sequences as in (10.17). Limit periodic sequences are particular examples of almost periodic ones as one easily verifies. The definition can be carried over to higher dimensions, but we consider only sequences indexed by $\mathbb{Z}^1$ here.

L and L_0 are closed subspaces of the Banach space l^∞, so that topological notions like dense, closed and G_δ (countable intersection of open sets) make sense.

Theorem 10.14 (SCAM).

(i) For a dense G_δ in L, the spectrum $\sigma(H_0 + V)$ is a Cantor set.
(ii) The same is true for a dense G_δ in L_0.

Remark. The name SCAM-theorem is a (linguistic) permutation of initials: *Avron* and *Simon* [29], *Chulaevsky* [64] and *Moser* [252]. Those authors actually worked in the continuous case, i.e. with Schrödinger operators on $L^2(\mathbb{R})$.

Avron and *Simon* [29] and *Chulaevsky* [64] proved—for a perhaps smaller set—the occurrence of recurrent absolutely continuous spectrum:

Theorem 10.15 (Avron and Simon, Chulaevsky). For a dense subset of L, the spectrum $\sigma(H_0 + V)$ is both a Cantor set and absolutely continuous. The same is true for L_0.

Notice that the above theorem does not claim that the dense set in question is a G_δ. We do not even believe that this is true.

We learn from Theorem 10.15 and Proposition 10.13 that the spectrum of $H_0 + V$ is recurrent absolutely continuous.

There is another result by *Bellissard-Simon* [42] establishing Cantor spectrum, this time for the almost Mathieu equation:

Theorem 10.16 (Bellissard-Simon). The set of pair (λ, α) for which $\sigma(H_0 + \lambda\cos(2\pi\alpha n + \theta))$ is a Cantor set is a dense G_δ in $\mathbb{R}^2$.

The physical significance of the distinction between recurrent and transient absolutely continuous spectrum comes from the intuitive connection of the long-time behavior of $\exp(-itH)$ and transport phenomena in the almost periodic structure. Fast decay of $\langle\varphi, \exp(-itH)\varphi\rangle$ means that the wave packet φ will spread out rapidly, while slow decay means that it will have anomalous long-time behavior. Hence, fast decay of $F(t) = \int\exp(-itx)\,d\mu_\varphi$ for the spectral measure μ_φ indicates good transport properties of the medium (think of electric transport via electrons moving in an imperfect crystal); slow or no decay of $F(t)$ indicates bad transport.

In this respect, recurrent absolutely continuous spectrum behaves much more like singular continuous spectrum than like a transient absolutely continuous one.

11. Witten's Proof of the Morse Inequalities

Thus far, we have described the study of Schrödinger operators for their own sake. In this chapter and the next, we will discuss some rather striking applications of the Schrödinger operators to analysis on manifolds. In a remarkable paper, *Witten* [370] showed that one can obtain the strong Morse inequalities from the semiclassical analysis of the eigenvalues of some appropriately chosen Schrödinger operators on a compact manifold M. The semiclassical eigenvalues theorems are discussed in Sect. 11.1, and Witten's choice of operators in Sect. 11.4. The Morse inequalities are stated in Sect. 11.2 and proven in Sect. 11.5. Some background from Hodge theory is described in Sect. 11.3.

Supersymmetric ideas play a role in the proof of the Morse index theorem, and played an even more significant role in Witten's motivation.

11.1 The Quasiclassical Eigenvalue Limit

We begin by discussing the quasiclassical eigenvalue limit for Schrödinger operators acting in $L^2(\mathbb{R}^\nu)$. We will consider self-adjoint operators of the form

$$H(\lambda) = -\Delta + \lambda^2 h + \lambda g$$

defined as the closure of the differential operator acting on $C_0^\infty(\mathbb{R}^\nu)$. Here h, $g \in C^\infty(\mathbb{R}^\nu)$, g is bounded, $h \geq 0$ and $h > \text{const} > 0$ outside a compact set. Furthermore, we assume that h vanishes at only finitely many points $\{x^{(a)}\}_{a=1}^k$, and that the Hessian

$$[A_{ij}^{(a)}] = \frac{1}{2}\left[\frac{\partial^2 h}{\partial x_i \partial x_j}(x^{(a)})\right]$$

is strictly positive definite for every a. The goal is to estimate the eigenvalues of $H(\lambda)$ for large λ. The idea is that for large λ the potential $\lambda^2 h + \lambda g$ should look like finitely many harmonic oscillator wells centered at the zeros of h and separated by large barriers. Thus, one expects that for large λ the spectrum of $H(\lambda)$ should look like the spectrum of a direct sum of operators of the form

$$H^{(a)}(\lambda) = -\Delta + \lambda^2 \sum_{ij} A_{ij}^{(a)}(x - x^{(a)})_i (x - x^{(a)})_j + \lambda g(x^{(a)}) \; .$$

for $\varphi \in D_0$, a dense set in $L^2(\mathbb{R}^\nu)$. Consequently, the wave operators $\Omega^\pm$ exist, and Cook's estimate (5.5) holds.

Proof. We take $D_0 = \{g(H_0)P_{(-\infty,a)}\psi \mid g \in C_0^\infty(\mathbb{R}),\ \operatorname{supp} g \subset [\alpha^2, \beta^2]$ for some $\alpha, \beta > 0,\ a \in \mathbb{R},\ \psi \in L^2(\mathbb{R}^\nu)\}$, $P_{(-\infty,a)}$ being the spectral projections corresponding to A. For $\varphi \in D_0$, i.e. $\varphi = g(H_0)P_{(-\infty,a)}\psi$,

$$\begin{aligned}\|Ve^{-itH_0}\varphi\| \le &\ \|V(H_0+1)^{-1}F(|x| > \delta t)\|\ \|(H_0+1)\varphi\| \\ &+ \|V(H_0+1)^{-1}\|\ \|F(|x| < \delta t)e^{-itH_0}(H_0+1)\varphi\| \ . \end{aligned} \tag{5.13}$$

The first term is integrable by Lemma 5.4. The second one can be estimated by

$$C\|F(|x| < \delta t)e^{-itH_0}(H_0+1)g(H_0)P_{(-\infty,a)}\psi\| \le C'(1+|t|)^{-2}$$

by Perry's estimate (Theorem 5.3); hence it is integrable. □

The following rather technical looking result will be a key to our proof of asymptotic completeness in Sect.5.6.

Proposition 5.6. Let φ_n be a sequence of vectors converging weakly to zero, with $\|\varphi_n\| = 1$. Then

$$\|(\Omega^- - 1)g(H_0)P_+\varphi_n\| \to 0 \ .$$

As usual, g denotes a C_0^∞-function with support on the (strictly) positive half-axis.

Proof. By Cook's estimate (5.5), we have

$$\|(\Omega^- - 1)g(H_0)P_+\varphi_n\| \le \int_0^\infty \|Ve^{-itH_0}g(H_0)P_+\varphi_n\|\, dt$$

$$\|Ve^{-itH_0}g(H_0)P_+\varphi_n\| = \|V(H_0+1)^{-1}e^{-itH_0}(H_0+1)g(H_0)P_+\varphi_n\|$$

goes to zero since $\varphi_n \xrightarrow{w} 0$, and by our short-range assumption, $V(H_0+1)^{-1}$ is compact. By (5.13), the integrand is bounded by an L^1-function. Therefore the assertion of the proposition follows from Lebesgue's theorem on dominated convergence. □

Proposition 5.6 says that $(\Omega^- - 1)g(H_0)P_+$ is compact. From this fact, one can prove asymptotic completeness fairly quickly (*Mourre* [255], *Perry* [277]). We will give a longer proof which is more intuitive, and which will serve as an introduction to the work of Enss on the three-body problem. We require two detours before returning to Proposition 5.6 in Sect. 5.6.

5.4 RAGE Theorems

In this section, we will prove three versions of the celebrated RAGE theorem. The theorem was originally proven by *Ruelle* [300], and extended by *Amrein* and *Georgescu* [14] and *Enss* [95] (hence the name "RAGE" theorem). The RAGE theorem states that the time mean of certain observables will tend to zero on the continuous subspace H_{cont}.

The theorems are based on the following result on time mean of Fourier transforms:

Theorem 5.7 (Wiener's Theorem). Let μ be a finite (signed) measure on $\mathbb{R}$, and let

$$F(t) = \int e^{-ixt}\, d\mu(x)$$

be its Fourier transform. Then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |F(t)|^2\, dt = \sum_{x\in\mathbb{R}} |\mu(\{x\})|^2 .$$

We remark that the sum $\sum |\mu(\{x\})|^2$ is finite, since μ is finite. Since we will, in essence, give the proof of Wiener's theorem while proving Theorem 5.8 below, we do not give it now.

Theorem 5.8 (RAGE). Let A be a self-adjoint operator.

(1) If C is a compact operator and $\varphi \in \mathrm{H}_{\text{cont}}$, then

$$\frac{1}{T}\int_0^T \|Ce^{-itA}\varphi\|^2\, dt \to 0 \quad \text{as } T\to\infty .$$

(2) If C is bounded and $C(A + i)^{-1}$ is compact, and $\varphi \in \mathrm{H}_{\text{cont}}$, then still

$$\frac{1}{T}\int_0^T \|Ce^{-itA}\varphi\|^2\, dt \to 0 .$$

(3) If C is compact, then

$$\left\| \frac{1}{T}\int_0^T e^{+itA} C P_{\text{cont}}(A) e^{-itA}\, dt \right\| \to 0 \quad \text{as } T\to\infty .$$

The integral in (3) is meant in the strong sense.

If we take $C = F(|x| \leq R)$ [in (2)], then the RAGE theorem tells us that any state in H_{cont} will "infinitely often leave" the ball of radius R. This is indeed what we expect physically.

Proof. We first prove that (1) and (2) follow from (3). Let $\varphi \in \mathrm{H}_{\text{cont}}$. Then

$$\frac{1}{T}\int_0^T \|Ce^{-itA}\varphi\|^2\, dt = \frac{1}{T}\int_0^T \langle \varphi, e^{itA}C^*Ce^{-itA}\varphi\rangle\, dt$$

$$= \left\langle \varphi, \frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}tA} C^* C P_{\mathrm{cont}}(A) \mathrm{e}^{-\mathrm{i}tA}\, dt \varphi \right\rangle$$

$$\leq \left\| \frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}tA} C^* C P_{\mathrm{cont}}(A) \mathrm{e}^{-\mathrm{i}tA}\, dt \right\| \|\varphi\|^2 \to 0$$

by (3), since C^*C is compact. For $\varphi \in D(A) \cap \mathrm{H}_{\mathrm{cont}}(A)$, we write $\varphi = (A + \mathrm{i})^{-1}\psi$ $[\psi \in \mathrm{H}_{\mathrm{cont}}(A)]$. Therefore,

$$\frac{1}{T}\int_0^T \|C\mathrm{e}^{-\mathrm{i}tA}\varphi\|^2\, dt = \frac{1}{T}\int_0^T \|C(A + \mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tA}\psi\|^2\, dt$$

converges to zero, given (1). This implies (2), since C is bounded and $D(A) \cap \mathrm{H}_{\mathrm{cont}}(A)$ is dense in $\mathrm{H}_{\mathrm{cont}}(A)$.

We now come to the proof of (3). Since the compact operator C can be approximated in norm by finite rank operators, it suffices to prove (3) for those operators. Since any operator of finite rank is a (finite) sum of rank 1 operators, we may restrict ourselves to rank 1 operators. Thus, let $C\varphi = \langle \rho, \varphi \rangle \psi$ (the most general operator of rank 1). Then $C^*\varphi = \langle \psi, \varphi \rangle \rho$. Define

$$Q(T) := \frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}tA} C P_{\mathrm{cont}}(A) \mathrm{e}^{-\mathrm{i}tA}\, dt$$

$$= \frac{1}{T}\int_0^T \langle \mathrm{e}^{\mathrm{i}tA} P_{\mathrm{cont}}(A)\rho, \cdot \rangle \mathrm{e}^{\mathrm{i}tA}\psi\, dt \;,$$

we have

$$Q(T)^* = \frac{1}{T}\int_0^T \langle \mathrm{e}^{\mathrm{i}tA}\psi, \cdot \rangle \mathrm{e}^{\mathrm{i}tA} P_{\mathrm{cont}}(A)\rho\, dt \;,$$

and therefore

$$Q(T)Q(T)^*\varphi = \frac{1}{T}\int_0^T \langle \mathrm{e}^{\mathrm{i}tA} P_{\mathrm{cont}}(A)\rho, Q(T)^*\varphi \rangle \mathrm{e}^{\mathrm{i}tA}\psi\, dt$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \langle \mathrm{e}^{\mathrm{i}tA} P_{\mathrm{cont}}\rho, \mathrm{e}^{\mathrm{i}sA} P_{\mathrm{cont}}\rho \rangle \langle \mathrm{e}^{\mathrm{i}sA}\psi, \varphi \rangle \mathrm{e}^{\mathrm{i}tA}\psi\, ds\, dt \;.$$

Therefore,

$$\left\| \frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}tA} C \mathrm{e}^{-\mathrm{i}tA} P_{\mathrm{cont}}(A)\, dt \right\|^2$$

$$= \|Q(T)\|^2 = \|Q(T)Q(T)^*\|$$

$$\leq \frac{1}{T^2}\int_0^T\int_0^T |\langle e^{itA}\rho, e^{isA}P_{\text{cont}}(A)\rho\rangle|\, ds\, dt\, \|\psi\|^2$$

$$\leq \|\psi\|^2\left(\frac{1}{T^2}\int_0^T\int_0^T |\langle P_{\text{cont}}\rho, e^{-i(t-s)A}P_{\text{cont}}\rho\rangle|^2\, ds\, dt\right)^{1/2} .$$

Let μ denote the spectral measure for $P_{\text{cont}}\rho$. Then

$$\frac{1}{T^2}\int_0^T\int_0^T |\langle P_{\text{cont}}\rho, \exp[-i(t-s)A]P_{\text{cont}}\rho\rangle|^2\, ds\, dt$$

$$\leq \frac{1}{T^2}\int_0^T\int_0^T \left|\int \exp[-i(t-s)\lambda]\, d\mu(\lambda)\right|^2 ds\, dt$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \iint \exp[-i(t-s)(\lambda-\kappa)]\, d\mu(\lambda)\, d\mu(\kappa)\, ds\, dt$$

$$= \iint\left[\frac{1}{T}\int_0^T \exp[-it(\lambda-\kappa)]\, dt \right.$$

$$\left. \times \frac{1}{T}\int_0^T \exp[+is(\lambda-\kappa)]\, ds\right] d\mu(\lambda)\, d\mu(\kappa) \quad \text{(by Fubini)} . \tag{5.14}$$

Computing

$$\frac{1}{T}\int_0^T \exp[i(\lambda-\kappa)s]\, ds \frac{1}{T}\int_0^T \exp[-i(\lambda-\kappa)t]\, dt$$

$$= \frac{1}{T^2(\lambda-\kappa)^2}\{\exp[i(\lambda-\kappa)T]-1\}\{\exp[-i(\lambda-\kappa)T]-1\}$$

$$= \frac{1}{T^2(\lambda-\kappa)^2}\{\exp[i(\lambda-\kappa)T/2]-\exp[-i(\lambda-\kappa)T/2]\}$$

$$\cdot\{\exp[-i(\lambda-\kappa)T/2]-\exp[i(\lambda-\kappa)T/2]\}$$

$$= \frac{4\sin^2\{(\lambda-\kappa)T/2\}}{T^2(\lambda-\kappa)^2}$$

with the convention that $\sin 0/0 = 1$. Since

$$\frac{4\sin^2\{(\lambda-\kappa)T/2\}}{T^2(\lambda-\kappa)^2} \leq 1$$

[which is in $L^2(d\mu)$], and since furthermore

$$\frac{4\sin^2(\lambda-\kappa)T/2}{T^2(\lambda-\kappa)^2}$$

tends to zero for $\lambda \neq \kappa$, and to one for $\lambda = \kappa$ as $T \to \infty$, we have that (5.14) tends to

$$\int \mu(\{\kappa\})\,d\mu(\kappa) = \sum_{\kappa\in\mathbf{R}} \mu(\{\kappa\})^2$$

by Lebesgue's theorem on dominated convergence. Since the measure μ (the spectral measure for $P_{\text{cont}}\rho$) is continuous, i.e. does not have atoms, we know that $\sum_{\kappa\in\mathbf{R}}\mu(\{\kappa\})^2 = 0$. $\square$

We will make use of the RAGE theorem in Sect.5.5, as well as in the chapter on random Jacobi matrices.

5.5 Asymptotics of Observables

In this section, we are concerned with recent developments of time-dependent scattering theory due to *Enss* [98, 100]. These new ideas present, in the two-body case, more physical insight and simplify the proof of asymptotic completeness for long-range forces. Furthermore, they are an essential ingredient for Enss' three-body proof.

The main result of this section states that some observables, $B(t) = \exp(\mathrm{i}Ht)B\exp(-\mathrm{i}Ht)$, behave on H_{cont} asymptotically in time in a similar way as they would under the free time evolution, more precisely: $(x(t)/t)^2 \sim 2H$, $A(t)/t \sim 2H$, $H_0(t) \sim H$.

Theorem 5.9. For $f\in C_\infty(\mathbb{R})$ and any $\varphi\in\mathrm{H}_{\text{cont}}(H)$:

(i) $f\left(\left(\dfrac{x(t)}{t}\right)^2\right)\varphi \to f(2H)\varphi$

(ii) $f\left(\dfrac{A(t)}{t}\right)\varphi \to f(2H)\varphi$

(iii) $f(H_0(t))\varphi \to f(H)\varphi$ as $t\to\pm\infty$.

Remark. The only assumptions on V we need for the proof below are $D(H) = D(H_0)$, and $[A, V]$ is a compact operator from H_{+2} to H_{-2}. For a proof under very weak assumptions allowing long-range forces, see [98]. Before proving the theorem, we first state and prove two of its consequences.

