Four Tasks to Catalyze Change: Perimeter, Area, Surface Area, and Volume

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For some, finding new ways to communicate mathematics recently became an imperative as teachers pivoted their teaching pedagogies during the pandemic. Simultaneously, the march continues to ensure the highest quality mathematical learning experiences for all students. *Catalyzing Change in Middle School Mathematics* (NCTM 2020), one of three recent books for initiating critical conversations on policies, practices, and issues in mathematics education, reminds us of a shared belief: "Each and every student is capable of engaging in rigorous and challenging mathematics … if provided access to a high-quality mathematics program" (p. 1). At the same time, "disparities in learning opportunities based on race, class, language, gender, and perceived mathematical ability are far too prevalent in school mathematics in North America." We must address these disparities.

The four key recommendations needed to create high-quality middle school mathematics described in *Catalyzing Change* include:

- ✔ **Broaden the purposes of learning mathematics:** Each and every student should develop a deep mathematical understanding, understand and critique the world through mathematics, and experience the wonder, joy, and beauty of mathematics, which all contribute to a positive mathematical identity.
- ✔ **Create equitable structures in mathematics:** Middle school mathematics should dismantle inequitable structures, including tracking teachers and the practice of ability grouping and tracking students into qualitatively different courses.
- ✔ **Implement equitable mathematics instruction:** Mathematics instruction should be consistent with research-informed and equitable teaching practices that foster students' positive mathematical identities and a strong sense of agency.
- ✔ **Develop a deep mathematical understanding:** Middle schools should offer a common shared pathway grounded in the use of mathematical practices and processes to coherently develop deep mathematical understanding, ensuring the highest quality mathematics education for each and every student (pp. 5–6).

In this article, I describe rich geometry tasks that can be used to connect multiple representations of mathematics using several technologies. The NCTM *Principles to Actions: Ensuring Mathematical Success for All* (2014) document lists using and connecting mathematical representations as one of their eight effective mathematics teaching practices.

Through visualizing, building, reasoning, and arguing, students can use words, numbers, pictures, and contexts when learning (Lesh, Post, and Behr 1987). Doing so allows a student to develop what Huinker (2015) calls *representational competence*, which involves working successfully with different representations (***Figure 1***).

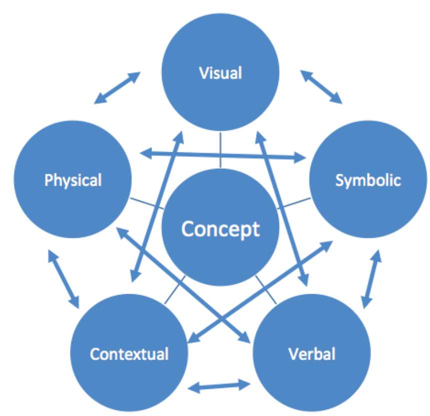

Figure 1: Mathematical representations and their important connections

For example, the concept of area might include: a diagram showing an 8 inch by 12 inch rectangle, the context of designing a quilt, the symbols in the expression 8 x 12, the spoken words "eight rows with 12 square inches per row," and the physical array of 8 rows with 12 square tiles per row.

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As Smith, diSessa, and Roschelle (1993) argued, we can reconceive misconceptions, which they define as student conceptions that produce a systematic pattern of errors. Rather than think that students have misconceptions, we can reframe it as students share ideas about mathematics through their own experiences before learning expert concepts. Holding such a view assumes that student conceptions play an important and productive role in developing their understandings. By addressing the difference between students' conceptions and expert conceptions, students will see the advantages and efficiency of the expert way over their initial ways of thinking.

The following four tasks illustrate concrete examples of ways we can provide students with quality geometry learning experiences. There is also discussion of tips for implementation. I use Google Slides to communicate mathematics with pre-service teachers in a mathematics content course focused on statistics and geometry. I modeled my process on Theresa Wills' experiences teaching mathematics online (AMTE 2020; Wills 2020). While tasks were developed in a course for future elementary teachers at a university course taught synchronously via distance-learning, teachers can adapt tasks for elementary, middle, or high school settings.