Corollary 1. For $\varphi\in\mathrm{H}_{\text{cont}}$

$$\|P_-\mathrm{e}^{-\mathrm{i}tH}\varphi\| \to 0 \quad \text{as } t\to\infty \quad \text{and}$$

$$\|P_+\mathrm{e}^{-\mathrm{i}tH}\varphi\| \to 0 \quad \text{as } t\to -\infty\ .$$

IAS/Park City Mathematics Series
Volume 00, Pages 000–000
S 1079-5634(XX)0000-0

Arithmetic spectral transitions: a competition between hyperbolicity and the arithmetics of small denominators

Svetlana Jitomirskaya, Wencai Liu, and Shiwen Zhang

Abstract. These lectures first cover the basics of discrete ergodic Schrodinger operators, with a focus on the 1D quasiperiodic case and the interplay between arithmetics and hyperbolicity. We then present two methods that led to sharp arithmetic spectral transition results. On the localization side, we present a method to prove 1D Anderson localization in the regime of positive Lyapunov exponents, that has, in particular, allowed to solve the arithmetic spectral transition (from absolutely continuous to singular continuous to pure point spectrum) problem for the almost Mathieu operator, in coupling, frequency and phase. On the other end of the arithmetic hierarchy, we present a method to prove quantitative delocalization for 1D operators, leading to sharp arithmetic criterion for the transition to full spectral dimensionality in the singular continuous regime, for the entire class of analytic quasiperiodic potentials.

Contents

2010 *Mathematics Subject Classification.* Primary: 47B36; Secondary: 81Q10,37D25.
Key words and phrases. Park City Mathematics Institute.

1. Introduction

Unlike random, one-dimensional quasiperiodic operators feature spectral transitions with changes of parameters. The transitions between absolutely continuous and singular spectra are governed by vanishing/non-vanishing of the Lyapunov exponent. In the regime of positive Lyapunov exponents there are also more delicate transitions: between localization (point spectrum with exponentially decaying eigenfunctions) and singular continuous spectrum, and dimensional/quantum dynamics transitions within the regime of singular continuous spectrum, governed by the arithmetics. Delicate dependence of spectral properties on the arithmetics is perhaps the most mathematically fascinating feature of quasiperiodic operators, made particularly prominent by Douglas Hofstadter's famous plot of spectra of the almost Mathieu operators, the Hofstadter's butterfly [21], see Figure 1.0.1, demonstrating their self-similarity governed by the continued fraction expansion of the magnetic flux.

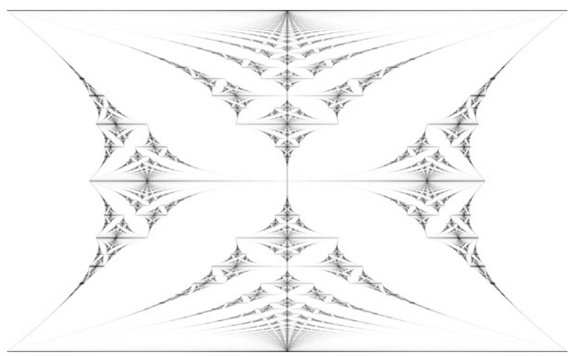

FIGURE 1.0.1. Hofstadter's butterfly

This self-similarity is even more remarkable because it appears even in various experimental and quantum computing contexts, see e.g. Figure 1.0.2.

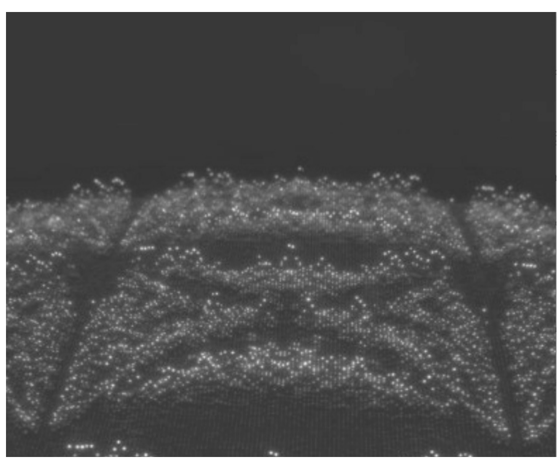

FIGURE 1.0.2. Photon spectrum simulated using a chain of 9 super-conducting quantum qubits [42]

Remarkably, such self-similarity of both spectra and eigenfunctions were predicted a dozen years before Hofstadter in the work of Mark Azbel [11], which, according to Hofstadter, was way ahead of its time. The self-similar behavior of eigenfunctions reflects the self-similar nature of resonances that are in competition with hyperbolicity provided by the Lyapunov growth. This competition also leads to the sharp transition between pure point (hyperbolicity wins) and singular continuous (resonances win) spectra in the positive Lyapunov exponent regime.

In the first three lectures we will outline a method to prove 1D Anderson localization in the regime of positive Lyapunov exponents that has allowed to solve the sharp arithmetic spectral transition problem (from absolutely continuous to singular continuous to pure point spectrum) for the almost Mathieu operator, in coupling, frequency and phase, and to describe the self-similar structure of localized eigenfunctions. The method is an adaptation of [24, 30], but has its roots in [34] and even [32], with an important development in [4]. The last lecture will be devoted to the opposite goal: a method to prove certain delocalization within the regime of singular continuous spectrum (after [27]), that allowed to obtain a sharp arithmetic spectral transition result for the entire class of analytic quasiperiodic potentials.

2. The basics

2.1. Spectral measure of a selfadjoint operator Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$. The time evolution of a wave function is described in the Schrödinger picture of quantum mechanics by

$$i\frac{\partial\psi}{\partial t} = H\psi.$$

The solution with initial condition $\psi(0) = \psi_0$ is given by

$$\psi(t) = e^{-itH}\psi_0.$$

By the spectral theorem, for any $\psi_0 \in \mathcal{H}$, there is a unique spectral measure μ_{ψ_0} such that

$$(e^{-itH}\psi_0, \psi_0) = \int_{\mathbb{R}} e^{-it\lambda} d\mu_{\psi_0}(\lambda). \tag{2.1.1}$$

2.2. Spectral decompositions Let $\mathcal{H} = \mathcal{H}_{pp} \bigoplus \mathcal{H}_{sc} \bigoplus \mathcal{H}_{ac}$, where

$$\mathcal{H}_\gamma = \{\phi \in \mathcal{H} : \mu_\phi \text{ is } \gamma\}$$

and $\gamma \in \{pp, sc, ac\}$. Here pp (sc, ac) are abbreviations for pure point (singular continuous, absolutely continuous).

The operator H preserves each $\mathcal{H}_\gamma$, where $\gamma \in \{pp, sc, ac\}$. We may then define: $\sigma_\gamma(H) = \sigma(H|_{\mathcal{H}_\gamma})$, $\gamma \in \{pp, sc, ac\}$. The set $\sigma_{pp}(H)$ admits a direct characterization as the closure of the set of all eigenvalues

$$\sigma_{pp}(H) = \overline{\sigma_p(H)},$$

where

$$\sigma_p(H) = \{\lambda : \text{ there exists a nonzero vector } \psi \in \mathcal{H} \text{ such that } H\psi = \lambda\psi\}.$$

2.3. Ergodic operators We are going to study discrete one-dimensional Schrödinger operators with potentials related to dynamical systems. Let $H = \Delta + V$ be defined by

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n) \tag{2.3.1}$$

on a Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$. Here $V : \mathbb{Z} \to \mathbb{R}$ is the potential. Let (Ω, P) be a probability space. A measure-preserving bijection $T : \Omega \to \Omega$ is called ergodic, if any T-invariant measurable set $A \subset \Omega$ has either $P(A) = 1$ or $P(A) = 0$. By a dynamically defined potential we understand a family $V_\omega(n) = \nu(T^n\omega), \omega \in \Omega$, where $\nu : \Omega \to \mathbb{R}$ is a measurable function. The corresponding family of operators $H_\omega = \Delta + V_\omega$ is called an ergodic family. More precisely,

$$(H_\omega u)(n) = u(n+1) + u(n-1) + \nu(T^n\omega)u(n). \tag{2.3.2}$$

Theorem 2.3.3 (Pastur [41]; Kunz-Souillard [36])**.** *There exists a full measure set Ω_0 and $\sum$, $\sum_{pp}$, $\sum_{sc}$, $\sum_{ac}$ such that for all $\omega \in \Omega_0$, we have $\sigma(H_\omega) = \sum$, and $\sigma_\gamma(H_\omega) = \sum_\gamma$, $\gamma = pp, sc, ac$.*

Theorem 2.3.4. *[Avron-Simon* [10]*,Last-Simon* [38]*] If T is minimal, then $\sigma(H_\omega) = \sum$, and $\sigma_{ac}(H_\omega) = \sum_{ac}$ for all $\omega \in \Omega$.*

Theorem 2.3.4 does not hold for $\sigma_\gamma(H_\omega)$ with $\gamma \in \{sc, pp\}$ [26], but whether it holds for $\sigma_{\text{sing}}(H_\omega) = \sigma_{pp}(H_\omega) \cup \sigma_{sc}(H_\omega)$ is an interesting and difficult open problem.

2.4. Schnol's theorem Let $H = \Delta + V$ be a Schrödinger operator on $\ell^2(\mathbb{Z})$. We say u is a generalized eigenfunction and E is the corresponding generalized eigenvalue if $Hu = Eu$ and $|u(n)| \leqslant C(1+|n|)^{\frac{1}{2}+\epsilon}$ for some $C, \epsilon > 0$.

Theorem 2.4.1 (Schnol's theorem)**.** *Let S be the set of all generalized eigenvalues. For any $\psi \in \ell^2(\mathbb{Z})$, the spectral measure μ_ψ gives full weight to S and $\sigma(\Delta + V) = \overline{S}$.*

Here we modify the definition a little bit to avoid unnecessary notations. We will say that ϕ is a generalized eigenfunction of H with generalized eigenvalue E, if

$$H\phi = E\phi, \text{ and } |\phi(k)| \leqslant \hat{C}(1+|k|). \tag{2.4.2}$$

In the following, we usually normalize $\phi(k)$ so that

$$\phi^2(0) + \phi^2(-1) = 1. \tag{2.4.3}$$

2.5. Anderson Localization We say a self-adjoint operator H on $\ell^2(\mathbb{Z})$ satisfies Anderson localization if H only has pure point spectrum and all the eigenfunctions decay exponentially. By Schnol's theorem, in order to show the Anderson localization of H, it suffices to prove that all polynomially bounded eigensolutions are exponentially decaying.

This can be done by establishing exponential off-diagonal decay of Green's functions. Block-resolvent expansion, a form of which we are about to see, is the backbone of Fröhlich-Spencer's multi-scale analysis, allowing to pass from smaller to larger scales and from local to global decay. The form we present, first developed for the almost Mathieu operator [32, 34], includes an important modification of multi-scale analysis type arguments, in simultaneously considering shifted boxes. This is the central ingredient in nonperturbative proofs for deterministic potentials [12].

For an interval $I \subset \mathbb{Z}$, let $G_I = (R_I(H_x - I)R_I)^{-1}$ if well defined (G_I is called the Green's function).

Definition 2.5.1. Fix $\tau > 0$, $0 < \delta < 1/2$. A point $y \in \mathbb{Z}$ will be called (τ, k, δ) regular if there exists an interval $[x_1, x_2]$ containing y, where $x_2 = x_1 + k - 1$, such that

$$|G_{[x_1,x_2]}(y, x_i)| \leqslant e^{-\tau|y-x_i|} \text{ and } |y - x_i| \geqslant \delta k \text{ for } i = 1, 2.$$

This definition can be easily made multi-dimensional, with obvious modifications. The following argument is also multi-dimensional but we present a 1D version for simplicity.

First note that for $H\phi = E\phi$, we have $\phi = G_I \Gamma_I \phi$ where Γ_I is the decoupling operator at the boundary of I. In one dimensional case this reads

$$\phi(x) = -G_{[x_1,x_2]}(x_1, x)\phi(x_1 - 1) - G_{[x_1,x_2]}(x, x_2)\phi(x_2 + 1), \tag{2.5.2}$$

where $x \in I = [x_1, x_2] \subset \mathbb{Z}$.

Theorem 2.5.3. *Let* $h(k) \to \infty$ *as* $k \to \infty$. *Suppose* $H\phi = E\phi$ *and* ϕ *satisfies* (2.4.2). *Suppose for any large* $k \in \mathbb{Z}$, k *is* (τ, y, δ) *regular for some* $h(k) \leqslant y \leqslant k$. *Then* H *satisfies Anderson localization. Moreover for any eigenfunction,*

$$\limsup_{n} \frac{\ln|\phi(n)|}{n} \leqslant -\tau\,.$$

Proof. : Under the assumptions, there is some $\hat{k} \geqslant \delta \min_{y\in[\sqrt{k},2k]} h(y)$ such that for any $y \in [\sqrt{k}, 2k]$, there exists an interval $I(y) = [x_1, x_2] \subset [-4k, 4k]$ with $y \in I(y)$ such that

$$\operatorname{dist}(y, \partial I(y)) \geqslant \hat{k} \tag{2.5.4}$$

and

$$|G_{I(y)}(y, x_i)| \leqslant e^{-\tau|y-x_i|},\ i = 1, 2. \tag{2.5.5}$$

Denote by $\partial I(y)$ the boundary of the interval $I(y)$. For $z \in \partial I(y)$, let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 < 2k$ or $x_1 - 1 > \sqrt{k}$, we can expand $\phi(x_2 + 1)$ or $\phi(x_1 - 1)$ as (2.5.2). We can continue this process until we arrive to z such that $z+1 \geqslant 2k$ or $z-1 \leqslant \sqrt{k}$, or the iterating number reaches $[\frac{2k}{\hat{k}}]$, where $[t]$ denotes the greatest integer less than or equal to t.

By (2.5.2),

$$\Big|\phi(k)\Big| = \Big|\sum_{s}\sum_{z_{i+1}\in\partial I(z_i')} G_{I(k)}(k,z_1)\prod_{i=1}^{s}\Big(G_{I(z_i')}(z_i',z_{i+1})\Big)\phi(z_{s+1}')\Big|, \tag{2.5.6}$$

where in each term of the summation we have $\sqrt{k}+1<z_i<2k-1$, $i=1,\cdots,s$, and either $z_{s+1}\notin[\sqrt{k}+2,2k-2]$, $s+1<[\frac{2k}{\hat{k}}]$; or $s+1=[\frac{2k}{\hat{k}}]$.

If $z_{s+1}\notin[\sqrt{k}+2,2k-2]$, $s+1<[\frac{2k}{\hat{k}}]$, by (2.5.5) and noting that we have $|\phi(z_{s+1}')|\leqslant(1+|z_{s+1}'|)^C\leqslant k^C$, one has

$$\begin{aligned}
&\Big|\sum_{z_{i+1}\in\partial I(z_i')} G_{I(k)}(k,z_1)\prod_{i=1}^{s}\Big(G_{I(z_i')}(z_i',z_{i+1})\Big)\phi(z_{s+1}')\Big|\\
&\leqslant e^{-\tau(|k-z_1|+\sum_{i=1}^{s}|z_i'-z_{i+1}|)}\,k^C\\
&\leqslant e^{-\tau(|k-z_{s+1}|-(s+1))}\,k^C\\
&\leqslant \max\{e^{-\tau(k-\sqrt{k}-4-\frac{2k}{\hat{k}})}\,k^C, e^{-\tau(2k-k-4-\frac{2k}{\hat{k}})}\,k^C\}.
\end{aligned} \tag{2.5.7}$$

If $s+1=[\frac{2k}{\hat{k}}]$, using (2.5.4) and (2.5.5), we obtain

$$|G_{I(k)}(k,z_1)G_{I(z_1')}(z_1',z_2)\cdots G_{I(z_s')}(z_s',z_{s+1})\phi(z_{s+1}')|\leqslant k^C e^{-\tau\hat{k}[\frac{2k}{\hat{k}}]}. \tag{2.5.8}$$

Finally, notice that the total number of terms in (2.5.6) is at most $2^{[\frac{2k}{\hat{k}}]}$. Combining with (2.5.7) and (2.5.8), since $k/\hat{k}=o(k)$, we obtain for any $\varepsilon>0$,

$$|\phi(k)|\leqslant e^{-(\tau-\varepsilon)k}$$

for large enough k . For $k<0$, the proof is similar. Thus one has

$$|\phi(k)|\leqslant e^{-(\tau-\varepsilon)|k|} \text{ if } |k| \text{ is large enough.} \tag{2.5.9}$$

□

Therefore we only need to prove that large $k\in\mathbb{Z}$, are $(\tau,h(k),\delta)$ regular for some τ,h,δ.

Lemma 2.5.10. *Suppose* $H\phi=E\phi$ *and* ϕ *satisfies* (2.4.2) *and* (2.4.3). *Then* 0 *is* (τ,k,δ) *singular for any* $\tau,\delta>0$.

Proof. It follows from (2.5.2) immediately. □

Thus it suffices to show that (τ,k,δ) singular points are sufficiently far apart.

2.6. Cocycles and Lyapunov exponents By a cocycle, we mean a pair (T,A), where an invertible $T:\Omega\to\Omega$ is ergodic, A is a measurable 2×2 matrix valued function on Ω and $\det A=1$. This is what is usually called an $SL_2(\mathbb{R})$ cocycle, but we will simply say "a cocycle".

We can regard it as a dynamical system on $\Omega\times\mathbb{R}^2$ with

$$(T,A):(x,f)\longmapsto(Tx,A(x)f),\ (x,f)\in\Omega\times\mathbb{R}^2.$$

For $k>0$, we define the k-step transfer matrix as

$$A_k(x)=\prod_{l=k}^{1}A(T^{l-1}x).$$

For $k < 0$, define $A_k(x) = A_{-k}^{-1}(T^k x)$. Denote $A_0 = I$, where I is the 2×2 identity matrix. Then $f_k(x) = \ln \|A_k(x)\|$ is a subadditive ergodic process. The (non-negative) Lyapunov exponent (LE) for the cocycle (α, A) is given by

$$L(T, A) = \inf_n \frac{1}{n} \int_\Omega \ln \|A_n(x)\| dx \xlongequal{\text{a.e. } x} \lim_{n \to \infty} \frac{1}{n} \ln \|A_n(x)\| dx, \tag{2.6.1}$$

with both the existence and the second equality in (2.6.1) guaranteed by Kingman's subadditive ergodic theorem. Cocycles with positive Lyapunov exponents are called hyperbolic. Here one should distinguish uniform hyperbolicity where there exists a continuous splitting of $\mathbb{R}^2$ into expanding and contracting directions, and nonuniform, where $L > 0$ but such splitting does not exist. Nevertheless,

Theorem 2.6.2 (Oseledec)**.** *Suppose* $L(T, A) > 0$. *Then, for almost every* $x \in \Omega$, *there exist solutions* $v^+, v^- \in \mathbb{C}^2$ *such that* $\|A_k(x)v^\pm\|$ *decays exponentially at* $\pm\infty$, *respectively, at the rate* $-L(T, A)$. *Moreover, for every vector* w *which is linearly independent with* v^+ *(resp.,* v^-*),* $\|A_k(x)w\|$ *grows exponentially at* $+\infty$ *(resp.,* $-\infty$*) at the rate* $L(T, A)$.

Suppose u is an eigensolution of $H_x u = Eu$. Then

$$\begin{bmatrix} u(n+m) \\ u(n+m-1) \end{bmatrix} = A_n(T^m x) \begin{bmatrix} u(m) \\ u(m-1) \end{bmatrix}, \tag{2.6.3}$$

where $A_n(x)$ is the transfer matrix of $A(x)$ and

$$A(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

Such $(T, A(x))$ is called the Schrödinger cocycle. Denote by $L(E)$ the Lyapunov exponent of the Schrödinger cocycle (we omit the dependence on T and v). It turns out that (at least for uniquely ergodic dynamics) the resolvent set of H is precisely the set of uniform hyperbolicity of the Schrödinger cocycle. The set $\sigma \cap \{L(E) > 0\}$ is therefore the set of non-uniform hyperbolicity, and is our main interest. Then Oseledec theorem can be reformulated as

Theorem 2.6.4. *Suppose that* $L(E) > 0$. *Then, for every* $x \in \Omega_E$ *(*Ω_E *has full measure), there exist solutions* ϕ^+, ϕ^- *of* $H_x\phi = E\phi$ *such that* $\phi^\pm$ *decays exponentially at* $\pm\infty$, *respectively, at the rate* $-L(E)$. *Moreover, every solution which is linearly independent of* ϕ^+ *(resp.,* ϕ^-*) grows exponentially at* $+\infty$ *(resp.,* $-\infty$*) at the rate* $L(E)$.

It turns out that the set where the Lyapunov exponent vanishes fully determines the absolutely continuous spectrum.

Theorem 2.6.5 (Ishii-Pastur-Kotani)**.** $\sigma_{ac}(H_x) = \overline{\{E \in \mathbb{R} : L(E) = 0\}}^{ess}$ *for almost every* $x \in \Omega$.

The inclusion "$\subseteq$" was proved by Ishii and Pastur [22, 41]. The other inclusion was proved by Kotani [35, 43]. Here we give a proof of the Ishii-Pastur part.

Proof. Denote $\mathcal{Z} = \{E \in \mathbb{R} : L(E) = 0\}$. If $L(E) > 0$, Oseledec' Theorem says that for almost every x, the eigensolution $u(x, E)$ of $H_x u = Eu$ is either exponentially decaying or exponentially growing. Applying Fubini's theorem, we see that for almost every x (with respect to P), the set of $E \in \mathbb{R} \setminus \mathcal{Z}$ for which the property just described fails, has zero Lebesgue measure. In other words, let $S_1 \subset \mathbb{R} \setminus \mathcal{Z}$ be the set with the non-Oseledec behavior. Then S_1 has zero Lebesgue measure. It implies that S_1 has zero weight with respect to the absolutely continuous part of any spectral measure. Let $S_2 \subset \mathbb{R} \setminus \mathcal{Z}$ be the set with the Oseledec behavior. To prove the Theorem, it suffices to show S_2 has zero weight with respect to any ac spectral measure. Indeed, if the solution of $H_x u = Eu$ is exponentially growing at ∞ or $-\infty$, by Schnol's theorem, such E does not make any contribution to the spectral measure. If the solution of $H_x u = Eu$ is exponentially decaying at both ∞ and $-\infty$, then E is an eigenvalue. The collection of eigenvalues must be countable, which also gives zero weight with respect to the ac spectral measure. □

It may seem that positive Lyapunov exponent should imply pure point spectrum with exponentially localized eigenfunctions, since, as above, for every E and a.e. phase a solution, if polynomially bounded, must decay exponentially on both sides. However, this is a flawed argument because, for a given phase, spectral measures may potentially be supported on the zero measure set of E, excluded by the Fubini theorem, for which there may be no such behavior. It turns out this is not a nuisance to disprove in relevant situations, but actually does happen in some of the prominent examples.

2.7. Example: The Almost Mathieu Operator The almost Mathieu operator (AMO) is the (discrete) quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta} u)(n) = u(n+1) + u(n-1) + 2\lambda \cos 2\pi(\theta + n\alpha) u(n), \tag{2.7.1}$$

where λ is the coupling, α is the frequency, and θ is the phase.

For the AMO, $L(E)$ can be computed exactly for E on the spectrum, but for now we will just need an estimate $L(E) \geqslant \ln \lambda$ for all $\alpha \notin \mathbb{Q}, E$ (See Theorem 3.0.2 for details). Thus, for $\lambda > 1$, Lyapunov exponent is strictly positive on the spectrum. In fact, we will later see that it does not even feel the arithmetics and is constant in the spectrum in both E and α.

We now quickly review the basics of continued fraction approximations.

2.8. Continued fraction expansion Define, as usual, for $0 \leqslant \alpha < 1$,

$$a_0 = 0, \alpha_0 = \alpha,$$

and, inductively for $k > 0$,

$$a_k = \lfloor \alpha_{k-1}^{-1} \rfloor, \alpha_k = \alpha_{k-1}^{-1} - a_k.$$

We define

$$p_0 = 0, \quad q_0 = 1,$$
$$p_1 = 1, \quad q_1 = a_1,$$

and inductively,

$$p_k = a_k p_{k-1} + p_{k-2},$$
$$q_k = a_k q_{k-1} + q_{k-2}.$$

Recall that $\{q_n\}_{n\in\mathbb{N}}$ is the sequence of denominators of best rational approximants to irrational number α, since it satisfies

$$\text{for any } 1 \leqslant k < q_{n+1}, \|k\alpha\|_{\mathbb{R}/\mathbb{Z}} \geqslant \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}}. \tag{2.8.1}$$

Moreover, we also have the following estimate,

$$\frac{1}{2q_{n+1}} \leqslant \Delta_n \triangleq \|q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leqslant \frac{1}{q_{n+1}}. \tag{2.8.2}$$

Here, we give several arithmetic conditions on α:

- α is called Diophantine if there exists $\kappa, \nu > 0$ such that $\|k\alpha\| \geqslant \frac{\nu}{|k|^\kappa}$ for any $k \neq 0$, where $\|x\| = \min_{k\in\mathbb{Z}} |x-k|$.
- α is called Liouville if

$$\beta(\alpha) = \limsup_{k\to\infty} \frac{-\ln\|k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n} > 0 \tag{2.8.3}$$

- α is called weakly Diophantine if $\beta(\alpha) = 0$.

Clearly, Diophantine implies weakly Diophantine. By Borel-Cantelli lemma, Diophantine α form a set of full Lebesgue measure.

Lemma 2.8.4 (Gordon [18], Avron-Simon [9])**.** *Suppose* $v \in C^1(\mathbb{T})$*. There is some constant* C *such that if* $\beta(\alpha) > C$*, then* $\sigma_{pp}(H_{v,\alpha,\theta}) = \emptyset$*.*

Remark: The constant in Lemma 2.8.4 can be estimated in a sharp way [4,8].

Lemma 2.8.4 is the first indication of the role of arithmetics in the spectral theory of quasiperiodic operators in the regime of positive LE, as it demonstrates the necessity of imposing an arithmetic condition.

Let us now denote

$$P_k(x) = \det(R_{[0,k-1]}(H_x - E)R_{[0,k-1]}).$$

It is easy to check by induction that

$$A_k(x) = \begin{pmatrix} P_k(x) & -P_{k-1}(Tx) \\ P_{k-1}(x) & -P_{k-2}(Tx) \end{pmatrix}. \tag{2.8.5}$$

Thus in the regime of positive $L(E)$, P_k "typically" behaves as $e^{kL(E)}$.

By Cramer's rule, for given x_1 and $x_2 = x_1 + k - 1$, with $y \in I = [x_1, x_2] \subset \mathbb{Z}$, one has

$$|G_I(x_1, y)| = \left| \frac{P_{x_2-y}(T^{y+1}x)}{P_k(T^{x_1}x)} \right| \tag{2.8.6}$$

and

$$|G_I(y, x_2)| = \left| \frac{P_{y-x_1}(T^{x_1}x)}{P_k(T^{x_1}x)} \right|. \tag{2.8.7}$$

Thus if P_k indeed hadn't deviated much from $e^{kL(E)}$, we would immediately have exponential decay of both terms. It turns out that for uniquely ergodic T there are no bad deviations for the numerator.

Lemma 2.8.8 ([15]). *Suppose T is uniquely ergodic, continuous and A is continuous. Then*

$$L(T,A) = \lim_{n\to\infty} \sup_{x\in\Omega} \frac{1}{n} \ln \|A_n(x)\|. \tag{2.8.9}$$

Under the assumptions of Lemma 2.8.8, we have for $\varepsilon > 0$,

$$|P_k(\theta)|, \|A_k(x)\| \leqslant e^{(L+\varepsilon)k}, \text{for } k \text{ large enough.} \tag{2.8.10}$$

Thus all deviations can only happen on the lower side. We denote the large deviation set by $A_{k,\epsilon} = \{x : |P_k(x)| < \exp\big((k+1)(L-\epsilon)\big)\}$.

Lemma 2.8.11. *Assume x is $(L-\epsilon, k, \frac{1}{4})$-singular. Then, for large k, we can choose $j \in I_{k,x} = [x-3k/4, x-k/4]$ so that $T^{j+k-1}x \notin A_{k,\frac{1}{4}\epsilon+\epsilon_1}$ for any $\epsilon_1 > 0$.*

Thus, two $(L-\epsilon, k, \frac{1}{4})$-singular points x_1, x_2 such that I_{k,x_1} and I_{k,x_2} do not intersect, produce two long strings of consecutive iterations that fall into the large deviation set.

3. Basics for the Almost Mathieu Operators

It is easy to see that $P_k(\theta)$ is an even function of $\theta + \frac{1}{2}(k-1)\alpha$ and can be written as a polynomial of degree k in $\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)$:

$$P_k(\theta) = \sum_{j=0}^{k} c_j \cos^j 2\pi(\theta + \frac{1}{2}(k-1)\alpha) \triangleq Q_k(\cos 2\pi(\theta + \frac{1}{2}(k-1)\alpha)),$$

where Q_k is an algebraic polynomial of degree k.

For the almost Mathieu operator, the transfer matrix is given by

$$A_k(\theta) = \prod_{j=k-1}^{0} A(\theta + j\alpha) = A(\theta + (k-1)\alpha)A(\theta + (k-2)\alpha)\cdots A(\theta) \tag{3.0.1}$$

and $A(\theta) = \begin{pmatrix} E - 2\lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}$.

By Herman's trick [12,20], we get the following lower bound estimate for $\lambda > 1$,

Theorem 3.0.2.

$$\int_{\mathbb{T}} (\ln |P_k|) d\theta \geqslant k \ln \lambda; \int_{\mathbb{T}} (\ln \|A_k\|) d\theta \geqslant k \ln \lambda. \tag{3.0.3}$$

For the AMO, the Lyapunov exponent on the spectrum actually can be obtained explicitly.

Theorem 3.0.4 ([13]). *For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\lambda \in \mathbb{R}$ and $E \in \sigma(H_{\lambda,\alpha,\theta})$, one has $L_{\lambda,\alpha}(E) = \max\{\ln \lambda, 0\}$.*

Moreover, one can even compute the Lyapunov exponent L^{ϵ} of a complexified cocycle $A(x+i\epsilon)$. It leads to the following three cases (see [2,3] for more general definitions).

Subcritical: $\lambda < 1$. In this case, we have $L^{\epsilon}(E) = 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$ and $\epsilon \leqslant \frac{-\ln\lambda}{2\pi}$. $H_{\lambda,\alpha,\theta}$ has purely ac spectrum [1,5].

Critical: $\lambda = 1$. In this case, it can be shown that $L(E) = 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$, but $L^{\epsilon}(E) > 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$ and $\epsilon > 0$. $H_{\lambda,\alpha,\theta}$ has purely sc spectrum [6,7,23,37].

Supercritical: $\lambda > 1$. $L(E) = \ln\lambda > 0$ for $E \in \sigma(H_{\lambda,\alpha,\theta})$.

In these lectures, we are interested only in the supercritical regime, $\lambda > 1$. In the following we always assume $E \in \sigma(H_{\lambda,\alpha,\theta})$.

The fact that $P_k(\theta) = Q_k\Big(\cos 2\pi\big(\theta + \frac{1}{2}(k-1)\alpha\big)\Big)$, hence is a polynomial in $\cos 2\pi\big(\theta + \frac{1}{2}(k-1)\alpha\big)$ allows the use of the following Lagrange interpolation trick. Note that by Lagrange interpolation, $Q_k(x) = \sum_{j=1}^{k} \prod_{i\neq j} Q_k(x_j)\frac{x-x_i}{x_j-x_i}$. Thus if $\theta_i, i = 1, ..., k+1$, are in the large deviation set, we must have for some i,

$$\max_{x\in[-1,1]} \prod_{j=1, j\neq i}^{k+1} \frac{|x-\cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} > e^{k\epsilon}.$$

This motivates

Definition 3.0.5. We say that the set $\{\theta_1, \cdots, \theta_{k+1}\}$ is ϵ-uniform if

$$\max_{x\in[-1,1]} \max_{i=1,\cdots,k+1} \prod_{j=1, j\neq i}^{k+1} \frac{|x-\cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \leqslant e^{k\epsilon}. \tag{3.0.6}$$

This is a convenient way of stating that the θ_i have low discrepancy since $\int \ln|a - \cos 2\pi x| dx = -\ln 2$ for any $a \in [-1, 1]$.

We have the following Lemma.

Lemma 3.0.7. *Suppose* $\{\theta_1, \cdots, \theta_{k+1}\}$ *is* ϵ_1*-uniform. Then there exists a* θ_i *in the set* $\{\theta_1, \cdots, \theta_{k+1}\}$ *such that* $\theta_i - \frac{k-1}{2}\alpha \notin A_{k,\epsilon}$, *if* $\epsilon > \epsilon_1$ *and* k *is sufficiently large.*

We also have

Lemma 3.0.8. [4, Lemma 9.7] *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $x \in \mathbb{R}$ *and* $0 \leqslant \ell_0 \leqslant q_n - 1$ *be such that* $|\sin\pi(x+\ell_0\alpha)| = \inf_{0\leqslant\ell\leqslant q_n-1} |\sin\pi(x+\ell\alpha)|$, *then for some absolute constant* $C > 0$,

$$-C\ln q_n \leqslant \sum_{\ell=0,\ell\neq\ell_0}^{q_n-1} \ln|\sin\pi(x+\ell\alpha)| + (q_n-1)\ln 2 \leqslant C\ln q_n. \tag{3.0.9}$$

4. First transition line for Diophantine frequencies and phases

We already know that non-Diophantine frequencies are trouble for localization, so let's fix a Diophantine α. It turns out, somewhat surprisingly, that the phase θ matters as well.

- θ is called Diophantine with respect to α (or just $\alpha-$Diophantine) if there exists $\kappa, \nu > 0$ such that $\|2\theta + k\alpha\| \geqslant \frac{\nu}{|k|^\kappa}$ for any $k \neq 0$.
- θ is called Liouville with respect to α if
$$\delta(\alpha, \theta) = \limsup_{k\to\infty} \frac{-\ln \|2\theta + k\alpha\|_{\mathbb{R}/\mathbb{Z}}}{|k|} > 0$$
- θ is called weakly Diophantine with respect to α if $\delta(\alpha, \theta) = 0$.

By Borel-Cantelli lemma, for fixed α, the set of $\alpha-$Diophantine has full Lebesgue measure. We have

Lemma 4.0.1 (J.-Simon [26]). *For even functions $v \in C^1(\mathbb{T})$, there exists some constant $C > 0$ such that if $\delta(\alpha, \theta) > C$, then $\sigma_{pp}(H_{v,\alpha,\theta}) = \emptyset$.*

Thus we need Diophantine-type conditions on both α and θ. In this section, we will prove

Theorem 4.0.2. *Suppose α is Diophantine and θ is Diophantine with respect to α. Then the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization.*

Remark 4.0.3.
- Theorem 4.0.2 was proved in [34]. Here the frame of the proof follows [34], with some modifications from [30, 39, 40].
- Actually, the proof of Theorem 4.0.2 holds also for weakly Diophantine frequencies and phases.

Let E be a generalized eigenvalue with generalized eigenfunction ϕ. Without loss of generality, assume $\phi(0) = 1$ (sometimes we assume $\phi^2(0) + \phi^2(1) = 1$). Take $k > 0$. Let n be such that $q_n \leqslant \frac{k}{4} < q_{n+1}$. Set I_1 and I_2 as follows:

$$I_1 = [-q_n, q_n - 1] \tag{4.0.4}$$

and

$$I_2 = [k - q_n, k + q_n - 1]. \tag{4.0.5}$$

The set $\{\theta_j\}_{j\in I_1\cup I_2}$ consists of $4q_n$ elements, where $\theta_j = \theta + j\alpha$ and j ranges through $I_1 \cup I_2$.