Task 1: Katherine and Dorothy

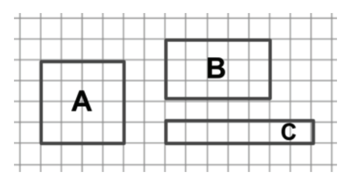
We began with the concept of perimeter and area of rectangles. Students worked in virtual groups, shared audio in real-time, and used their Google Slides to communicate a solution. Reflect on the following task before continuing:

> **Task 1:** *Katherine says, "given the perimeter of a rectangle, we can determine its area." Dorothy disagrees, and says, "there are many possible areas for a given perimeter of a rectangle."*
>
> *The answer to their debate matters because they plan to sew borders on different sized quilts and have limited resources.*
>
> ***Who is correct, Katherine or Dorothy? Explain in a way that helps both understand what is going on geometrically. Justify your answer using graph paper, numbers, words, and technology.***

Katherine believes that, given the perimeter of a rectangle, say 24 inches, we can determine the area of that same rectangle. An important word in the question is "the," which suggests that there is only one corresponding area.

Dorothy disagrees. Maybe Dorothy is thinking that a 10'' by 2'' rectangle has a perimeter of 24 inches, and so does a ½'' and 48'' rectangle. By inviting students into the Katherine and Dorothy debate, students explore several mathematical representations. Since testing several scenarios is a problem-solving technique, students might begin by selecting a special perimeter. I chose 24 inches because 24 has many factors (1, 2, 3, 4, 6, 8, 12, 24). By reasoning about the area and perimeter using graph paper, students might type the following in a text-box and create the following three rectangles on their Google Slide:

> **Dorothy is Correct:** Rectangles can have the same perimeters but have different areas. These three rectangles each have perimeters of 16 units. When finding the area of each rectangle, we get different values: 16 square units for rectangle A, 15 square units for rectangle B, and 7 square units for rectangle C.



The stumbling point intentionally targeted here is a that some students believe a perimeter identifies the dimensions of the rectangle. Students need to recognize that a perimeter is a single value that describes the distance around the outside length of a shape. Dimensions, on the other hand, refer to quantities such as length, height, and width, which do not directly correlate to the perimeter. A student who assumes that a perimeter of 24 units is the same as the rectangle with dimensions 10 units by 2 units is missing that there are other rectangles possible. Students with these beliefs might write the following:

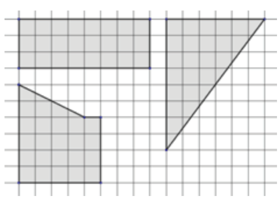
> **Katherine is Correct:** Given the perimeter of the rectangle, we can find the length and width of all four sides. To find an area of a rectangle it is LW = area. So area = $a \cdot b$ and perimeter = $2 \cdot (a + b)$.

This group's incorrect explanation assumes that perimeter means dimensions. The students showed an understanding of how to find the perimeter of a rectangle. Yet the deep understanding of this concept involves recognizing that there are an infinite number of rectangles for any given perimeter. Consequently, when the teacher finds students holding this concept, they can ask, "Is the perimeter of a shape the same thing as its dimensions?"

To challenge students further, the teacher could swap the rectangle for a trapezoid or six-sided polygon that can be decomposed into shapes and recombined to find perimeter and area.

Task 2: Twenty-four Unit Squares

The next task involves flipping a traditional area question on its head. Rather than provide the shape and ask students to determine the area, we provide the area and students determine a shape. The task is adapted from Illustrative Mathematics (**https://tasks.illustrativemathematics.org/content-standards/6/G/A/1/tasks/2131**). Take a moment to play around with some potential shapes before reading on.

> **Task 2:** *The area of each shape shown below is 24 square units. On grid-paper, find a way to draw more figures with an area of 24 square units.*
>
> *Other than the shapes shown, can you draw the following shapes?*
>
> 
>
> - ✔ *A different rectangle?*
> - ✔ *A different polygon with more than 4 sides?*
> - ✔ *A different right triangle?*
> - ✔ *A different parallelogram?*
> - ✔ *A wild and creative shape of your own design?*
>
> *For each figure you draw, convince us how you know its area is exactly 24 square units!*

Note that this problem could be asked in a more open-ended manner: "Draw as many different shapes as you can that have 24 square units. Convince us how you know each shape has an area of 24 squares."