Since α is Diophantine, one has

$$q_{n+1} \leqslant k^C, k \leqslant q_n^C.$$

Theorem 4.0.6. *For any $\varepsilon > 0$, the set $\{\theta_j\}_{j\in I_1\cup I_2}$ is ε-uniform if n is sufficiently large.*

Proof. We first estimate the numerator in (3.0.6). In (3.0.6), let $x = \cos 2\pi a$ and take the logarithm. One has

$$\begin{aligned}
&\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \\
&= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a + \theta_j)| + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(a - \theta_j)| + (4q_n - 1)\ln 2 \\
&= \sum\nolimits_+ + \sum\nolimits_- + (4q_n - 1)\ln 2,
\end{aligned} \tag{4.0.7}$$

where, in the final line, $\sum_{+}$ and $\sum_{-}$ are the corresponding sums from the second line. Both $\sum_{+}$ and $\sum_{-}$ consist of 4 terms of the form of (3.0.9), plus 4 terms of the form

$$\ln \min_{j=0,1,\cdots,q_n-1} |\sin \pi(x+j\alpha)|, \tag{4.0.8}$$

minus $\ln|\sin\pi(a\pm\theta_i)|$. There exists an interval of length q_n containing i, in both sums. By the minimality, the minimum over this interval is not more than $\ln|\sin\pi(a\pm\theta_i)|$ (). Thus, using (3.0.9) 4 times for each of $\sum_{+}$ and $\sum_{-}$, one has

$$\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j| \leqslant -4q_n \ln 2 + C\ln q_n. \tag{4.0.9}$$

The estimate of the denominator of (3.0.6) requires a bit more work. Without loss of generality, assume $i\in I_1$.

In (4.0.7), let $a=\theta_i$. We obtain

$$\begin{aligned}
&\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j| \\
&= \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(\theta_i+\theta_j)| + \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(\theta_i-\theta_j)| + (4q_n-1)\ln 2 \qquad (4.0.10)\\
&= \sum\nolimits_{+} + \sum\nolimits_{-} + (4q_n-1)\ln 2,
\end{aligned}$$

where now

$$\sum\nolimits_{+} = \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(2\theta+(i+j)\alpha)|, \tag{4.0.11}$$

and

$$\sum\nolimits_{-} = \sum_{j\in I_1\cup I_2, j\neq i} \ln|\sin\pi(i-j)\alpha|. \tag{4.0.12}$$

We first estimate $\sum_{+}$. First $I_1\cup I_2$ can be represented as a disjoint union of four segments B_j, each of length q_n. Applying (3.0.9) to each B_j, we obtain

$$\sum\nolimits_{+} > -4q_n\ln 2 + \sum_{j\in J_1\cup J_2} \ln|\sin\pi\hat{\theta}_j| - C\ln q_n - \ln|\sin 2\pi(\theta+i\alpha)|, \tag{4.0.13}$$

where

$$|\sin\pi\hat{\theta}_j| = \min_{\ell\in B_j} |\sin\pi(2\theta+(\ell+i)\alpha)|. \tag{4.0.14}$$

By the fact that θ is Diophantine with respect to α, we have

$$\ln|\sin\pi\hat{\theta}_j| \geqslant -C\ln|k| \geqslant -C\ln q_n. \tag{4.0.15}$$

Putting (4.0.13)–(4.0.15) together, we have

$$\sum\nolimits_{+} > -4q_n\ln 2 - C\ln q_n. \tag{4.0.16}$$

Now let us estimate $\sum_{-}$. By the fact that α is Diophantine , we have for $i\neq j$, and $i,j\in I_1\cup I_2$,

$$\ln|\sin\pi(\theta_i-\theta_j)| \geqslant \ln|k|^{-C} \geqslant -C\ln q_n. \tag{4.0.17}$$

Replacing (4.0.15) with (4.0.17) and using the same argument as for $\sum_{+}$, we get a similar estimate,

$$\sum_{-} > -4q_n \ln 2 - C \ln q_n. \tag{4.0.18}$$

From (4.0.10), (4.0.16) and (4.0.18), we have for any $\varepsilon > 0$,

$$\max_{i \in I_1 \cup I_2} \left(\prod_{j \in I_1 \cup I_2, j \neq i} \frac{|\cos 2\pi a - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \right) < e^{(4q_n - 1)\varepsilon}, \tag{4.0.19}$$

for n large enough. □

Theorem 4.0.20. *Fix any $\varepsilon > 0$. For any large $k \in \mathbb{Z}$, k is $(\ln\lambda - \varepsilon, y, \frac{1}{4})$ regular for some $k^{\frac{1}{C}} \leqslant y \leqslant k$.*

Proof. Define I_1 and I_2 as in (4.0.4) and (4.0.5). Take $y = 4q_n$. Then, $k^{\frac{1}{C}} \leqslant y \leqslant k$. By Lemma 3.0.7, there exists some j_0 with $j_0 \in I_1 \cup I_2$ such that

$$\theta_{j_0} - \frac{4q_n - 1}{2}\alpha \notin A_{4q_n - 1, \varepsilon}.$$

By Lemma 2.8.11, for all $j \in I_1$, $\theta_j - \frac{4q_n-1}{2}\alpha \notin A_{4q_n-1,\varepsilon}$. Thus we have $j_0 \in I_2$.

Set $I = [j_0 - 2q_n + 1, j_0 + 2q_n - 1] = [x_1, x_2]$. By (2.8.6), (2.8.7) and (2.8.10), it is easy to verify

$$|G_I(k, x_i)| \leqslant \exp\{(\ln\lambda + \varepsilon)(4q_n - 1 - |k - x_i|) - 4q_n(\ln\lambda - \varepsilon)\}.$$

Notice that $|k - x_i| \geqslant q_n$, so we obtain

$$|G_I(k, x_i)| \leqslant \exp\{-(\ln\lambda - \varepsilon)|k - x_i|\}. \tag{4.0.21}$$

□

Proof of Theorem 4.0.2. This Theorem now follows by combining Theorems 2.5.3 and 4.0.20. □

5. Asymptotics of the eigenfunctions and proof of the second spectral transition line conjecture

By Theorem 2.6.5, $H_{\lambda,\alpha,\theta}$ does not have ac spectrum for $\lambda > 1$. Lemmas 2.8.4 and 4.0.1 imply that $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $\delta(\alpha, \theta)$ or $\beta(\alpha)$ is large, and we proved that there is Anderson localization if $\beta = \delta = 0$. Is there a sharp transition? The reason large β or δ are trouble is because they lead to resonances: eigenvalues of box restrictions that are too close to each other in relation to the distance between the boxes, leading to small denominators in various expansions. Indeed, large β leads to almost repetitions of the potential, and large δ to almost reflections.

In both these cases, the strength of the resonances is in competition with the exponential growth controlled by the Lyapunov exponent. It was conjectured by the author in 1994 [33] that for the almost Mathieu family the two types of resonances discussed above are the only ones that appear, and that the competition between the Lyapunov growth and resonance strength resolves, in both cases, in a sharp way.

Conjecture 1:

1a: (Diophantine phase) $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\lambda > e^{\beta(\alpha)}$ and $\delta(\alpha,\theta) = 0$, and $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for all θ if $1 < \lambda < e^{\beta(\alpha)}$.

1b: (Diophantine frequency) Suppose $\beta(\alpha) = 0$. Then $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\lambda > e^{\delta(\alpha,\theta)}$, and has purely singular continuous spectrum if $1 < \lambda < e^{\delta(\alpha,\theta)}$.

Conjecture 1a says that without phase resonances, if the Lyapunov exponent beats the frequency resonance, Anderson localization follows. . Conjecture 1b says that without frequency resonances, if the Lyapunov exponent beats the phase resonance, then Anderson localization follows. Otherwise, in both cases, $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum.

In order to simplify the presentation, we assume

$$\lim_{n\to\infty} \frac{\ln q_{n+1}}{q_n} = \beta(\alpha). \tag{5.0.1}$$

Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ we define functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$ in the following way. Let $\frac{p_n}{q_n}$ be the continued fraction approximants to α. For any $\frac{q_n}{2} \leqslant k < \frac{q_{n+1}}{2}$, define $f(k), g(k)$ as follows: for $\ell \geqslant 1$, let

$$\bar{r}^n_\ell = e^{-(\ln\lambda - \frac{\ln q_{n+1}}{q_n} + \frac{\ln \ell}{q_n})\ell q_n}.$$

Set also $\bar{r}^n_0 = 1$ for convenience. If $\ell q_n \leqslant k < (\ell+1)q_n$ with $\ell \geqslant 0$, set

$$f(k) = \left(e^{-|k-\ell q_n|\ln\lambda}\right)\bar{r}^n_\ell + \left(e^{-|k-(\ell+1)q_n|\ln\lambda}\right)\bar{r}^n_{\ell+1}, \tag{5.0.2}$$

and

$$g(k) = \left(e^{-|k-\ell q_n|\ln\lambda|}\right)\frac{q_{n+1}}{\bar{r}^n_\ell} + \left(e^{-|k-(\ell+1)q_n|\ln\lambda|}\right)\frac{q_{n+1}}{\bar{r}^n_{\ell+1}}. \tag{5.0.3}$$

The graphs of these functions are shown in Figures 5.0.4 and 5.0.5.

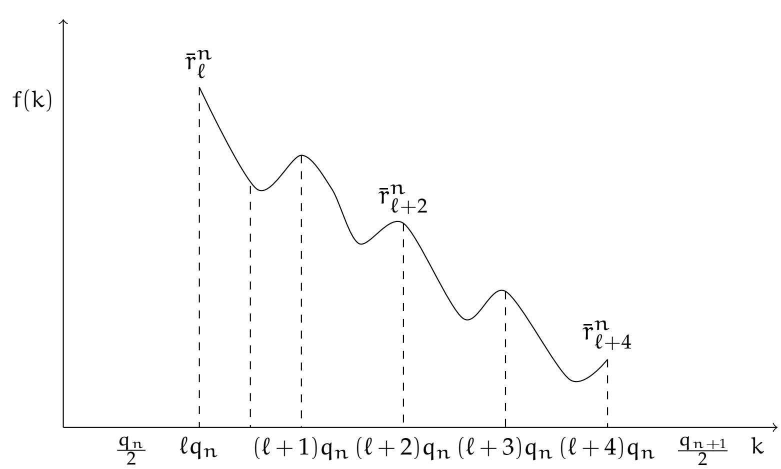

Figure 5.0.4. Graph of $f(k)$.

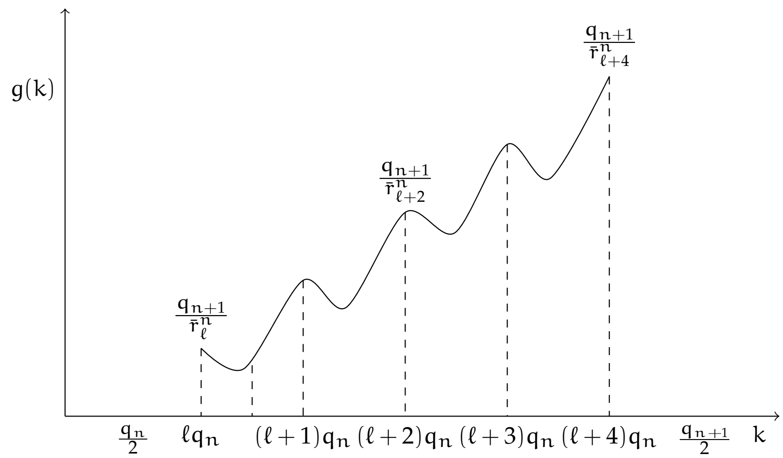

FIGURE 5.0.5. Graph of $g(k)$.

Theorem 5.0.6. [30] *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $\lambda > e^{\beta(\alpha)}$. Suppose θ is Diophantine with respect to α, E is a generalized eigenvalue of $H_{\lambda,\alpha,\theta}$ and ϕ is the generalized eigenfunction. Let $U(k) = \binom{\phi(k)}{\phi(k-1)}$. Then for any $\varepsilon > 0$, there exists K (depending on $\lambda, \alpha, \hat{C}, \varepsilon$ and Diophantine constants κ, ν) such that for any $|k| \geqslant K$, $U(k)$ and A_k satisfy*

$$f(|k|)e^{-\varepsilon|k|} \leqslant \|U(k)\| \leqslant f(|k|)e^{\varepsilon|k|}, \tag{5.0.7}$$

and

$$g(|k|)e^{-\varepsilon|k|} \leqslant \|A_k\| \leqslant g(|k|)e^{\varepsilon|k|}. \tag{5.0.8}$$

By (2.8.3), Theorem 5.0.6 implies the following Theorem.

Theorem 5.0.9. [30]*Suppose θ is Diophantine with respect to α. Then*

1. *$H_{\lambda,\alpha,\theta}$ has Anderson localization if $\lambda > e^{\beta(\alpha)}$.*
2. *$H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum if $1 < \lambda < e^{\beta(\alpha)}$.*
3. *$H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum if $\lambda < 1$.*

Remark 5.0.10. (1) Part 1 of Theorem 5.0.6 holds for $\delta(\alpha, \theta) = 0$.
(2) Part 2 is known for all α, θ [1] and is included here for completeness.
(3) Part 3 is known for all α, θ [8] and is included here for completeness.
(4) Parts 1 and 2 of Theorem 5.0.6 verify the frequency half of the conjecture in [33]. The measure theoretic version was proved in [8,28].

Corollary 5.0.11. *Under the conditions of Theorem 5.0.6, we have*

(I) $\limsup_{k\to\infty} \frac{\ln\|A_k\|}{k} = \ln\lambda.$

(II) $\liminf_{k\to\infty} \frac{\ln\|A_k\|}{k} = \ln\lambda - \beta.$

(III) $\limsup_{k\to\infty} \frac{-\ln\|U(k)\|}{k} = \ln\lambda.$

(IV) $\liminf_{k\to\infty} \frac{-\ln\|U(k)\|}{k} = \ln\lambda - \beta.$

Now let us move to the Diophantine frequency case.

Theorem 5.0.12. [24] *Suppose α is Diophantine. We have*

1. $H_{\lambda,\alpha,\theta}$ *has Anderson localization if* $\lambda > e^{\delta(\alpha,\theta)}$.
2. $H_{\lambda,\alpha,\theta}$ *has purely singular continuous spectrum if* $1 < \lambda < e^{\delta(\alpha,\theta)}$.
3. $H_{\lambda,\alpha,\theta}$ *has purely absolutely continuous spectrum if* $\lambda < 1$.

Remark

(1) Parts 1 and 2 of Theorem 5.0.12 hold for weakly Diophantine α.
(2) We can prove part 2 for all irrational α, and general Lipschitz ν.
(3) Parts 1 and 2 of Theorem 5.0.12 verify the phase half of the conjecture stated in [33].

For the Diophantine frequencies case, we can also get the asymptotics of the eigenfunctions and transfer matrices. For simplicity, we only give the asymptotics of eigenfunctions. For any ℓ, let x_0 (we can choose any one if x_0 is not unique) be such that

$$|\sin\pi(2\theta + x_0\alpha)| = \min_{|x|\leqslant 2|\ell|} |\sin\pi(2\theta + x\alpha)|.$$

Let $\eta = 0$ if $2\theta + x_0\alpha \in \mathbb{Z}$, otherwise let $\eta \in (0,\infty)$ be given by the following equation,

$$|\sin\pi(2\theta + x_0\alpha)| = e^{-\eta|\ell|}. \tag{5.0.13}$$

Define $\hat{f} : \mathbb{Z} \to \mathbb{R}^+$ as follows.

Case 1: If $x_0 \cdot \ell \leqslant 0$, set $\hat{f}(\ell) = e^{-|\ell|\ln\lambda}$.
Case 2: If $x_0 \cdot \ell > 0$, set $\hat{f}(\ell) = e^{-\left((|x_0|+|\ell-x_0|)\ln\lambda\right)} e^{\eta|\ell|} + e^{-|\ell|\ln\lambda}$.

Theorem 5.0.14. [24] *Suppose α is Diophantine. Assume* $\ln\lambda > \delta(\alpha,\theta)$. *If* E *is a generalized eigenvalue and* ϕ *is the corresponding generalized eigenfunction of* $H_{\lambda,\alpha,\theta}$, *then for any* $\varepsilon > 0$, *there exists* K *such that for any* $|\ell| \geqslant K$, $U(\ell)$ *satisfies*

$$\hat{f}(\ell)e^{-\varepsilon|\ell|} \leqslant ||U(\ell)|| \leqslant \hat{f}(\ell)e^{\varepsilon|\ell|}. \tag{5.0.15}$$

6. Universal hierarchical structure for Diophantine phases and universal reflective-hierarchical structure for Diophantine frequencies

In this section, we will describe the universal hierarchical structure of the eigenfunctions in the Diophantine phase case. For Diophantine frequencies there is another, also universal, structure, conjectured to hold, for a.e. phase for all even functions, that features reflective-hierarchy. We refer the readers to [24] for the description of universal relective-hierarchical structure.

Note that Theorem 5.0.6 holds around arbitrary point $k = k_0$. This implies the self-similar nature of the eigenfunctions: $U(k)$ behaves as described at scale q_n but when seen in windows of size $q_k, q_k < q_{n-1}$ will demonstrate the same universal behavior around appropriate local maxima/minima.

To make the above precise, let ϕ be an eigenfunction, and $U(k) = \binom{\phi(k)}{\phi(k-1)}$. Let $I^j_{\sigma_1,\sigma_2} = [-\sigma_1 q_j, \sigma_2 q_j]$, for some $0 < \sigma_1, \sigma_2 \leqslant 1$. We will say k_0 is a local j-maximum of ϕ if $\|U(k_0)\| \geqslant \|U(k)\|$ for $k - k_0 \in I^j_{\sigma_1,\sigma_2}$. Occasionally, we will also use terminology (j,σ)-maximum for a local j-maximum on an interval $I^j_{\sigma,\sigma}$.

We will say a local j-maximum k_0 is *nonresonant* if

$$\|2\theta + (2k_0 + k)\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{{q_{j-1}}^{\nu}},$$

for all $|k| \leqslant 2q_{j-1}$ and

$$\|2\theta + (2k_0 + k)\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{6.0.1}$$

for all $2q_{j-1} < |k| \leqslant 2q_j$.

We will say a local j-maximum is *strongly nonresonant* if

$$\|2\theta + (2k_0 + k)\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{6.0.2}$$

for all $0 < |k| \leqslant 2q_j$.

An immediate corollary of (the proof of) Theorem 5.0.6 is the universality of behavior at all (strongly) nonresonant local maxima.

Theorem 6.0.3. *Given $\varepsilon > 0$, there exists $j(\varepsilon) < \infty$ such that if k_0 is a local j-maximum for $j > j(\epsilon)$, then the following two statements hold:*

If k_0 is nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leqslant \frac{\|U(k_0+s)\|}{\|U(k_0)\|} \leqslant f(|s|)e^{\varepsilon|s|}, \tag{6.0.4}$$

for all $2s \in I^j_{\sigma_1,\sigma_2}$, $|s| > \frac{q_{j-1}}{2}$.

If k_0 is strongly nonresonant, then

$$f(|s|)e^{-\varepsilon|s|} \leqslant \frac{\|U(k_0+s)\|}{\|U(k_0)\|} \leqslant f(|s|)e^{\varepsilon|s|}, \tag{6.0.5}$$

for all $2s \in I^j_{\sigma_1,\sigma_2}$.

Theorem 5.0.6 also guarantees an abundance (and a hierarchical structure) of local maxima of each eigenfunction. Let k_0 be a global maximum .

We first describe the hierarchical structure of local maxima informally. We will say that a scale n_{j_0} is exponential if $\ln q_{n_{j_0}+1} > cq_{n_{j_0}}$.[1] Then there is a constant scale $\hat{n}_0$ thus a constant $C := q_{\hat{n}_0+1}$, such that for any exponential scale n_j and any eigenfunction there are local n_j-maxima within distance no more than C of $k_0 + sq_{n_{j_0}}$ for each $0 < |s| < e^{cq_{n_{j_0}}}$. Moreover, these are the only local n_{j_0}-maxima in the interval $[k_0 - e^{cq_{n_{j_0}}}, k_0 + e^{cq_{n_{j_0}}}]$. The exponential behavior of the eigenfunction in the local neighborhood (of size $q_{n_{j_0}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Note that only exponential behavior at the corresponding scale is determined by f and fluctuations of much smaller size are invisible.

[1]Note that per our simplifying assumption (5.0.1) all scales n are exponential.

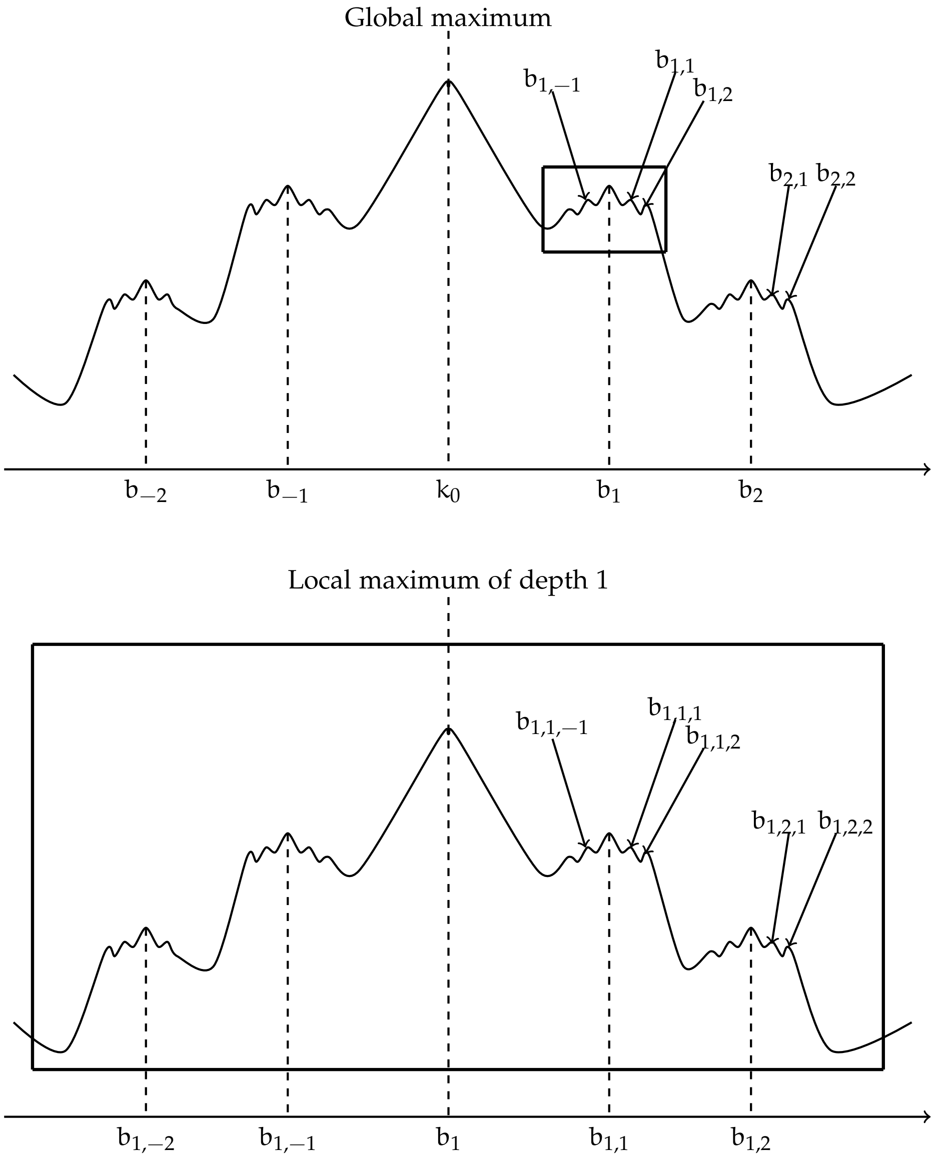

FIGURE 6.0.6. Universal hierarchical structure of an eigenfunction. Above, a view centered on a global maximum with a window centered on a local maximum of depth 1. Below, a magnified view of this local window looks very much like the global view.

Now, let $n_{j_1} < n_{j_0}$ be another exponential scale. Denote the “depth 1” local maximum located near $k_0 + a_{n_{j_0}} q_{n_{j_0}}$ by $b_{a_{n_{j_0}}}$. Near it, we then have a similar picture: there are local n_{j_1}-maxima in the vicinity of $b_{a_{n_{j_0}}} + sq_{n_{j_1}}$ for each $0 < |s| < e^{cq_{n_{j_1}}}$. Again, this describes all the local $q_{n_{j_1}}$-maxima within an exponentially large interval. And again, the exponential (for the n_{j_1} scale) behavior in the local neighborhood (of size $q_{n_{j_1}}$) of each such local maximum, normalized by the value at the local maximum is given by f. Denoting those "depth 2" local maxima located near $b_{a_{n_{j_0}}} + a_{n_{j_1}} q_{n_{j_1}}$, by $b_{a_{n_{j_0}}, a_{n_{j_1}}}$ we then get the same picture taking the magnifying glass another level deeper and so on. At the end we obtain a complete hierarchical structure of local maxima that we denote by $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ with each "depth $s+1$" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ being in the corresponding vicinity of the "depth s" local maximum $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}}$ and with universal behavior at the corresponding scale around each. The accuracy of thge approximations gets lower with each level, yet the depth of the hierarchy that can be so achieved is at least $j/2 - C$. The upper half of Figure 6.0.6 schematically illustrates the structure of local maxima of depth one and two; the lower half shows that the view around a local maximum appropriately magnified looks like a view of the global maximum.

We now describe the hierarchical structure precisely. Suppose

$$\|2(\theta + k_0\alpha) + k\alpha\|_{\mathbb{R}/\mathbb{Z}} > \frac{\kappa}{|k|^{\nu}}, \tag{6.0.7}$$

for any $k \in \mathbb{Z}\backslash\{0\}$. Fix $0 < \sigma, \epsilon$ with $\sigma + 2\epsilon < 1$. Let $n_j \to \infty$ be such that $\ln q_{n_j+1} \geqslant (\sigma + 2\epsilon) \ln \lambda q_{n_j}$. Let

$$\mathfrak{c} = \mathfrak{c}_j = \frac{(\ln q_{n_j+1} - \ln |a_{n_j}|)}{\ln \lambda q_{n_j}} - \epsilon\,.$$

We have $\mathfrak{c}_j > \epsilon$ for $0 < a_{n_j} < e^{\sigma \ln \lambda q_{n_j}}$. Then we have

Theorem 6.0.8. *There exists* $\hat{n}_0(\alpha, \lambda, \kappa, \nu, \epsilon) < \infty$ *such that for any* $j_0 > j_1 > \cdots > j_k$, $n_{j_k} \geqslant \hat{n}_0 + k$, *and* $0 < a_{n_{j_i}} < e^{\sigma \ln \lambda q_{n_{j_i}}}$, $i = 0, 1, \ldots, k$, *for all* $0 \leqslant s \leqslant k$ *there exists a local* n_{j_s}*-maximum* $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ *on the interval* $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} + I^{n_{j_s}}_{\mathfrak{c}_{j_s}, 1}$ *for all* $0 \leqslant s \leqslant k$ *such that the following holds:*

I: $|b_{a_{n_{j_0}}} - (k_0 + a_{n_{j_0}} q_{n_{j_0}})| \leqslant q_{\hat{n}_0+1}$,

II: *For any* $1 \leqslant s \leqslant k$,

$$|b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}} - (b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + a_{n_{j_s}} q_{n_{j_s}})| \leqslant q_{\hat{n}_0+s+1}.$$

III: *If* $2(x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}) \in I^{n_{j_k}}_{\mathfrak{c}_{j_k}, 1}$ *and* $|x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_k}}}| \geqslant q_{\hat{n}_0+k}$, *then for each* $s = 0, 1, \ldots, k$,

$$f(x_s) e^{-\varepsilon |x_s|} \leqslant \frac{\|U(x)\|}{\|U(b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}})\|} \leqslant f(x_s) e^{\varepsilon |x_s|}, \tag{6.0.9}$$

where $x_s = |x - b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}|$ *is large enough.*

Moreover, every local n_{j_s}*-maximum on the interval*

$$b_{a_{n_j}, a_{n_{j_1}}, \ldots, a_{n_{j_{s-1}}}} + [-e^{\epsilon \ln \lambda q_{n_{j_s}}}, e^{\epsilon \ln \lambda q_{n_{j_s}}}]$$

is of the form $b_{a_{n_{j_0}}, a_{n_{j_1}}, \ldots, a_{n_{j_s}}}$ *for some* $a_{n_{j_s}}$.

7. Proof of Theorem 5.0.6

Define $b_n = q_n^t$ with $\frac{8}{9} \leqslant t < 1$ (t will be defined later). For any $k > 0$, we will distinguish two cases with respect to n:

(i) $|k - \ell q_n| \leqslant b_n$ for some $\ell \geqslant 1$, called $n-$*resonance*.

(ii) $|k - \ell q_n| > b_n$ for all $\ell \geqslant 0$, called $n-$*nonresonance*.

Let s be the largest integer such that $4sq_{n-1} \leqslant \mathrm{dist}(y, q_n\mathbb{Z})$.

Theorem 7.0.1. *Assume $\lambda > e^{\beta(\alpha)}$ and that θ is Diophantine with respect to α. Suppose that either*

(1) $b_n \leqslant |y| < Cb_{n+1}$, *where $C > 1$ is a fixed constant, or,*

(2) $0 \leqslant |y| < q_n$.

Then for any $\varepsilon > 0$ and n large enough, if y is $n-$nonresonant, we have y is $(\ln \lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} - \varepsilon, 4sq_{n-1} - 1, \frac{1}{4})$ regular.

Proof. We again assume for simplicity $\lim \frac{\ln q_{n+1}}{q_n} = \beta(\alpha) > 0$. Then we have $s > 0$ for large n. For an n-nonresonant y in the Theorem, one has

$$\min_{j,i \in I_1 \cup I_2} \ln|\sin \pi(2\theta + (j+i)\alpha)| \geqslant -C \ln q_n. \tag{7.0.2}$$

and

$$\min_{i \neq j; i,j \in I_1 \cup I_2} \ln|\sin \pi(j-i)\alpha)| \geqslant -C \ln q_n. \tag{7.0.3}$$

The idea modeled on the proof of Theorem 4.0.6 so we use the same notations.

The upper bound of $\sum_{j \in I_1 \cup I_2, j \neq i} \ln|\cos 2\pi a - \cos 2\pi\theta_j|$ is the same as (4.0.9). (4.0.10)-(4.0.12) also hold. However the estimate of $\sum_{j \in J_1} \ln|\sin \pi\hat{\theta}_j|$ is much more difficult in the non-Diophantine case. Here we sketch the argument.

Assume that $\hat{\theta}_{j+1} = \hat{\theta}_j + q_n\alpha$ for every $j, j+1 \in J_1$. Applying the Stirling formula and (7.0.2), one has

$$\begin{aligned} \sum_{j \in J_1} \ln|\sin 2\pi\hat{\theta}_j| &> 2\sum_{j=1}^{s} \ln \frac{j\Delta_n}{C} - C \ln q_n \\ &> 2s \ln \frac{s}{q_{n+1}} - Cs \ln q_n. \end{aligned} \tag{7.0.4}$$

In the other cases, decompose J_1 in maximal intervals T_κ such that for $j, j+1 \in T_\kappa$ we have $\hat{\theta}_{j+1} = \hat{\theta}_j + q_n\alpha$. Notice that the boundary points of an interval T_κ are either boundary points of J_1 or satisfy $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_n \geqslant \frac{\Delta_{n-1}}{2}$. This follows from the fact that if $0 < |z| < q_n$, then $\|\hat{\theta}_j + q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \leqslant \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} + \Delta_n$, and $\|\hat{\theta}_j + (z+q_n)\alpha\|_{\mathbb{R}/\mathbb{Z}} \geqslant \|z\alpha\|_{\mathbb{R}/\mathbb{Z}} - \|\hat{\theta}_j + q_n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geqslant \Delta_{n-1} - \|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} - \Delta_n$. Assuming $T_\kappa \neq J_1$, then there exists $j \in T_\kappa$ such that $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} \geqslant \frac{\Delta_{n-1}}{2} - \Delta_n$.

If T_κ contains some j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \frac{\Delta_{n-1}}{10}$, then

$$\begin{aligned} |T_\kappa| &\geqslant \frac{\frac{\Delta_{n-1}}{2} - \Delta_n - \frac{\Delta_{n-1}}{10}}{\Delta_n} \\ &\geqslant \frac{1}{4}\frac{\Delta_{n-1}}{\Delta_n} - 1 \geqslant s - 1, \end{aligned} \tag{7.0.5}$$

where $|T_\kappa| = b - a + 1$ for $T_\kappa = [a, b]$. For such T_κ, a similar estimate to (7.0.4) gives

$$\begin{aligned}\sum_{j\in T_\kappa} \ln|\sin\pi\hat{\theta}_j| &> |T_\kappa| \ln\frac{|T_\kappa|}{q_{n+1}} - Cs\ln q_n \\ &> |T_\kappa|\ln\frac{s}{q_{n+1}} - Cs\ln q_n.\end{aligned} \tag{7.0.6}$$

If T_κ does not contain any j with $\|\hat{\theta}_j\|_{\mathbb{R}/\mathbb{Z}} < \frac{\Delta_{n-1}}{10}$, then by (2.8.2)

$$\begin{aligned}\sum_{j\in T_\kappa} \ln|\sin\pi\hat{\theta}_j| &> -|T_\kappa|\ln q_n - C|T_\kappa| \\ &> |T_\kappa|\ln\frac{s}{q_{n+1}} - C|T_\kappa|.\end{aligned} \tag{7.0.7}$$

By (7.0.6) and (7.0.7), one has

$$\sum_{j\in J_1} \ln|\sin\pi\hat{\theta}_j| \geqslant 2s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \tag{7.0.8}$$

Similarly,

$$\sum_{j\in J_2} \ln|\sin\pi\hat{\theta}_j| \geqslant 2s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \tag{7.0.9}$$

We now turn to estimating the quantities $\sum_+$ and $\sum_-$ defined in (4.0.11) and (4.0.12). Putting (4.0.13), (7.0.8) and (7.0.9) together, we have

$$\sum\nolimits_+ > -4sq_n\ln 2 + 6s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \tag{7.0.10}$$

Replacing (7.0.2) with (7.0.3) and proceeding as for $\sum_+$, we have the similar estimate,

$$\sum\nolimits_- > -4sq_n\ln 2 + 4s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \tag{7.0.11}$$

From (4.0.10), (7.0.10) and (7.0.11), it follows that

$$\sum_{j\in I_1\cup I_2, j\neq i} \ln|\cos 2\pi\theta_i - \cos 2\pi\theta_j| > -4sq_n\ln 2 + 8s\ln\frac{s}{q_{n+1}} - Cs\ln q_n. \tag{7.0.12}$$

Combining with (4.0.9), we have for any $\varepsilon > 0$,

$$\begin{aligned}&\max_{i\in I_1\cup I_2} \prod_{j\in I_1\cup I_2, j\neq i} \frac{|\cos 2\pi a - \cos 2\pi\theta_j|}{|\cos 2\pi\theta_i - \cos 2\pi\theta_j|} \\ &\leqslant \exp\left((4sq_n - 1)\left(\ln\lambda + \frac{8\ln(\frac{sq_{n-1}}{q_n})}{q_{n-1}} - \varepsilon\right)\right).\end{aligned}$$

□

Remark 7.0.13. In the nonresonant case, for any $\varepsilon > 0, \frac{8}{9} \leqslant t < 1$, one has $\ln\lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} \geqslant \ln\lambda - 8(1-t)\beta - \varepsilon > 0$. In addition, we have $\ln\lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} \geqslant \ln\lambda - 2\varepsilon$ if t is close to 1.