Like the first task, this task was completed in small virtual groups who shared audio in real-time while problem solving with Google Slides. Students used the shape feature to create polygons on graph paper, which can be copied and pasted on a slide or inserted as the background of a slide.

Through this task, students' conceptions of an area are addressed and developed. The task requires students to decompose a 2-dimensional shape into smaller familiar shapes, find the corresponding areas, then add them all together. Decomposing and recomposing is a familiar idea learned through place value: 1,234 can be decomposed into 1 group of 1,000 plus 2 groups of 100 plus 3 groups of 10 plus 4 groups of 1. When adding 79 to 1,234, a student recomposes the 9 ones and 4 ones into one group of 10 and one group of 3 ones. Decomposing shapes builds on a familiar idea of mathematics in a connected way.

Students might reflect that "breaking down shapes helps me understand how to design my own shapes with certain areas," "I didn't know that you can find the area without a formula," or "there are so many ways to find area rather than just memorize a formula." By building their shapes, students understand the concept of an area rather than memorize a formula without meaning.

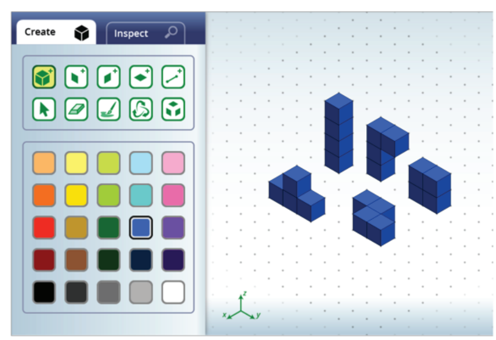


Figure 2: Screenshot from the NCTM Isometric Drawing Tool showing objects with a volume of 4 cubes

The next two tasks make use of a website-based tool (***Figure 2***) created by NCTM called the Isometric Drawing Tool (**www.nctm.org/Classroom-Resources/Illuminations/Interactives/Isometric-Drawing-Tool/**). Just like with any tool, students will need time to explore before using the tool in the following two tasks. I encourage you to navigate to the website and explore the tool. The ten tools in the top left section of the screen include: Create Cube, Create *X*-Face, Create *Y*-Face, Create *Z*-Face, Create Line, Pointer, Eraser, Paint, Rotate, and Explode. Together these tools assist students in visualizing, building, and moving objects around virtually.

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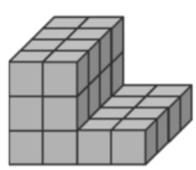
As a warm-up, ask each student to create an object using only four connected blocks (i. e., with a volume of four cubes.) The blocks must be connected by combining faces, rather than edges (see ***Figure 2*** on page 27). These arrangements are similar to two-dimensional tetrominoes. Students can then try to make an object using only five connected blocks.

Task 3: Volume of an L-based Prism

A prism is a solid figure that has two opposite and parallel bases with the lateral sides all forming parallelograms, we name the prism by the characteristics of its base. I like to physically hold a prism and touch the two opposite and parallel bases while describing a prism. Doing so helps to connect the new vocabulary words to a physical object.

The third task can be completed with students in small virtual groups or individually. Students communicate their mathematics using text-boxes and screenshots from NCTM's Isometric Drawing Tool website while speaking mathematics together. Reflect on the following task before continuing. (This may involve navigating to the website to play around with the tool!)