Remark 7.0.14. Here, we only use Theorem 7.0.1 with $C = 50C_\star$, where $C_\star$ is given by (7.0.15) (see below).

Clearly, it is enough to consider $k > 0$. In this section we study the resonant case. Suppose there exists some $k \in [b_n, b_{n+1}]$ such that k is $n-$resonant. For any $\varepsilon > 0$, choose $\eta = \frac{\varepsilon}{C}$, where C is a large constant (depending on λ, α).

Let

$$C_* = 2(1 + \lfloor \frac{\ln \lambda}{\ln \lambda - \beta} \rfloor), \tag{7.0.15}$$

where $\lfloor m \rfloor$ denotes the smallest integer not exceeding m.

For an arbitrary solution φ satisfying $H\varphi = E\varphi$, let

$$r_j^{n,\varphi} = \sup_{|r| \leqslant 10\eta} |\varphi(jq_n + rq_n)|,$$

where $|j| \leqslant 50C_* \frac{b_{n+1}}{q_n}$.

Let ϕ be the generalized eigenfunction. Denote by

$$r_j^n = r_j^{n,\phi}.$$

Since we keep n fixed in this section we omit the dependence on n from the notation and write r_j^φ, R_j, and r_j.

Note that below we always assume n is large enough.

Lemma 7.0.16. *Let* $k \in [jq_n, (j+1)q_n]$ *with* dist$(k, q_n\mathbb{Z}) \geqslant 10\eta q_n$, *and suppose further that* $|j| \leqslant 48C_* \frac{b_{n+1}}{q_n}$. *Then for sufficiently large* n,

$$\begin{aligned} |\varphi(k)| \leqslant \max\Big\{ & r_j^\varphi \exp\big(-(\ln\lambda - 2\eta)(d_j - 3\eta q_n)\big), \\ & r_{j+1}^\varphi \exp\big(-(\ln\lambda - 2\eta)(d_{j+1} - 3\eta q_n)\big)\Big\}, \end{aligned} \tag{7.0.17}$$

where $d_j = |k - jq_n|$ *and* $d_{j+1} = |k - jq_n - q_n|$.

Proof. The proof builds on the ideas akin to those used in the proof of Theorem 2.5.3. However it requires a more careful approach.

For any $y \in [jq_n + \eta q_n, (j+1)q_n - \eta q_n]$, apply (i) of Theorem 7.0.1 taking $C = 50C_\star$. Notice that in this case, we have

$$\ln \lambda + 8\ln(sq_{n-1}/q_n)/q_{n-1} - \eta \geqslant \ln\lambda - 2\eta.$$

Thus y is regular with $\tau = \ln\lambda - 2\eta$. Therefore there can choose an interval $I(y) = [x_1, x_2] \subset [jq_n, (j+1)q_n]$ such that $y \in I(y)$,

$$\text{dist}(y, \partial I(y)) \geqslant \frac{1}{4}|I(y)| \geqslant q_{n-1} \tag{7.0.18}$$

and

$$|G_{I(y)}(y, x_i)| \leqslant e^{-(\ln\lambda - 2\eta)|y - x_i|},\ i = 1, 2, \tag{7.0.19}$$

where $\partial I(y)$ is the boundary of the interval $I(y)$ (i.e. $\{x_1, x_2\}$), and $|I(y)|$ is the size of $I(y) \cap \mathbb{Z}$ (i.e., $|I(y)| = x_2 - x_1 + 1$). For $z \in \partial I(y)$, let z' be the neighbor of z, (i.e., $|z - z'| = 1$) not belonging to $I(y)$.

If $x_2 + 1 \leqslant (j+1)q_n - \eta q_n$ or $x_1 - 1 \geqslant jq_n + \eta q_n$, we can expand $\varphi(x_2 + 1)$ or $\varphi(x_1 - 1)$ using (2.5.2). We can continue this process until we arrive to z such that $z + 1 > (j+1)q_n - \eta q_n$ or $z - 1 < jq_n + \eta q_n$, or we have iterated $\lfloor \frac{2q_n}{q_{n-1}} \rfloor$ times.

Thus, by (2.5.2)

$$|\varphi(k)| = \Big| \sum_{s; z_{i+1} \in \partial I(z_i')} G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}') \Big|, \tag{7.0.20}$$

where we have $jq_n + \eta q_n + 1 \leqslant z_i \leqslant (j+1)q_n - \eta q_n - 1$, $i = 1, \cdots, s$, in each term of the summation and either $z_{s+1} \notin [jq_n + \eta q_n + 1, (j+1)q_n - \eta q_n - 1]$, $s + 1 < \lfloor \frac{2q_n}{q_{n-1}} \rfloor$, or $s + 1 = \lfloor \frac{2q_n}{q_{n-1}} \rfloor$. We should mention that $z_{s+1} \in [jq_n, (j+1)q_n]$.

If $z_{s+1} \in [jq_n, jq_n + \eta q_n]$, $s + 1 < \lfloor \frac{2q_n}{q_{n-1}} \rfloor$, this implies

$$|\varphi(z_{s+1}')| \leqslant r_j^{\varphi}.$$

By (7.0.19), we have

$$\begin{aligned} &|G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}')| \\ &\leqslant r_j^{\varphi} e^{-(\ln\lambda - 2\eta)\left(|k - z_1| + \sum_{i=1}^{s} |z_i' - z_{i+1}|\right)} \\ &\leqslant r_j^{\varphi} e^{-(\ln\lambda - 2\eta)\left(|k - z_{s+1}| - (s+1)\right)} \\ &\leqslant r_j^{\varphi} e^{-(\ln\lambda - 2\eta)\left(d_j - 2\eta q_n - 4 - \frac{2q_n}{q_{n-1}}\right)}. \end{aligned} \tag{7.0.21}$$

If $z_{s+1} \in [(j+1)q_n - \eta q_n, (j+1)q_n]$, $s + 1 < \lfloor \frac{2q_n}{q_{n-1}} \rfloor$, by the same arguments, we have

$$\begin{aligned} &|G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}')| \\ &\leqslant r_{j+1}^{\varphi} e^{-(\ln\lambda - 2\eta)\left(d_{j+1} - 2\eta q_n - 4 - \frac{2q_n}{q_{n-1}}\right)}. \end{aligned} \tag{7.0.22}$$

If $s + 1 = \lfloor \frac{2q_n}{q_{n-1}} \rfloor$, using (7.0.18) and (7.0.19), we obtain

$$\begin{aligned} &|G_{I(k)}(k, z_1) G_{I(z_1')}(z_1', z_2) \cdots G_{I(z_s')}(z_s', z_{s+1}) \varphi(z_{s+1}')| \\ &\leqslant e^{\left(-(\ln\lambda - 2\eta) q_{n-1} \lfloor \frac{2q_n}{q_{n-1}} \rfloor\right)} |\varphi(z_{s+1}'|. \end{aligned} \tag{7.0.23}$$

Notice that the total number of terms in (7.0.20) is at most $2^{\lfloor \frac{2q_n}{q_{n-1}} \rfloor}$ and that d_j and d_{j+1} are both at least $10\eta q_n$. By (7.0.21)–(7.0.23), we have

$$\begin{aligned} |\varphi(k)| \leqslant \max\Big\{ & r_j^{\varphi} e^{-\left((\ln\lambda - 2\eta)(d_j - 3\eta q_n)\right)}, \\ & r_{j+1}^{\varphi} e^{-\left((\ln\lambda - 2\eta)(d_{j+1} - 3\eta q_n)\right)}, \\ & \max_{p \in [jq_n, (j+1)q_n]} \big\{ e^{-(\ln\lambda - 2\eta) q_n} |\varphi(p)| \big\} \Big\}. \end{aligned} \tag{7.0.24}$$

Now we will show that $p \in [jq_n, (j+1)q_n]$ implies $|\varphi(p)| \leqslant \max\{r_j^{\varphi}, r_{j+1}^{\varphi}\}$. Then (7.0.24) implies case (i) of Lemma 7.0.16. Otherwise, by the definition of r_j^{φ}, if $|\varphi(p')|$ is maximum over $z \in [jq_n + 10\eta q_n + 1, (j+1)q_n - 10\eta q_n - 1]$ of $|\varphi(z)|$, then $|\varphi(p')| > \max\{r_j^{\varphi}, r_{j+1}^{\varphi}\}$. Applying (7.0.24) to $\varphi(p')$ and noticing that $\operatorname{dist}(p', q_n\mathbb{Z}) \geqslant 10\eta q_n$, we get

$$|\varphi(p')| \leqslant e^{\left(-7(\ln\lambda - 2\eta)\eta q_n\right)} \max\big\{ r_j^{\varphi}, r_{j+1}^{\varphi}, |\varphi(p')| \big\}.$$

This is impossible because $|\varphi(p')| > \max\{r_j^{\varphi}, r_{j+1}^{\varphi}\}$. □

By the properties of continued fractions and since θ is α-Diophantine, one can obtain the following estimates:

Lemma 7.0.25. *For any* $|i|,|j| \leqslant 50C_* b_{n+1}$, *the following estimate holds,*

$$\ln|\sin\pi(2\theta+(j+i)\alpha)| \geqslant -C\ln q_n. \tag{7.0.26}$$

Lemma 7.0.27. *Assume* $|i|,|j| \leqslant 50C_* b_{n+1}$, *and* $i-j \neq q_n\mathbb{Z}$. *Then*

$$\ln|\sin\pi(j-i)\alpha| \geqslant -C\ln q_n. \tag{7.0.28}$$

We then have

Theorem 7.0.29. *For* $1 \leqslant j \leqslant 46C_\star \frac{b_{n+1}}{q_n}$, *the following holds*

$$r_j^\varphi \leqslant \max\{r_{j\pm1}^\varphi \frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}\}. \tag{7.0.30}$$

Proof. Fix j with $1 \leqslant j \leqslant 46C_* \frac{b_{n+1}}{q_n}$ and $|r| \leqslant 10\eta q_n$.

Next, define subsets $I_1, I_2 \subset \mathbb{Z}$ as follows

$$I_1 = [-\lfloor\frac{1}{2}q_n\rfloor, q_n - \lfloor\frac{1}{2}q_n\rfloor - 1],$$

$$I_2 = [jq_n - \lfloor\frac{1}{2}q_n\rfloor, (j+1)q_n - \lfloor\frac{1}{2}q_n\rfloor - 1].$$

Let $\theta_m = \theta + m\alpha$ for $m \in I_1 \cup I_2$. The set $\{\theta_m\}_{m\in I_1\cup I_2}$ will thus consist of $2q_n$ elements.

By Lemmas 7.0.25 and 7.0.27, and following the proof of Theorem 4.0.6, one obtains that $\{\theta_m\}$ is $\frac{\ln q_{n+1}-\ln j}{2q_n}+\varepsilon$ uniform for any $\varepsilon > 0$. Combining with Lemma 3.0.7, there exists some j_0 with $j_0 \in I_1 \cup I_2$ such that

$$\theta_{j_0} \notin A_{(2q_n-1),(\ln\lambda - \frac{\ln q_{n+1}-\ln j}{2q_n} - \eta)}.$$

First, we assume $j_0 \in I_2$. Set $I = [j_0 - q_n + 1, j_0 + q_n - 1] = [x_1, x_2]$. In (2.8.10), let $\varepsilon = \eta$. Combining with (2.8.6) and (2.8.7), it is easy to verify

$$\begin{aligned}
&|G_I(jq_n + r, x_i)| \\
\leqslant\ & exp\Big((\ln\lambda+\eta)\big(2q_n - 1 - |jq_n + r - x_i|\big) \\
&\qquad - (2q_n-1)\big(\ln\lambda - \frac{\ln q_{n+1}-\ln j}{2q_n} - \eta\big)\Big).
\end{aligned}$$

Using (2.5.2), we obtain

$$|\varphi(jq_n + r)| \leqslant \sum_{i=1,2} \frac{q_{n+1}}{j} e^{5\eta q_n} |\varphi(x_i')|\, e^{-|jq_n+r-x_i|\ln\lambda}, \tag{7.0.31}$$

where $x_1' = x_1 - 1$ and $x_2' = x_2 + 1$.

Let $d_j^i = |x_i - jq_n|$, $i = 1,2$. It is easy to check that

$$|jq_n + r - x_i| + d_j^i, |jq_n + r - x_i| + d_{j\pm1}^i \geqslant q_n - |r|, \tag{7.0.32}$$

and

$$|jq_n + r - x_i| + d_{j\pm2}^i \geqslant 2q_n - |r|. \tag{7.0.33}$$

If $\mathrm{dist}(x_i, q_n\mathbb{Z}) \geqslant 10\eta q_n$, then we bound $\varphi(x_i)$ in (7.0.31) using (7.0.17). If $\mathrm{dist}(x_i, q_n\mathbb{Z}) \leqslant 10\eta q_n$, then we bound $\varphi(x_i)$ in (7.0.31) by some proper r_j. Combining with (7.0.32), (7.0.33), we have

$$\begin{aligned} r_j^\varphi \leqslant \max\Big\{ & r_{j\pm1}^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}, \\ & r_j^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}, \\ & r_{j\pm2}^\varphi \frac{q_{n+1}}{j} \exp\{-2(\ln\lambda - C\eta)q_n\}\Big\}. \end{aligned}$$

However, we cannot have

$$\begin{aligned} r_j^\varphi &\leqslant r_j^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\} \\ &\leqslant r_j^\varphi \exp\{-(\ln\lambda - \beta - C\eta)q_n\} \end{aligned}$$

so we must have

$$\begin{aligned} r_j^\varphi \leqslant \max\Big\{ & r_{j\pm1}^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}, \\ & r_{j\pm2}^\varphi \frac{q_{n+1}}{j} \exp\{-2(\ln\lambda - C\eta)q_n\}\Big\}. \end{aligned} \tag{7.0.34}$$

In particular,

$$r_j^\varphi \leqslant \exp\{-(\ln\lambda - \beta - C\eta)q_n\} \max\{r_{j\pm1}^\varphi\, r_{j\pm2}^\varphi\}. \tag{7.0.35}$$

If $j_0 \in I_1$, then (7.0.35) holds for $j = 0$. Let $\varphi = \phi$ in (7.0.35). We get

$$|\phi(0)|, |\phi(-1)| \leqslant \exp\{-(\ln\lambda - \beta - C\eta)q_n\},$$

which is in contradiction with $|\phi(0)|^2 + |\phi(-1)|^2 = 1$. Therefore $j_0 \in I_2$, so (7.0.34) holds for any φ.

By (2.6.3) and (2.8.10), we have

$$\left\|\begin{pmatrix} \varphi(k_1) \\ \varphi(k_1 - 1) \end{pmatrix}\right\| \geqslant Ce^{-(\ln\lambda+\varepsilon)|k_1-k_2|}\left\|\begin{pmatrix} \varphi(k_2) \\ \varphi(k_2 - 1) \end{pmatrix}\right\|. \tag{7.0.36}$$

This implies

$$r_{j\pm2}^\varphi \leqslant r_{j\pm1}^\varphi \exp\{(\ln\lambda + C\eta)q_n\},$$

thus (7.0.34) becomes

$$r_j^\varphi \leqslant \max\{r_{j\pm1}^\varphi \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}\}, \tag{7.0.37}$$

for any $1 \leqslant j \leqslant 46C_* \frac{b_{n+1}}{q_n}$. □

We now show that by Theorem 2.4.1 exponential growth is not allowed, r_j must actually decay.

Theorem 7.0.38. *For $1 \leqslant j \leqslant 10\frac{b_{n+1}}{q_n}$, the following holds*

$$r_j \leqslant r_{j-1} \exp\{-(\ln\lambda - C\eta)q_n\}\frac{q_{n+1}}{j}. \tag{7.0.39}$$

Proof. Let $\varphi = \phi$ in Lemma 7.0.29. We must have

$$r_j \leqslant \max\{r_{j\pm1} \frac{q_{n+1}}{j} \exp\{-(\ln\lambda - C\eta)q_n\}\}, \tag{7.0.40}$$

for any $1 \leqslant j \leqslant 46C_* \frac{b_{n+1}}{q_n}$.

Suppose for some $1 \leqslant j \leqslant 10\frac{b_{n+1}}{q_n}$, the following holds,

$$r_j \leqslant r_{j+1}\frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\} \leqslant r_{j+1}\exp\{-(\ln\lambda-\beta-C\eta)q_n\}. \tag{7.0.41}$$

Applying (7.0.40) to $j+1$, we obtain

$$r_{j+1} \leqslant \max\{r_j, r_{j+2}\}\frac{q_{n+1}}{j+1}\exp\{-(\ln\lambda - C\eta)q_n\}. \tag{7.0.42}$$

Combining with (7.0.41), we must have

$$r_{j+1} \leqslant r_{j+2}\exp\{-(\ln\lambda-\beta-C\eta)q_n\}. \tag{7.0.43}$$

Generally, for any $0 < p \leqslant (C_*+1)j-1$, we obtain

$$r_{j+p} \leqslant r_{j+p+1}\exp\{-(\ln\lambda-\beta-C\eta)q_n\}. \tag{7.0.44}$$

Thus

$$r_{(C_*+1)j} \geqslant r_j\exp\{(\ln\lambda-\beta-C\eta)C_* j q_n\}. \tag{7.0.45}$$

Clearly, by (7.0.36), one has

$$r_j \geqslant \exp\{-(\ln\lambda + C\eta)jq_n\}.$$

Then

$$r_{(C_*+1)j} \geqslant \exp\{((C_*-1)\ln\lambda - C_*\beta - C\eta)jq_n\}. \tag{7.0.46}$$

By the definition of C_*, one has

$$(C_*-1)\ln\lambda - C_*\beta > 0.$$

Thus (7.0.46) is in contradiction with the fact that $|\phi(k)| \leqslant 1+|k|$.

Now that (7.0.41) can not happen, from (7.0.40), we must have

$$r_j \leqslant r_{j-1}\frac{q_{n+1}}{j}\exp\{-(\ln\lambda - C\eta)q_n\}. \tag{7.0.47}$$

□

Theorem 7.0.48. *For $1 \leqslant j \leqslant 10\frac{b_{n+1}}{q_n}$, the following holds*

$$r_j \geqslant r_{j-1}\exp\{-(\ln\lambda-\varepsilon)q_n\}\frac{q_{n+1}}{j}. \tag{7.0.49}$$

Proof. See [30] for details. □

We are now ready to complete the proof of Theorem 5.0.6.

Proof of Theorem 5.0.6. Set $t_0 = 1 - \frac{\varepsilon}{8\beta}$. Let $t = t_0$ in the definition of resonance, i.e. $b_n = q_n^{t_0}$.

Case I: $\ell \geqslant q_{n+1}^{t_0}$:

By case II of Theorem 7.0.1, we know that any $y \in (\varepsilon q_{n+1}^{t_0}, q_{n+1} - \varepsilon q_{n+1}^{t_0})$ is $(\ln\lambda + 8\ln(sq_n/q_{n+1})/q_n - \varepsilon, 4sq_n - 1)$ regular with $\delta = \frac{1}{4}$. Notice that

$$(s+1)q_n \geqslant \varepsilon q_{n+1}^{t_0} \geqslant \varepsilon q_{n+1}^{t_0},$$

thus we have

$$\ln\lambda + \frac{8\ln(s\frac{q_n}{q_{n+1}})}{q_n} \geqslant \ln\lambda - 8(1-t_0)\beta - \varepsilon \geqslant \ln\lambda - 2\varepsilon.$$

Thus for any $y \in (\varepsilon q_{n+1}^{t_0}, q_{n+1} - \varepsilon q_{n+1}^{t_0})$, y is $(\ln \lambda - 2\varepsilon, 4sq_n - 1)$ regular. Following the proof of Lemma 7.0.16, one has for $\ell q_n \leqslant k \leqslant (\ell+1)q_n$,

$$\|U(k)\| \leqslant e^{-(\ln \lambda - \varepsilon)|k|},$$

which implies Theorem 5.0.6 in this case.
Case II: $0 \leqslant \ell \leqslant q_{n+1}^{t_0}$:
By Theorems 7.0.48 and 7.0.38, and Stirling's formula,

$$\bar{r}_j^n e^{-\varepsilon j q_n} \leqslant r_j \leqslant \bar{r}_j^n e^{\varepsilon j q_n}.$$

Now Theorem 5.0.6 follows from Lemma 7.0.16. □

8. Arithmetic criteria for spectral dimension

We know that in the regime of positive Lyapunov exponent the spectrum is always singular. Now that we also know (Lemma 2.8.4) that large β implies continuous (and therefore singular continuous) spectrum, it's natural to ask whether even larger β implies increased continuity. "Continuity" of singular continuous spectrum can be quantified through fractal dimensions. The most popular object is Hausdorff dimension. However Hausdorff dimension is a poor tool for characterizing the singular continuous spectral measures arising in the regime of positive Lyapunov exponents, as it is always equal to zero (a very general theorem of Simon that holds for general ergodic potentials and a.e. phase, see Theorem 8.2.6 [44] (and for every phase for the zero entropy dynamical systems [19] (see also [29, 31]). It turns out that some other dimensions do present good tools to finely distinguish between different kinds of singular continuous spectra appearing in the supercritical regime. The main goal of this lecture is to briefly present a simultaneous quantitative version of two well known statements

(1) *Periodicity implies absolute continuity.* We prove that a quantitative weakening (near periodicity that holds sufficiently long) implies quantitative continuity of the (fractal) spectral measure.
(2) *Gordon condition (a single/double almost repetition) implies continuity of the spectral measure.* Indeed, we prove that a strengthening (with multiple almost repetitions) implies quantitative continuity of the spectral measure.

This will allow us to establish a sharp arithmetic criterion for certain dimension of the spectral measure in terms of β, for *general* analytic potentials.

8.1. m-function and subordinacy theory Let μ be a finite Borel measure on $\mathbb{R}$. Define the Borel transform of μ to be:

$$m(z) := \int \frac{1}{E - z} d\mu(E), \quad z \in \mathbb{C}. \tag{8.1.1}$$

It is easy to check that for any finite Borel measure μ on $\mathbb{R}$, its m-function is holomorphic in the upper half plane and satisfies

$$m^*(z) = m(z^*), \quad |m(z)| \leqslant \frac{\mu(\mathbb{R})}{\operatorname{Im} z} \quad z \in \mathbb{C}_+.$$

Remark 8.1.2. Functions with this property are known as Herglotz, Pick or R functions. They map the upper half-plane into itself, but are not necessarily injective or surjective. Note that such an m is holomorphic in $\mathbb{C}\backslash\sigma(\mu)$, where $\sigma(\mu) := \{E \in \mathbb{R} : \mu(E-\varepsilon, E+\varepsilon) > 0 \text{ for all } \varepsilon > 0\}$.

The boundary behavior of m is linked to the Radon-Nikodym derivative $D\mu$ of μ, which in turn determines the decomposition of μ, see e.g. [45].

Theorem 8.1.3. *Let μ be a finite Borel measure and m its Borel transform. Then the limit*

$$\mathrm{Im}\big(m(E)\big) = \lim_{\varepsilon \downarrow 0} \mathrm{Im}\big(m(E+i\varepsilon)\big) \tag{8.1.4}$$

exists a.e. with respect to both μ and Lebesgue measure (finite or infinite) and

$$D\mu(E) = \frac{1}{\pi}\mathrm{Im}\big(m(E)\big) \tag{8.1.5}$$

whenever $D\mu(E)$ exists. Moreover, the set $\{E|\mathrm{Im}\big(m(E)\big) = \infty\}$ is a support for the singular continuous part and $\{E|\mathrm{Im}\big(m(E)\big) < \infty\}$ is a minimal support for the absolutely continuous part.

Fractal properties of μ can also be characterized through m. In the rest of this subsection, we briefly review the power-law extension of the Gilbert-Pearson subordinacy theory [16, 17], developed in [29].

For simplicity, consider the right half line operator (2.3.1) on $\ell^2(\mathbb{Z}^+)$ with boundary condition $u(1) = \cos\varphi, u(0) = \sin\varphi$ for some $\varphi \in (-\pi/2, \pi/2]$. Let μ be the spectral measure. In this case, the Borel transform of μ is also called the Weyl-Titchmarsh m-function.

For any function $u : \mathbb{Z}^+ \to \mathbb{C}$ and $\ell \in \mathbb{R}^+$, define

$$\|u\|_\ell := \Big[\sum_{n=1}^{[\ell]} |u(n)|^2 + (\ell - [\ell])|u([\ell]+1)|^2\Big]^{1/2}. \tag{8.1.6}$$

Suppose u and v solve $Hu = Eu$ with orthogonal boundary conditions

$$\begin{pmatrix} u(1) & v(1) \\ u(0) & v(0) \end{pmatrix} = R_\varphi,$$

where R_φ is a matrix of rotation by φ. Now given any $\varepsilon > 0$, we define a length $\ell(\varepsilon) \in (0, \infty)$ by requiring the equality

$$\|u\|_{\ell(\varepsilon)} \cdot \|v\|_{\ell(\varepsilon)} = \frac{1}{2\varepsilon}. \tag{8.1.7}$$

The function $\ell(\varepsilon)$ is a well defined monotonically decreasing continuous function which goes to infinity as ε goes to 0, and we also have $\frac{1}{2\varepsilon} \geqslant \frac{1}{2}([\ell]-1)$. It turns out that the boundary behavior of $m(E+i\varepsilon)$ is linked in a quantitative way to $\frac{\|u\|_{\ell(\varepsilon)}}{\|v\|_{\ell(\varepsilon)}}$, thus to the power-law behavior of solutions.

Lemma 8.1.8 (J.-Last inequality, [29]). *For $E \in \mathbb{R}$ and $\varepsilon > 0$,*

$$\frac{5-\sqrt{24}}{|m(E+i\varepsilon)|} < \frac{\|u\|_\ell}{\|v\|_\ell} < \frac{5+\sqrt{24}}{|m(E+i\varepsilon)|}. \tag{8.1.9}$$

From Lemma 8.1.8, one can easily recover the results of Gilbert-Pearson [17] with a simpler proof, while strengthening their theory. The above inequality links the power-law behavior of the generalized eigenfunctions of $Hu = Eu$ and the boundary behavior of the Borel transform of the spectral measure μ in a quantitative way. A particular consequence of Lemma 8.1.8 is

Lemma 8.1.10. *For any* $E \in \mathbb{R}$ *and* $0 < \gamma < 1$, *suppose there is a sequence of positive numbers* $\varepsilon_k \to 0$ *and an absolute constant* $C > 0$ *so that both* u, v *satisfy*

$$C^{-1}\ell_k^{\gamma} \leqslant \|u\|_{\ell_k}^2 \leqslant C\ell_k^{2-\gamma} \tag{8.1.11}$$

where $\ell_k = \ell(\varepsilon_k)$ *is given by* (8.1.7). *Then*

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{1-\gamma}|m(E + i\varepsilon)| < \infty. \tag{8.1.12}$$

8.2. Spectral continuity Fix $0 < \gamma < 1$. If (8.1.12) holds for μ a.e. E, we say measure μ is (upper) γ-spectral continuous. Define the (upper) spectral dimension of μ to be

$$s(\mu) = \sup\left\{\gamma \in (0,1) : \ \mu \text{ is } \gamma\text{-spectral continuous}\right\}. \tag{8.2.1}$$

In this part, we focus on the quantitative spectral continuity and the lower bound of the spectral dimension. Our spectral continuity result does not necessarily require quasiperiodic structure of the potential and can be generalized to wider contexts (so-called β-almost periodic potential, see [27], - a class, that includes, for example, some skew shift potentials). The general result of [27] only goes in one direction. However, in the important context of analytic quasiperiodic operators this leads to a sharp if-and-only-if result. Let H be defined as in (2.3.2) with quasiperiodic potential:

$$(Hu)(n) = u(n+1) + u(n-1) + v(\theta + n\alpha)u(n), \quad \theta, \alpha \in \mathbb{T}, \quad v : \mathbb{T} \mapsto \mathbb{R}. \tag{8.2.2}$$

Theorem 8.2.3 ([27])**.** *Let* H *be as in* (8.2.2) *with real analytic potential* v *and* μ *be the spectral measure*[2]. *Assume* $L(E) > 0$ *for all* $E \in \mathbb{R}$. *For any* $\theta \in \mathbb{T}$, $s(\mu) = 1$ *if and only if* $\beta(\alpha) = \infty$.

Remark 8.2.4. The theorem also holds locally for any spectral projection onto the subset where the Lyapunov exponent is positive.

Remark 8.2.5. The 'if' part will be a consequence of Theorem 8.2.7 which can be viewed as a quantitative strengthening of the results of Gordon type (Lemma 2.8.4). The 'only if' part follows from the general analytic Theorem 8.3.1 which can be viewed as a weakening/extension of localization type results for large β.

Spectral continuity captures the lim inf power-law behavior of $m(E + i\varepsilon)$, while the corresponding lim sup behavior is linked to the Hausdorff dimension [14].

[2]Discrete Schrödinger operators may have multiplicity two. However, δ_0, δ_1 always form a cyclic pair, so it is enough to consider the so called maximal spectral measure given by $\mu = \mu_{\delta_0} + \mu_{\delta_1}$, where μ_{δ_0} and μ_{δ_1} are defined as in (2.1.1).

One can easily check that $\dim_H(\mu) \leqslant s(\mu) \leqslant \dim_P(\mu)$, where $\dim_H(\mu)/\dim_P(\mu)$ denote the Hausdorff/packing dimension of a measure in the usual sense.

Theorem 8.2.6 (Simon, [44])**.** *Suppose* H *is an ergodic Schrödinger operator as in* (2.3.2) *with positive Lyapunov exponent. For a.e. phase* ω*,* $\dim_H(\mu) = 0$*.*

Let H be as in (8.2.2) and μ be the spectral measure. We have the following quantitative lower bound of the spectral dimension.

Theorem 8.2.7. *Suppose* v *is Lipschitz continuous. Let*

$$\Lambda := \sup_{E\in\sigma(H),n,\theta} \frac{1}{n} \ln \|A_n(\theta)\|. \tag{8.2.8}$$

There exists an absolute constant $C > 0$ *such that for any* $\theta \in \mathbb{T}$*,*

$$s(\mu) \geqslant 1 - \frac{C\Lambda}{\beta(\alpha)}.$$

The general version of Theorem 8.2.7 is actually more robust and only requires some regularity of v, which allows us to obtain new results for other popular models, such as the critical almost Mathieu operator, Sturmian potentials, and others. Lower bounds on spectral dimension also have immediate applications to the lower bounds on packing/box counting dimensions and on quantum dynamics(upper transport exponents). The method developed in [27] for the bounded $SL(2,\mathbb{R})$ case generalizes to the unbounded case (e.g. the Maryland model) and the non-Schrödinger case (e.g. the extended Harper's model) [46].

For simplicity, we only prove the right half line case and we also assume (5.0.1) holds. According to Lemma 8.1.10, to prove spectral continuity, it is enough to obtain power-law estimate (8.1.11) for half-line solution u of $Hu = Eu$ with any boundary condition φ.

First, for β large, the system can be approximated by a periodic one exponentially fast, in the following sense.

Lemma 8.2.9. *Let* q_n *be given as in* (5.0.1)*. For any* $\beta < \beta(\alpha)$*, any* $\theta \in \mathbb{T}$*, we have*

$$\|A_{q_n}(\theta) - A_{q_n}(\theta + q_n\alpha)\| \leqslant e^{(-\beta+2\Lambda)q_n}. \tag{8.2.10}$$

The ultimate goal is to estimate $\|A_{Nq_n}\|$ by the size of q_n for $N \sim e^{c\beta q_n}$. This eventually leads to the desired power-law for u by (2.6.3). We will conclude this in the end of this part. The standard rational approximation fails here since the error terms may reach the size of $e^{N\Lambda} \sim e^{e^{c\beta q_n}}$. We need some quantitative telescoping arguments.

Lemma 8.2.11. *Suppose* G *is a two by two matrix satisfying*

$$\|G^j\| \leqslant M < \infty, \quad \textit{for all } 0 < j \leqslant N \in \mathbb{N}^+, \tag{8.2.12}$$

where $M \geqslant 1$ *only depends on* N*. Let* $G_j = G + \Delta_j$*,* $j = 1, \cdots, N$*, be a sequence of two by two matrices with*

$$\delta = \max_{1\leqslant j\leqslant N} \|\Delta_j\|. \tag{8.2.13}$$

If

$$NM\delta < 1/2, \tag{8.2.14}$$

then for any $1 \leqslant n \leqslant N$

$$\|\prod_{j=1}^{n} G_j - G^n\| \leqslant 2NM^2\delta. \tag{8.2.15}$$

Combining (8.2.10) with this lemma, one can show that A_{Nq_n} is close to $A_{q_n}^N$ up to the size of $\|A_{q_n}^N\|$. Now the question is reduced from the dynamical behavior of A_{Nq_n} to the algebraic properties of $A_{q_n}^N$. We need some additional linear algebraic facts about $SL(2, \mathbb{R})$ matrices.

Lemma 8.2.16. *Suppose* $G \in SL(2, \mathbb{R})$ *with* $2 < |\mathrm{Tr}\, G| \leqslant 6$. *There exists an invertible matrix* B *such that*

$$G = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1} \tag{8.2.17}$$

where $\rho^{\pm 1}$ *are the two conjugate real eigenvalues of* G, $|\det B| = 1$ *and*

$$\|B\| = \|B^{-1}\| < \frac{\sqrt{\|G\|}}{\sqrt{|\mathrm{Tr}\, G| - 2}}. \tag{8.2.18}$$

If $|\mathrm{Tr}\, G| > 6$, *then* $\|B\| \leqslant \frac{2\sqrt{\|G\|}}{\sqrt{|\mathrm{Tr}\, G| - 2}}$.

Lemma 8.2.19. *Suppose* $G \in SL(2, \mathbb{R})$ *has eigenvalues* $\rho^{\pm 1}$, $\rho > 1$. *For any* $k \in \mathbb{N}$, *if* $\mathrm{Tr}\, G \neq 2$, *then*

$$G^k = \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} \cdot \Big(G - \frac{\mathrm{Tr} G}{2} \cdot I\Big) + \frac{\rho^k + \rho^{-k}}{2} \cdot I. \tag{8.2.20}$$

Otherwise, $G^k = k(G - I) + I$.