> **Task 3:** *Josy is finding the volume of the L-based prism. She broke the prism into two rectangular prisms. Josy found the volume using this expression: (4 x 4 x 1) + (2 x 4 x 2).*
>
> *Charlie also broke the same L-based prism into two rectangular prisms, but differently than Josy. She found the volume using this expression: (2 x 4 x 3) + (2 x 4 x 1).*
>
> 
>
> ***Who is correct?***

Students use the Drawing Tool to recreate the decomposition method for both Josy and Charlie (***Figure 3***). This tells us that Josy's volume is 24 cubes plus 8 cubes, while Charlie's volume is 16 cubes plus 16 cubes. This task supports multiple correct methods and consequently supports students who may be in different places. It sends the message that there are many ways to promote a deep understanding of volume, which contrasts with memorizing the formula for the volume of a prism. A task like this builds a student's understanding of volume as counting layers and the number of cubes per layer (see ***Figure 3***).

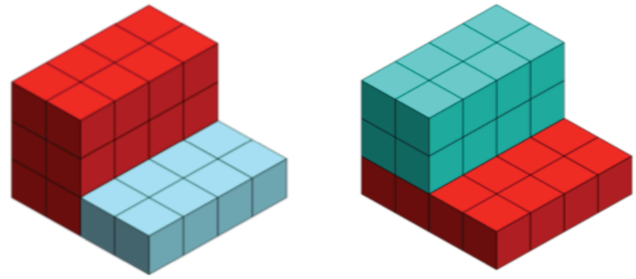


Figure 3: Josy's method of decomposing the L-based prism (on the left) and Charlie's method (on the right)

When students work with measurement problems, it is important to reiterate what unit we use to measure. While solving problems involving the perimeter, the unit of measure is a one-dimensional unit. Area uses a two-dimensional unit, the standard square unit. Volume measurements require a three-dimensional unit, the standard cubic unit. We can say that the volume is 32 cubes, or equivalently, 32 cubic units. (You might hear students say "units cubed" or "cubed units," which may result in reading left-to-right the symbols 32 units^3).

Cubic units can be translated more generally into cubic inches, cubic centimeters, cubic feet, etc. Whether students choose to say cubes, cubic units, or units cubed, they must understand that volume measures the space within a container with a standard unit of measure, rather than area or length. Together, these tips will help prepare teachers to facilitate the task successfully.

Task 4: Priya's Prisms

The last task involves thinking about the outside area that covers an object, called the surface area. Contexts such as wrapping presents, painting all faces of the object, and thinking about cereal boxes serve as important connections. Additionally, surface area connects to area of two-dimensional shapes, a strong springboard for students. While surface area can be challenging to understand at first, due to the need to visualize what exactly it is they are counting, instructors can anticipate this and design a task to uncover important ideas of surface area.

Before presenting the task, I provided a brief warm-up to the concept of surface area.

Surface areas can be discussed using nets, which are two-dimensional figures or unfoldings that can be folded into a three-dimensional object. I think about the cardboard cereal box that can be folded into a prism. I like using NCTM's interactive tool called "Cube Nets" to visually warm-up students folding shapes into cubes (NCTM 2020)

The question posed on this webpage (see ***Figure 4***) is, "Which of the nets below will form a cube?"

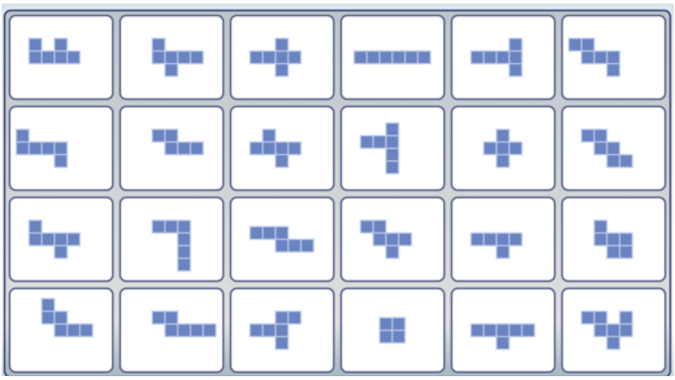


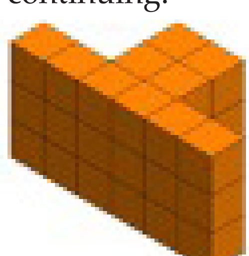
Figure 4: Twenty four interactive nets of 3-D shapes, only some of which can be folded to form a cube

The interactive tool then allows the user to click on a net to see if their prediction on whether it will fold into a cube is correct (see ***Figure 5***).