Assume further that $\big||\mathrm{Tr}\, G| - 2\big| < \tau < 1$. *Then there exist universal constants* $1 < C_1 < \infty, c_1 > 1/3$ *such that for* $1 \leqslant k \leqslant \tau^{-1}$, *we have*

$$c_1 < \frac{\rho^k + \rho^{-k}}{2} < C_1\ , \quad c_1 k < \frac{\rho^k - \rho^{-k}}{\rho - \rho^{-1}} < C_1 k. \tag{8.2.21}$$

By Lemma 8.2.16, if the trace of A_{q_n} is away from 2, we have the decomposition

$$A_{q_n} = B \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} B^{-1}, \ \ \|B\| = \|B^{-1}\| \leqslant \frac{2\sqrt{\|A_{q_n}\|}}{\sqrt{|\mathrm{Tr}\, A_{q_n}| - 2}} \tag{8.2.22}$$

and the matrix product turns into a scalar product,

$$A_{q_n}^N = B \begin{pmatrix} \rho^N & 0 \\ 0 & \rho^{-N} \end{pmatrix} B^{-1}, \ \ \|A_{q_n}^N\| \leqslant \|B\|^2 |\rho|^N. \tag{8.2.23}$$

By Lemma 8.2.19, when the trace of A_{q_n} is close to 2, $A_{q_n}^N$ behaves almost linearly in N:

$$A_{q_n}^N \sim N\big(A_{q_n} - \frac{1}{2}\mathrm{Tr}\, A_{q_n}\big) + I. \tag{8.2.24}$$

The above two asymptotic behaviors of $A_{q_n}^N$ allow us to study the spectral measure of the following two sets:

$$S_1 = \limsup_{n\to\infty}\{E: \ |\mathrm{Tr}A_{q_n}| > 2 + e^{-10\Lambda q_n}\}, \tag{8.2.25}$$

$$S_2 = \limsup_{n\to\infty}\{E: \big||\mathrm{Tr}A_{q_n}| - 2\big| < e^{-10\Lambda q_n}\}. \tag{8.2.26}$$

To estimate the spectral measure of S_1, we use the idea of a Gordon-type argument to estimate the lower bound of the solution. Recall that the key step to prove Lemma 2.8.4 is that for $G \in SL(2,\mathbb{R})$ and $X \in \mathbb{C}^2$,

$$\max\{\|GX\|, \|G^{-1}X\|\} \geqslant \frac{1}{2}|\mathrm{Tr}G| \cdot \|X\|. \tag{8.2.27}$$

If $E \in S_1$, roughly speaking, we have a sequence of scales q_n such that the trace of A_{q_n} is large. Putting (8.2.10),(8.2.15),(8.2.22) and (8.2.27) together, we can show that there are integer sequences $x_{q_n} \to \infty$ *independent of* $E \in S_1$, such that

$$|u(x_{q_n})| > e^{q_n}, \tag{8.2.28}$$

where u solves the half-line problem $Hu = Eu$ with any boundary condition. The following extended Schnol's Theorem shows that such E must have spectral measure zero.

Lemma 8.2.29 (Extended Schnol's Theorem, [27])**.** *Fix any $y > 1/2$. For any fixed sequence $|x_k| \to \infty$, for spectrally a.e. E, there is a generalized eigenvector u of $Hu = Eu$, such that*

$$|u(x_k)| < C(1+|k|)^y.$$

For S_2, note that $A_q(E)$ is a polynomial in E with degree at most q. If the trace is close to 2, the following preimage estimate of a polynomial reduces the set in S_2 to several small intervals of width at most $e^{-5\Lambda q_n}$.

Lemma 8.2.30 ([25])**.** *Let $p \in \mathcal{P}_{n;n}(\mathbb{R})$ with $y_1 < \cdots < y_{n-1}$ the local extrema of p. Let*

$$\zeta(p) := \min_{1\leqslant j\leqslant n-1} |p(y_j)| \tag{8.2.31}$$

and $0 \leqslant a < b$. Then,

$$|p^{-1}(a,b)| \leqslant 2\mathit{diam}(z(p-a)) \max\left\{\frac{b-a}{\zeta(p)+a}, \left(\frac{b-a}{\zeta(p)+a}\right)^{\frac{1}{2}}\right\} \tag{8.2.32}$$

where $z(p)$ is the zero set of p and $|\cdot|$ denotes the Lebesgue measure.

The definition of m-function implies $\mu(E-\varepsilon, E+\varepsilon) \leqslant 2\varepsilon\, \mathrm{Im}M(E+i\varepsilon)$, where the right-hand side can be estimated again by subordinacy theory (Lemma 8.1.8) with the help of (8.2.24). Together, these ideas can be used to show that for β large enough, $\mu(\{E: \big||\mathrm{Tr}A_{q_n}| - 2\big| < e^{-10\Lambda q_n}\}) < e^{-\Lambda q_n}$. Then the Borel-Cantelli lemma immediately implies $\mu(S_2) = 0$.

In conclusion, we have the following key estimate for the trace of the transfer matrices.

Theorem 8.2.33. *For $\beta > 40\Lambda$ and μ a.e. E, there is $K(E)$ such that*

$$|\mathrm{Trace} A_{q_n}(E)| < 2 - e^{-10\Lambda q_n}, \quad n \geqslant K(E). \tag{8.2.34}$$

Combining this trace estimate with previous algebraic facts (8.2.15) and (8.2.24), one has

Lemma 8.2.35. *There is a sequence of positive integers $N_k \to \infty$ such that for $0 < \gamma < 1$, if*

$$\beta > \frac{100\Lambda}{1-\gamma}, \tag{8.2.36}$$

then

$$\sum_{n=1}^{N_k \cdot q_k} \|A_n(E)\|^2 \leqslant (N_k \cdot q_k)^{2-\gamma}, \quad k \geqslant K(E). \tag{8.2.37}$$

Now (8.1.11) follows from (2.6.3) for any boundary condition φ.

8.3. Arithmetic criteria In this part, we focus on the spectral singularity and the quantitative upper bound of $s(\mu)$. For simplicity, we only state and prove the following upper bound for the right half line AMO. The same result holds for general analytic potentials with positive Lyapunov exponent, which together with Theorem 8.2.7 will complete the proof of Theorem 8.2.3.

Theorem 8.3.1. *Let H be the AMO given as in* (2.7.1). *Assume that $\lambda > 1$. There exists $\varphi \in (-\pi/2, \pi/2]$ and an absolute constant $c > 0$ such that for any $\theta \in \mathbb{T}$ if $\beta(\alpha) < \infty$ then for the associated half line spectral measure μ, we have that*

$$s(\mu) \leqslant \frac{1}{1 + c/\beta} < 1. \tag{8.3.2}$$

Lemma 8.3.3. *For any E there is a n_0 such that for any $n > n_0$, there exists an interval $\Delta_n \subset \mathbb{T}$ satisfying*

$$\mathrm{Leb}(\Delta_n) \geqslant \frac{1}{8n}, \quad \inf_{\theta \in \Delta_n} \frac{1}{n} \ln \|A_n(\theta)\| > \frac{1}{4} \ln \lambda. \tag{8.3.4}$$

Moreover, for all q_n large (depending on n_0), for any θ, and any $N \in \mathbb{N}$, there is $j_N \in [2Nq_n, 2(N+1)q_n)$ such that

$$\|A_{j_N}(\theta, E)\| > e^{\frac{1}{36} q_n \ln \lambda}. \tag{8.3.5}$$

Lemma 8.3.6. *For any $E \in \mathbb{R}$ and $\beta = \beta(\alpha) < \infty$, there is a $\ell_0 = \ell_0(E, \beta)$ such that for $\ell > \ell_0$, and any $\theta \in \mathbb{T}$, the following holds:*

$$\sum_{k=1}^{\ell} \|A_k(\theta, E)\|^2 \geqslant \ell^{1 + \frac{2c}{\beta}}. \tag{8.3.7}$$

Proof of Theorem 8.3.1: For any φ, we have

$$\|u^{\varphi}\|_{\ell}^2 + \|v^{\varphi}\|_{\ell}^2 \geqslant \frac{1}{2} \sum_{k=1}^{\ell} \|A_k(\theta)\|^2. \tag{8.3.8}$$

Therefore, (8.3.7) implies that $\|u^{\varphi}\|_{\ell}^2 + \|v^{\varphi}\|_{\ell}^2 \geqslant \ell^{1+\frac{2c}{\beta}}$ for ℓ large.

On the other hand, Last and Simon showed in [38] that, for μ-a.e. E, there exist φ and $C = C(E) < \infty$, such that for large ℓ,

$$\|u^{\varphi}\|_{\ell} \leqslant C\ell^{1/2} \ln \ell. \tag{8.3.9}$$

Combining (8.3.8) and (8.3.9), we have

$$\|v^{\varphi}\|_{\ell} \geqslant \ell^{1/2+c/\beta} \tag{8.3.10}$$

provided $\beta < \infty$ and $\ell > \ell_0(E, \beta)$. For any $\varepsilon > 0$ (small), let $\ell = \ell(\varepsilon)$ be given as in (8.1.7). By (8.1.9), one has for any $\gamma \in (0, 1)$,

$$\begin{aligned}\varepsilon^{1-\gamma}|m_{\varphi}(E + i\varepsilon)| &\geqslant \frac{1}{\left(2\|u^{\varphi}\|_{\ell}\|v^{\varphi}\|_{\ell}\right)^{1-\gamma}} \cdot (5 - \sqrt{24}) \frac{\|v^{\varphi}\|_{\ell}}{\|u^{\varphi}\|_{\ell}} \\ &\geqslant c_{\gamma}\ell^{(1+c/\beta)\gamma-1} \cdot \ln^{-2} \ell\end{aligned}$$

where $c_{\gamma} > 0$ only depends on γ. Now let $\gamma_0 = \frac{1}{1+c/\beta} < 1$. We have for any $\gamma > \gamma_0$,

$$\varepsilon^{1-\gamma}|m_{\varphi}(E + i\varepsilon)| \geqslant c_{\gamma}\ell^{\gamma/\gamma_0-1} \cdot \ln^{-2} \ell \to \infty$$

as $\varepsilon \to 0$. Therefore, $s(\mu) \leqslant \gamma_0$, according to the definition (8.2.1). $\square$

References

[1] A Avila, *The absolutely continuous spectrum of the almost Mathieu operator* (2008), available at arXiv:0810.2965. ←11, 16

[2] A Avila, *Almost reducibility and absolute continuity I* (2010), available at arXiv:1006.0704. ←11

[3] Artur Avila, *Global theory of one-frequency Schrödinger operators*, Acta Math. **215** (2015), no. 1, 1–54. MR3413976 ←11

[4] Artur Avila and Svetlana Jitomirskaya, *The Ten Martini Problem*, Ann. of Math. (2) **170** (2009), no. 1, 303–342. MR2521117 ←3, 9, 11

[5] Artur Avila and Svetlana Jitomirskaya, *Almost localization and almost reducibility*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 1, 93–131. MR2578605 ←11

[6] A. Avila, S. Jitomirskaya, and C. A. Marx, *Spectral theory of extended Harper's model and a question by Erdös and Szekeres*, Invent. Math. **210** (2017), no. 1, 283–339. MR3698344 ←11

[7] Artur Avila and Raphaël Krikorian, *Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles*, Ann. of Math. (2) **164** (2006), no. 3, 911–940. MR2259248 ←11

[8] Artur Avila, Jiangong You, and Qi Zhou, *Sharp phase transitions for the almost Mathieu operator*, Duke Math. J. **166** (2017), no. 14, 2697–2718. MR3707287 ←9, 16

[9] Joseph Avron and Barry Simon, *Singular continuous spectrum for a class of almost periodic Jacobi matrices*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 1, 81–85. MR634437 ←9

[10] Joseph Avron and Barry Simon, *Almost periodic Schrödinger operators. II. The integrated density of states*, Duke Math. J. **50** (1983), no. 1, 369–391. MR700145 ←4

[11] M. Y. Azbel., *Energy spectrum of a conduction electron in a magnetic field*, Sov. Phys. JETP **19** (1964), no. 3, 634–645. ←3

[12] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*, Annals of Mathematics Studies, vol. 158, Princeton University Press, Princeton, NJ, 2005. MR2100420 ←5, 10

[13] J. Bourgain and S. Jitomirskaya, *Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential*, J. Statist. Phys. **108** (2002), no. 5-6, 1203–1218. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. ←10

[14] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, *Operators with singular continuous spectrum. IV. Hausdorff dimensions, rank one perturbations, and localization*, J. Anal. Math. **69** (1996), 153–200. MR1428099 ←30

[15] Alex Furman, *On the multiplicative ergodic theorem for uniquely ergodic systems*, Ann. Inst. H. Poincaré Probab. Statist. **33** (1997), no. 6, 797–815. MR1484541 ←10

[16] D. J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*, Proc. Roy. Soc. Edinburgh Sect. A **112** (1989), no. 3-4, 213–229. MR1014651 ←29
[17] D. J. Gilbert and D. B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), no. 1, 30–56. MR915965 ←29, 30
[18] A. Ja. Gordon, *The point spectrum of the one-dimensional Schrödinger operator*, Uspehi Mat. Nauk **31** (1976), no. 4(190), 257–258. MR0458247 ←9
[19] Rui Han and Svetlana Jitomirskaya, *Quantum dynamical bounds for ergodic potentials with underlying dynamics of zero topological entropy*, Anal. PDE **12** (2019), no. 4, 867–902. MR3869380 ←28
[20] Michael-R. Herman, *Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractére local d'un théoréme d'Arnold et de Moser sur le tore de dimension* 2, Comment. Math. Helv. **58** (1983), no. 3, 453–502. MR727713 ←10
[21] D. R. Hofstadter, *Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*, Phys. Rev. B. **14** (1976), no. 6, 2239. ←2
[22] K. Ishii., *Localization of eigenstates and transport phenomena in the one-dimensional disordered system*, Progress of Theoretical Physics Supplement **53** (1973), 77–138. ←7
[23] S Jitomirskaya, *On point spectrum of critical almost Mathieu operators*, Submitted (2019). ←11
[24] S. Jitomirskaya and W. Liu, *Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase* (2018), available at arXiv:1802.00781. ←3, 17
[25] S. Jitomirskaya and C. A. Marx, *Analytic quasi-periodic Schrödinger operators and rational frequency approximants*, Geom. Funct. Anal. **22** (2012), no. 5, 1407–1443. MR2989438 ←33
[26] S. Jitomirskaya and B. Simon, *Operators with singular continuous spectrum. III. Almost periodic Schrödinger operators*, Comm. Math. Phys. **165** (1994), no. 1, 201–205. MR1298948 ←4, 12
[27] S. Jitomirskaya and S. Zhang, *Quantitative continuity of singular continuous spectral measures and arithmetic criteria for quasiperiodic Schrödinger operators* (2015), available at arXiv:1510.07086. ←3, 30, 31, 33
[28] Svetlana Jitomirskaya and Ilya Kachkovskiy, L^2*-reducibility and localization for quasiperiodic operators*, Math. Res. Lett. **23** (2016), no. 2, 431–444. MR3512893 ←16
[29] Svetlana Jitomirskaya and Yoram Last, *Power-law subordinacy and singular spectra. I. Half-line operators*, Acta Math. **183** (1999), no. 2, 171–189. MR1738043 ←28, 29
[30] Svetlana Jitomirskaya and Wencai Liu, *Universal hierarchical structure of quasiperiodic eigenfunctions*, Ann. of Math. (2) **187** (2018), no. 3, 721–776. MR3779957 ←3, 12, 16, 27
[31] Svetlana Jitomirskaya and Rajinder Mavi, *Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials*, Int. Math. Res. Not. IMRN **1** (2017), 96–120. MR3632099 ←28
[32] Svetlana Ya. Jitomirskaya, *Anderson localization for the almost Mathieu equation: a nonperturbative proof*, Comm. Math. Phys. **165** (1994), no. 1, 49–57. MR1298941 ←3, 5
[33] Svetlana Ya. Jitomirskaya, *Almost everything about the almost Mathieu operator. II*, XIth International Congress of Mathematical Physics (Paris, 1994), 1995, pp. 373–382. MR1370694 ←14, 16, 17
[34] Svetlana Ya. Jitomirskaya, *Metal-insulator transition for the almost Mathieu operator*, Ann. of Math. (2) **150** (1999), no. 3, 1159–1175. MR1740982 ←3, 5, 12
[35] Shinichi Kotani, *Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators*, Stochastic analysis (Katata/Kyoto, 1982), 1984, pp. 225–247. MR780760 ←7
[36] Herve Kunz and Bernard Souillard, *Sur le spectre des opérateurs aux différences finies aléatoires*, Comm. Math. Phys. **78** (1980/81), no. 2, 201–246. MR597748 ←4
[37] Y. Last, *Zero measure spectrum for the almost Mathieu operator*, Comm. Math. Phys. **164** (1994), no. 2, 421–432. MR1289331 ←11
[38] Yoram Last and Barry Simon, *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, Invent. Math. **135** (1999), no. 2, 329–367. MR1666767 ←4, 35
[39] W Liu, *Almost Mathieu operators with with completely resonant phases*, Ergodic Theory Dynam. Systems (to appear). ←12
[40] Wencai Liu and Xiaoping Yuan, *Anderson localization for the completely resonant phases*, J. Funct. Anal. **268** (2015), no. 3, 732–747. MR3292353 ←12
[41] L. A. Pastur, *Spectral properties of disordered systems in the one-body approximation*, Comm. Math. Phys. **75** (1980), no. 2, 179–196. MR582507 ←4, 7
[42] P. Roushan, C. Neill, J. Tangpanitanon, V.M. Bastidas, A. Megrant, R. Barends, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, A. Fowler, B. Foxen, M. Giustina, E. Jeffrey, J. Kelly, E. Lucero, J. Mutus, M. Neeley, C. Quintana, D. Sank, A. Vainsencher, J. Wenner, T. White, H. Neven, D. G. Angelakis,

and J. Martinis, *Spectral signatures of many-body localization with interacting photons*, Science (2017). ←2

[43] Barry Simon, *Kotani theory for one-dimensional stochastic Jacobi matrices*, Comm. Math. Phys. **89** (1983), no. 2, 227–234. MR709464 ←7

[44] Barry Simon, *Equilibrium measures and capacities in spectral theory*, Inverse Probl. Imaging **1** (2007), no. 4, 713–772. MR2350223 ←28, 31

[45] Gerald Teschl, *Mathematical methods in quantum mechanics*, Second, Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014. With applications to Schrödinger operators. MR3243083 ←29

[46] Fan Yang and Shiwen Zhang, *Singular continuous spectrum and generic full spectral/packing dimension for unbounded quasiperiodic Schrödinger operators*, Ann. Henri Poincaré **20** (2019), no. 7, 2481–2494. MR3962852 ←31

Department of Mathematics, University of California, Irvine, California 92697-3875, USA
Email address: szhitomi@math.uci.edu

Department of Mathematics, University of California, Irvine, California 92697-3875, USA

Current address: Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
Email address: liuwencai1226@gmail.com

Department of Mathematics, University of Minnesota, Minneapolis, MN, 55455
Email address: zhan7294@umn.edu