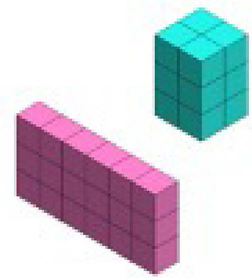
Figure 5: The unfolded picture on the left predicting a net and the folded picture on the right showing what happened after the user selected "yes"

Task 4 unpacks the idea of decomposing surface areas and the "hidden faces" that result when recomposing the objects. Reflect on the following task before continuing:

Task 4: *Priya is finding the surface area of the orange T-based prism (see right).*



Priya broke it into the teal and purple prisms (see right). The justification was:



- ✔ *The surface area of the teal prism is:* (4 x 6) + (2 x 4) = 32 *square units*
- ✔ *The surface area of the pink prism is:* (2 x 6) + (2 x 18) + (2 x 3) = 54 *square units*
- ✔ 38 *square units* + 54 *square units* = 86 *square units*

Is Priya Correct?

Some students may reason, "just like with two-dimensional shapes, where I decompose into smaller chunks, I can decompose the object into components and find the surface area of the individual components so I can, at last, add them together."

To address Priya's method, a student would have to consider that some of the pink and teal faces disappear when reconnecting the prisms to form the original object. That is, Priya would need to account for the 6 square units on the teal prism and the 6 square units on the purple prism that become the interior of the shape. A total of 12 square units must be subtracted from 86 square units to arrive at the correct surface area of 74 square units. Building productively on this idea involves subtracting any "hidden faces," which accounts for faces that were visible on the decomposition but become invisible when recomposed to the original object.

Connections between decompositions of two-dimensional shapes and three-dimensional objects can be made (see ***Figure 6***).

	Decompose and Recompose	Decompose, Recompose, and Compensate
2-D	I can decompose a 2-D shape to find sub-areas, then combine it to make one larger area.	I can decompose a 2-D shape to find sub-perimeters, but I need to think about the perimeter of the final shape when I recombine them.
3-D	I can decompose a 3-D object to find subvolumes, then combine it to make one larger object.	I can decompose a 3-D object to find subsurface areas, but I need to think about the surface area of the final object when I recombine them.

Figure 6

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Students can be challenged to generalize a way to find the surface area of any prism. This involves separating the prism's surface into two bases and a large rectangle. This rectangle results from the lateral sides, and has a height of however tall the prism is, and a length equal to the perimeter of the base. In our case, the perimeter of the base is 18 units, the height of the prism is 3 units, and the surface area of both bases combined is 20 square units, which together gives a surface area of 18 x 3 + 20 = 54 + 20, or 74 square units.

Conclusion

I once worked with several teachers from Southern California in a university and district partnership on a Math and Science Partnership three-year grant. Teachers from grades 3 through high school used lesson study and week-long summer institutes to learn about algebraic reasoning in the mid-2010s. After the grant ended in 2015, fifth-grade teacher, Kerry (pseudonym), contrasted how she formerly introduced the concept of volume to how she currently introduces the concept to her fifth-graders.

> *When I teach volume, I used to just pass out unifix cubes of boxes and have students estimate what they thought and then figure out what they got. But this time, I asked them if they were seeing any patterns. I asked them, "How tall is the box in cubes? How wide is the box? How many could fit there?" And I asked them to record that this time, along with their prediction and their actual count. And everybody saw the relationship. It was so neat! Because they had each measured different boxes of different sizes. They were seeing it – "oh, those numbers! Those numbers! We use those numbers to get this. We can predict how many blocks we're going to need because of how high it is, how long it is, how wide it is. That's gonna give me the answer!" Connecting the activity while developing the formula was powerful.*
>
> *I feel bad for all those years I didn't know those things that you thought kids understood. And all they were doing was manipulating numbers without meaning.*
>
> *And I mean, you should have seen their faces. The smile on their faces when they figured it out. They have as long as they needed to work with those materials [unifix cubes] until that relationship became clear to them. Because kids can do that! They can see a relationship. They can put things together. And I never give them formulas anymore. I say now, "Can you think of a rule? What rule could we write to figure out the volume of a box?" And they could figure it out.*

With physical limitations on the number of wooden cubes, along with a desire to improve inefficient methods, her students were motivated to determine a more efficient method. In reality, her students were thinking like mathematicians. Kerry had created rich opportunities for students to experience the wonder of mathematics. When students build their personal algorithms and methods for finding volume, it stays with them. As Kerry said,

> *Now it's [the understanding] theirs. They're never going to forget it. They already know it. So that's one of my great joys. That's why I love being a teacher. If I can give students the tools for them to be successful, but they develop the understanding through their own work and activity, then it's theirs. It's not something that they'll ever forget.*

Supporting the highest quality of mathematical learning experiences for all students involves stakeholders at many levels, from teachers to principals, to teacher educators. In this article, I introduced tasks that teachers can use to represent mathematics in multiple ways while using technology. Each of the four problems provides students with rich opportunities to engage deeply with multiple representations of mathematics and develop their representational competence. Doing so allows teachers to target foundational ideas of perimeter, area, volume, and surface area.

Several of the problems suggest a debate between two students, which models the practice of constructing, critiquing, and justifying valid arguments—important mathematical practices. Students benefit from using technology to communicate solutions. By engaging in effective mathematical teaching practices, such as implementing tasks that promote reasoning and problem solving, facilitating meaningful discussions, posing good questions, and supporting students in their sense-making activities and productive struggle, we can continue to achieve high quality mathematical learning for all students.

Together, we can support the purposes of learning mathematics to understand deeply, critique the world through mathematics, and experience the joy of mathematics. By creating a deeper understanding of mathematics, students become empowered to reason mathematically with their new set of tools. Through supported use of technology such as Google lides, NCTM's Isometric Drawing Tool, and shared audio in real-time, students can make strong mathematical connections. As Kerry noted, if we can give students tools for success, students will develop and refine their understandings as teachers address clear learning goals.

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Bring the Fun to Geometry

by Daniel Kline, Executive Director, Julia Robinson Mathematics Festival
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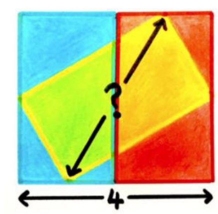

The Julia Robinson Mathematics Festival (JRMF) recently released a booklet featuring geometry puzzles created and designed by Catriona Agg (www.jrmf.org/booklets), a secondary school teacher in Essex, England. Inspired by Ed Southall's "Geometry Snacks," she began drawing her own puzzles, posting them on Twitter starting in August 2018 and quickly gained a following. She now has over 24,800 followers, and her puzzles generate lively discussions on multiple ways to solve each one. She loves to see how people solve the puzzles through the comments, photos, and even videos they post, and how they interact regarding the solutions.

Catriona's puzzles connect with the broader JRMF philosophy of encouraging collaboration and non-competitive mathematical explorations. In many K–12 mathematics classrooms, geometry is taught in a way that makes heavy use of memorization, making it difficult to play with geometric concepts. Contrarily, Catriona's puzzles make geometry accessible, malleable, and something that is fun to play with.

Ever since Euclid's *Elements*, geometry in the western world has often been taught through axiomatic constructions and abstract generalizations. Because of this, it is not clear for many students why they should care about learning geometry. Although one way to inspire motivation for students is to make mathematics more relatable and more "real-world," Catriona's take on geometry inspires motivation by providing students the opportunity to play and have fun with geometry.

The booklet we created with Catriona specifically focuses on activities that allow students to play with geometric transformations. Armed only with the tools of cutting, flipping, and rearranging, students are tasked to turn ugly shapes into beautifully simple squares, calculate unusual areas, and in general, create order from chaos. Unlike many of our experiences with K–12 geometry, students don't just memorize what a reflection is, they are able to do reflections. They don't just learn geometric theorems, they exert their creative powers on the objects of those theorems. Catriona's puzzles transform standard exercises in translations, rotations, and reflections into joyful, meaningful, memorable learning experiences.

What we love about Catriona's puzzles is

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